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Stress Analysis of Shrink-Fit Cylinders Under Torsion*

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In this paper, the stress distributions and displacements arising in a shrink-fit cylinder under torsion are analyzed by applying the point-matching method, which is very useful to analyse elastic contact problems. It is assumed that an infinite solid cylinder is shrink on to a finite hollow cylinder and subjected to torsion at $Z=\infty$, while the outer surface of this finite hollow cylinder is free or fixed by a rigid body.

The calculations are carried out for the following two cases:

- (i) The condition of the shrink-fit surface is complete adherence.
- (ii) The certain portions of the surface adhere, whereas other slide over one another.

The numerical results are obtained for models in the various values of the modulus of elasticity in shear.

1. Introduction

The problems of shrink-fit are found in an assembly of many practical members such as crankshaft, fly wheel and bushing, and have very important influences to increase their fatigue strength. Many studies to analyze the contact stresses of the shaft induced by shrink-fit have been made, in an experimental or a theoretical field. The author also reported in the previous paper a theoretical method to find the shrink-fit stresses and the results⁽¹⁾. In the practical cases, a bending or torsional load is superposed on the shrink-fit stresses and studies on this problem are very few except that Saito^(2,3) and Erdogan⁽⁴⁾ analysed the simplified models under the slipless condition.

In this paper, a stress analysis of a shrink-fit cylinder under torsion is made by applying the point-matching method⁽⁵⁾ and this analytical technique includes a case of the partial slip condition. A numerical calculation is performed on models in which an infinite solid cylinder shrunk on to a finite hollow cylinder is subjected to torsion at $Z=\infty$, while the outer surface of this hollow cylinder is free or fixed by a rigid body (as shown in Fig.1)

2. Theoretical analysis

The stresses in a shrink-fit cylinder under torsion are due to the effect of a fitted load and that of a torsional

moment. On the former effect, a theoretical method was already described in the previous paper, then here the technique to analyse the stresses induced in a shaft subjected only to torsional moment M at $Z=\infty$ as shown in Fig.1 will be analyzed in detail.

Figure 2 shows the co-ordinates and the stress notations. Assuming that the shaft in Fig.1 is subjected to M without the restriction of a boss, an angle of twist per unit length of this shaft is obtained as follows:

$$\varphi = M/G_I I_p \dots\dots\dots (1)$$

where G_I is the modulus in shear of the shaft and I_p is given by $\pi a^4/2$ as the polar moment of inertia.

Next, using Eq.(1) a circumferential displacement at $r=a$ of the shaft is

(a). No.1

(b) No.2

Fig.1 Shrink-fit cylinders under torsion

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$$V(z) = \varphi a z = 2zM / \pi a^3 G_I \quad \dots\dots\dots (2)$$

Fig. 2 Coordinates and stress notation

(a) Shrink-fitted shaft

(b) Contact area

Fig. 3 Surface displacement of shaft

(a) Shaft

(b) Boss

Fig. 4 Shaft and boss subjected to a band of uniform shear stress

But, when the shaft is shrunk by a boss this displacement decreases to $(V - v_i)$ from V owing to a restriction of the boss. On the other hand the inner surface of the boss deforms to v_i as shown in Fig. 3.

These stress distributions on the displacement are induced by the shear stress $\tau_{r\theta}$ at the contact surface $r=a$, and to obtain this stress the point-matching method is utilized as follows.

Now, assuming that a shaft is loaded by a contact pressure τ_i of an incremental length as shown in Fig. 4 (a) and the displacement at $r=a$ of this shaft is given by v_i , many incremental stresses $\tau_1, \dots, \tau_m$ will make the following surface displacement.

$$\sum_{i=1}^m v_i(h_j) \quad \dots\dots\dots (3)$$

The surface displacement of boss is also given as

$$\sum_{i=1}^m v_{II}(h_j) \quad \dots\dots\dots (4)$$

Next, considering a slipless condition at the contact surface, Eqs. (2), (3) and (4) must be related as follows:

$$\sum_{i=1}^m v_i(h_j) - \sum_{i=1}^m v_{II}(h_j) = V(h_j) \quad \dots\dots\dots (5)$$

As the relation of Eq. 5 is obtained at all the lines of $Z=h_1, \dots, h_m$, a series of simultaneous equations in the incremental pressure τ_i are derived. Hence the contact stress distribution will be determined.

As another condition, we consider the partial slip condition that the slipping is allowed at a contact surface. It is obvious that the slip will begin at the edge of contact parts as soon as $\tau_{r\theta}$ reaches a constant stress τ_c . Now, assuming that the condition of n parts near $Z=0$ as shown in Fig. 5 is complete adherence and on the other hand $(m-n)$ parts slide over one another, the following equation on $\tau_{r\theta}$ at the slipping parts will be obtained.

$$\tau_{n+1} = \tau_{n+2} = \dots = \tau_m = \tau_c \quad \dots\dots\dots (6)$$

And the unknown stresses $\tau_1, \dots, \tau_n$ can be obtained from the following equation.

$$\sum_{i=1}^n v_i(h_j) - \sum_{i=1}^n v_{II}(h_j) = V(h_j) - \left\{ \sum_{i=n+1}^m v_i(h_j) - \sum_{i=n+1}^m v_{II}(h_j) \right\} \quad \dots\dots\dots (7)$$

To determinate n in Eqs. (6) and (7), the point-matching method must be used by applying the trial and error method as follows: as the first step the stresses $\tau_1, \dots, \tau_m$ for the slipless condition are calculated and from these results the parts of $\tau_i > \tau_c$ are determined as the slipped parts in the second step, thus the calculations are repeated and the stress distribution will approach an exact value.

3. Numerical calculations

3.1 On the model No.1 The first model of numerical examples is No.1 in which the surface $r=b$ of a boss as shown in Fig. 1 (a) is free. To obtain $\tau_{r\theta}$ and v for this model, the form of $v_i(z)$ and $v_{II}(z)$ in Eqs. (5) and (7) must be determined first and then the point-

Fig.5 Shaft under partial slip condition

matching method is utilized.

3.1.1 Equations of displacements $v_{r1}(z)$ and $v_{\pi1}(z)$. From Fig.4 (a), the boundary condition at $r=a$ of the shaft is given by using the Fourier integral.

$$\tau_{r\theta} = \frac{2}{\pi} \int_0^\infty F(\alpha) \sin \alpha z d\alpha \quad (8)$$

where $F(\alpha) = -\tau_i (\cos \alpha(e_i + d_i) - \cos \alpha e_i) / \alpha$
Denoting now a displacement function as

$$G_1 v = \frac{2}{\pi} \int_0^\infty A(\alpha) I_1(\alpha r) \sin \alpha z d\alpha \quad (9)$$

v will satisfy the following equilibrium equation: $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0$ (10)

The relations between the stresses and the displacement are

$$\tau_{r\theta} = G_1 r \frac{\partial}{\partial r} \left(\frac{v}{r} \right), \quad \tau_{\theta z} = G_1 \frac{\partial v}{\partial z} \quad (11)$$

Substituting Eq. (9) in Eq. (11) and using Eq. (8), the function $v_{11}(z)$ is given in the following form.

$$G_1 v_{11}(z) = \frac{2}{\pi} \int_0^\infty \frac{I_1(\alpha a)}{\alpha I_2(\alpha a)} F(\alpha) \sin \alpha z d\alpha \quad (12)$$

On the other hand, the boundary condition of the boss is given by using the Fourier series.

$$\left. \begin{aligned} \tau_{r\theta} &= \sum_{s=1}^{\infty} a_s \sin \beta_s z & \text{at } r=a \\ \tau_{r\theta} &= 0 & \text{at } r=b \\ \tau_{\theta z} &= 0 & \text{at } z=\pm h \end{aligned} \right\} \quad (13)$$

where $\beta_s = (2s-1)\pi/2h$

$$a_s = -2\tau_i (\cos \beta_s(e_i + d_i) - \cos \beta_s e_i) / \beta_s h$$

And also denoting the displacement function as

$$G_1 v = \sum_{s=1}^{\infty} \frac{1}{\beta_s} \{A_s' I_1(\beta_s r) + B_s' K_1(\beta_s r)\} \sin \beta_s z \quad (14)$$

the forms of $\tau_{\theta z}$ and $\tau_{r\theta}$ will be determined from Eq. (11).

$$\left. \begin{aligned} \tau_{\theta z} &= \sum_{s=1}^{\infty} \{A_s' I_1(\beta_s r) + B_s' K_1(\beta_s r)\} \cos \beta_s z \\ \tau_{r\theta} &= \sum_{s=1}^{\infty} \{A_s' I_2(\beta_s r) - B_s' K_2(\beta_s r)\} \sin \beta_s z \end{aligned} \right\} \quad (15)$$

Comparing Eq. (15) with Eq. (13), it is easy to see that the condition of $\tau_{\theta z} = 0$ is already satisfied and the following relation is derived from the condition of $\tau_{r\theta}$.

$$\left. \begin{aligned} A_s' I_2(\beta_s a) - B_s' K_2(\beta_s a) &= a_s \\ A_s' I_2(\beta_s b) - B_s' K_2(\beta_s b) &= 0 \end{aligned} \right\} \quad (16)$$

Hence, the displacement function $v_{\pi1}(z)$ is

$$G_{\pi} v_{\pi1}(z) = \sum_{s=1}^{\infty} \frac{a_s \{I_1(\beta_s a) K_2(\beta_s b) + I_2(\beta_s b) K_1(\beta_s a)\}}{\beta_s \{I_2(\beta_s a) K_2(\beta_s b) - I_2(\beta_s b) K_2(\beta_s a)\}} \sin \beta_s z \quad (17)$$

In the above mentioned equations, $I_\mu(\beta_s r)$ and $K_\mu(\beta_s r)$ are the modified Bessel functions of first and second, respectively.

3.1.2 Contact stresses and displacements In application of the point-matching method by using Eqs. (5) and (7), the number m that divides the contact surface in to equal increments affects the accuracy of $\tau_{r\theta}$ and v . Hence the calculation for the model of $b/a = 2.0$ and $h/a = 1.0$ was carried out for $m=5, 10$ and 20 , the results being shown in Figs. 6 and 7. From these figures, it is known that v_{π} converges better than $\tau_{r\theta}$ but the values of $\tau_{r\theta}$ for $m=10$ are nearly exact. Also it is considered for the calculation that the stress concentrated part such as $z/h = 1.0$ ought to be divided especially finely. But here the equal increments division is used as $m=20$, because the problem contains the partial slip condition.

Now, the results on the slipless condition are shown in Figs. 8 to 10 for $\tau_{r\theta}$ and in Figs. 11 to 13 for $(V - v_{\pi})$. In these figures the transverse axis is z/h , the longitudinal axis is $\tau_{r\theta} / (\frac{2M}{\pi a^2})$ or $(V - v_{\pi}) / (\frac{2M}{\pi a^2})$ and the parameter is a_i / G_1 as a ratio of the rigidity of shaft and boss. In the figures, it is obvious that the stress $\tau_{r\theta}$ increases quickly in the part near $z/h = 1.0$. This tendency appears extremely with a decrease of a_i / G_1 . On the other hand, from the distributions of $(V - v_{\pi})$ it is recognized that the surface of shaft deforms almost in linear form except the part near $z/h = 1.0$.

Next, $\tau_{r\theta}$ and $(V - v_{\pi})$ under the partial slip condition are shown in Figs. 14 and 15. On account of that the critical shear stress τ_c , which is constant at all parts of contact surface, not being determined theoretically, the numerical results are calculated for the several values of τ_c as shown in the figures. From Fig. 14 it is known that the slip area increases with a decrease of τ_c and the stress distribution at the no slip area is nearly similar to Fig. 9. Figure 15 shows that $(V - v_{\pi})$ distributes linearly in the slip area.

3.2 On the model No.2 Here, the model No.2 as shown in Fig.1 (b) is analyzed and the outer surface of boss in No.2 is fixed by a rigid body. Since $v_{\pi1}(z)$ is equal to Eq. (12), $v_{\pi1}(z)$ only is proposed and then the results will be obtained.

3.2.1 Equation of displacement $v_{\pi1}(z)$

The boundary condition of the boss is as follows: $\tau_{r\theta} = \sum_{s=1}^{\infty} a_s \sin \beta_s z$ at $r=a$
 $v=0$ at $r=b$
 $\tau_{\theta z}=0$ at $z=\pm h$ (18)

where $\beta_s = (2s-1)\pi/2h$

Denoting the displacement function as Eq. (14) and considering the above mentioned condition,

$$\left. \begin{aligned} A_s' I_2(\beta_s a) - B_s' K_2(\beta_s a) &= a_s \\ A_s' I_1(\beta_s b) + B_s' K_1(\beta_s b) &= 0 \end{aligned} \right\} \dots\dots\dots (19)$$

Fig.6 Effect of m on the distribution of $\tau_{r\theta}$

Fig.7 Effect of m on the distribution of U_I

Fig.8 Distribution of $\tau_{r\theta}$ for $b/a=1.5$

Fig.9 Distribution of $\tau_{r\theta}$ for $b/a=2.0$

Fig.10 Distribution of $\tau_{r\theta}$ for $b/a=4.0$

Fig.11 Distribution of $(V-U_I)$ for $b/a=1.5$

and then the form of $\bar{v}_{II}(z)$ is obtained as follows:

$$G_{II} \bar{v}_{II}(z) = \sum_{s=1}^{\infty} \frac{a_s \{I_1(\beta_s a) K_1(\beta_s b) - I_1(\beta_s b) K_1(\beta_s a)\}}{\beta_s \{I_2(\beta_s a) K_1(\beta_s b) + K_2(\beta_s a) I_1(\beta_s b)\}} \sin \beta_s z. \quad (20)$$

3.2.2 Contact stresses and displacements Figs.16 to 21 show the results under the slipless condition. It is known from these figures that the distributions of $\tau_{r\theta}$ and $(V-\bar{v}_I)$ are very similar to the results of No.1, but a variation of b/a does not influence this distribution

Fig.12 Distribution of $(V-\bar{v}_I)$ for $b/a=2.0$

Fig.13 Distribution of $(V-\bar{v}_I)$ for $b/a=4.0$

Fig.14 $\tau_{r\theta}$ for $b/a=2.0$ under partial slip condition

Fig.15 $(V-\bar{v}_I)$ for $b/a=2.0$ under partial slip condition

Fig.16 Distribution of $\tau_{r\theta}$ for $b/a=1.5$

Fig.17 Distribution of $\tau_{r\theta}$ for $b/a=2.0$

because the condition of $b/a = \infty$ is nearly equal to $b/a = 0$. Figures 22 and 23 show $\tau_{r\theta}$ and $(V - v_i)$ under the partial slip condition respectively. It is known from these figures that the results of No.2 are very similar to those of No.1 except that the slip area of $\tau_c / (\frac{2M}{\pi a^2}) = 0.3$ for No.2 is a little larger than for No.1.

Fig.18 Distribution of $\tau_{r\theta}$ for $b/a = 4.0$

Fig.19 Distribution of $(V - v_i)$ for $b/a = 1.5$

Fig.20 Distribution of $(V - v_i)$ for $b/a = 2.0$

4. Conclusions

The shrink-fitted shafts under bending or torsional moment are very much used in the practical machine elements. Then in this paper the stress distributions and displacements arising in a shrink-fit cylinder under torsion are analyzed by applying the point-matching method. The analysed models are No.1 in which the outer surface of boss is free and No.2 in which the surface is fixed by a rigid body.

Fig.21 Distribution of $(V - v_i)$ for $b/a = 4.0$

Fig.22 $\tau_{r\theta}$ for $b/a = 2.0$ under partial slip condition

Fig.23 $(V - v_i)$ for $b/a = 2.0$ under partial slip condition

Fig.24 $\tau_{z\theta}$ for $b/a=2.0$ under slip condition (Model No.1)

Fig.25 $\tau_{z\theta}$ for $b/a=2.0$ under slip condition (Model No.2)

On these models, $\tau_{r\theta}$ and $(V-\psi_r)$ are calculated under the slipless or partial slip condition.

From the numerical results the following conclusions may be drawn.

(1). $\tau_{r\theta}$ under the slipless condition varies rapidly in the part near $z/h=1.0$ and this tendency is more conspicuous with a decrease of G_I/G_{II} .

(2). The variation of b/a affects and $(V-\psi_r)$ in the model No.1 but little in No.2.

(3). Under the slipless condition, $(V-\psi_r)$ varies rapidly in the part near $z/h=1.0$ but in the partial slip area the distribution is nearly linear.

(4). The slip area at the contact surface is determined by repeating the point-matching method and the area in No.2 is larger than in No.1.

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Appendix

For the design of a shaft, $\tau_{z\theta}$ is very important as well as the above obtained $\tau_{r\theta}$. This stress is obtained from the following equation by superposing M_r/I_p on the stress values $\tau_{r\theta}$ ($\tau_1, \dots, \tau_m$).

$$\tau_{z\theta} = \frac{2}{\pi} \int_0^\infty \frac{I_1(\alpha r)}{I_2(\alpha r)} F(\alpha) \cos \alpha z d\alpha \quad \dots\dots\dots(21)$$

where

$$\tau_{z\theta} = \frac{M_r}{I_p} + \sum_{i=1}^m \tau_{z\theta} \quad \dots\dots\dots(22)$$

By using Eq. (21), the stresses in the model of $b/a=2.0$, $h/a=1.0$ and $G_I/G_{II}=1.0$ are analyzed in Figs.24 and 25.

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