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Pressure Response to Linear Change of Flow Rate in Oil Pipeline*

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In this paper, the pressure response to the linear change of a flow rate in oil pipelines is treated. The theoretical expressions representing the shape of the pressure response to it have been derived, and the influences of the parameters in the pipeline-systems on the pressure response have been elucidated.

The theoretical expressions for the pressure response including nondimensional parameters have been indicated as a sum of series, and experiments have been performed by changing the inner diameter of pipeline, coefficient of kinematic viscosity of oil and linearly changing time of the flow rate, respectively. The theoretical expressions and the experimental results agreed with each other, and the results were examined closely. The expressions representing the maximum pressure and the damping properties of the shape of the pressure response have been derived approximately.

1. Introduction

Many studies on the pressure response to the step change of a flow rate in oil pipeline have been recently performed⁽¹⁾⁻⁽³⁾, but a study on the pressure response to the linear change of a flow rate has been hardly performed.

In the practical oil hydraulic circuits, when changing the displacement of oil pump or oil motor and operating the flow control valve, the flow rate in the pipeline often changes linearly. In their experiments, authors have linearly changed the flow rate by changing linearly the angle of inclination of a swash plate in a swash plate type variable displacement axial piston pump.

Many studies for the indicial response of pipeline by numerical analyses using an electronic computer have been recently performed⁽⁵⁾⁻⁽⁸⁾. But in this paper, authors have obtained analytical solutions of the pressure response to the linear change of a flow rate and have tried to clarify how each nondimensional parameter of the pipeline system exerts influence on the shape of the pressure response.

Authors have obtained the analytical solutions with the aid of the concept of the equivalent viscous resistance and have carried out an experiment by changing various conditions, namely the changing time of flow rate, diameter of pipeline and kinematic viscosity of oil, respectively. Moreover, they have made comparisons and considerations of the theoretical calculations and experimental results in this paper.

2. Theoretical analysis

2.1 Linear decrease of flow rate at the outlet of pipeline

In the pipeline-system such as shown in Fig.1, the transfer function, $G_0(s) = P_2(s)/Q_2(s)$, becomes as follows:⁽¹⁾

$$G_0(s) = \frac{P_2(s)}{Q_2(s)} = -\frac{\gamma(s)/\sinh[\gamma(s)(l-x)]}{Ks \cosh \gamma(s)l} \dots (1)$$

where, P and Q denote pressure and flow rate, respectively, and the suffixes 1, 2 and x show inlet of pipeline, outlet of pipeline and position x, respectively. P(s) and Q(s) denote Laplace transform of the pulsating pressure and flow rate, respectively.

$\gamma(s)l = T_s \sqrt{s^2 + R_f^2}$
 $R_f' = (R_f, (\chi > 3.817)) \quad R_f (\chi \leq 3.817) \quad R_f = \lambda \omega \sqrt{\chi^2 + 2}$
 $\lambda = \chi / (\chi^2 - \sqrt{2}\chi + 1) \quad \chi = r_0 \sqrt{\omega/\nu}$
 r_0 : inner radius of pipeline, ω : angular frequency
 ν : coefficient of kinematic viscosity of oil $R_f = 8\nu/r_0^2$
 $T_s = l/a$ l : length of pipeline $K = fl/\rho a^2$
 a : velocity of pressure wave ρ : density of oil
 x denotes distance from outlet of pipeline s : Laplace variable
 f : inner cross sectional area of pipeline

To decrease linearly the flow rate at the outlet of pipeline such as shown in Fig.2, we must give the flow rate change at the outlet of pipeline such as shown in Fig.3. Showing the function in Fig.3 by Laplace transform, we have the following expression.

$$Q_2(s) = -\frac{b}{s^2} (1 - e^{-As}) = -\frac{Q_0}{Ks^2} (1 - e^{-As}) \dots (2)$$

Fig.1 Pipeline-system

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Fig.2 Flow rate change at the outlet of pipeline

where, h denotes linearly the changing-time of a flow rate. Q_0 denotes the mean flow rate before the flow rate changes.

$$b=Q_0/h.$$

Obtaining $P_s(s)/\rho u_0 a$ from Eqs.(1) and (2), it becomes as follows:

$$\frac{P_s(s)}{\rho u_0 a} = \frac{(1-e^{-hs})\gamma(s)l \sinh[\gamma(s)(l-x)]}{hT_s s^2 \cosh \gamma(s)l} \quad (3)$$

For $\zeta_{nk} < 1$

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_k} [1-d(-\tau_k)] \left[c_1 \tau + c_2 - \sum_{k=1}^{\infty} \frac{2 \cos(\varepsilon_{nk} L_s)}{\varepsilon_{nk}^2 \sqrt{1-\zeta_{nk}^2}} e^{-\varepsilon_{nk}^2 \tau} ((1-4\zeta_{nk}^2) \sqrt{1-\zeta_{nk}^2} \times \cos(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau) + \zeta_{nk} (3-4\zeta_{nk}^2) \sin(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau)) \right] \quad (4)$$

For $\zeta_{nk} = 1$

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_k} [1-d(-\tau_k)] \left[c_1 \tau + c_2 + 2(3+\varepsilon_{nk} L_s) e^{-\varepsilon_{nk}^2 \tau} \frac{\cos(\varepsilon_{nk} L_s)}{\varepsilon_{nk}^2} - \sum_{k=2}^{\infty} \frac{2 \cos(\varepsilon_{nk} L_s)}{\varepsilon_{nk}^2 \sqrt{1-\zeta_{nk}^2}} \times e^{-\varepsilon_{nk}^2 \tau} ((1-4\zeta_{nk}^2) \sqrt{1-\zeta_{nk}^2} \cos(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau) + \zeta_{nk} (3-4\zeta_{nk}^2) \sin(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau)) \right] \quad (5)$$

For $\zeta_{nk} > 1$ (When $k=m$, $\zeta_{nm}=1$)

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_k} [1-d(-\tau_k)] \left[c_1 \tau + c_2 - \sum_{k=1}^{m-1} \frac{2 \cos(\varepsilon_{nk} L_s)}{\varepsilon_{nk}^2 \sqrt{\zeta_{nk}^2-1}} e^{-\varepsilon_{nk}^2 \tau} ((1-4\zeta_{nk}^2) \sqrt{\zeta_{nk}^2-1} \times \cosh(\varepsilon_{nk} \sqrt{\zeta_{nk}^2-1} \tau) + \zeta_{nk} (3-4\zeta_{nk}^2) \sinh(\varepsilon_{nk} \sqrt{\zeta_{nk}^2-1} \tau)) \right. \\ \left. + 2(3+\varepsilon_{nm} L_s) e^{-\varepsilon_{nm}^2 \tau} \cos(\varepsilon_{nm} L_s) / \varepsilon_{nm}^2 - \sum_{k=m+1}^{\infty} \frac{2 \cos(\varepsilon_{nk} L_s)}{\varepsilon_{nk}^2 \sqrt{1-\zeta_{nk}^2}} e^{-\varepsilon_{nk}^2 \tau} ((1-4\zeta_{nk}^2) \sqrt{1-\zeta_{nk}^2} \times \cos(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau) + \zeta_{nk} (3-4\zeta_{nk}^2) \sin(\varepsilon_{nk} \sqrt{1-\zeta_{nk}^2} \tau)) \right] \quad (6)$$

where, $\tau = t/T_s$, t : time, $\tau_k = h/T_s$, $L_s = x/l$, $c_1 = T_s R_f (1-L_s)$, $c_2 = (1-L_s)(1-T_s^2 R_f^2/2 + T_s^2 R_f^2 (1-L_s)^2/6)$,

$$\varepsilon_{nk} = (2k-1)\pi/2, \quad k=1, 2, 3, \dots, \quad \zeta_{nk} = T_s R_{nk}/2\varepsilon_{nk}$$

When $\chi > 3.817$, R_{nk} is the value of R_f , obtained by substituting $\omega_{nk} = \varepsilon_{nk}/T_s$ into ω in the expression of R_f . When $\chi \leq 3.817$, $R_{nk} = R_f$. $d(-\tau_k)f(\tau)$ means delaying the function $f(\tau)$ only by τ_k .

Therefore, the shapes of the pressure response are drawn and characterized by the nondimensional parameters $R_f T_s$, ζ_{nk} , τ_k and L_s .

Neglecting the viscous resistance, $P_s(\tau)/\rho u_0 a$ becomes as follows:

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_k} [1-d(-\tau_k)] \left[(1-L_s) - \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2 \pi^2} \cos\left\{\frac{(2k-1)\pi}{2} L_s\right\} \cos\left\{\frac{(2k-1)\pi}{2} \tau\right\} \right] \quad (7)$$

For $L_s=0$, Eq.(7) is rewritten as follows:

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_k} [1-d(-\tau_k)] \left[1 - \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2 \pi^2} \cos\left\{\frac{(2k-1)\pi}{2} \tau\right\} \right] \quad (8)$$

When $h \rightarrow 0$ and $\tau_k \rightarrow 0$, Eqs.(4)~(8) become the expressions of the oil hammer caused by closing the valve quickly.

2.2 Linear increase of flow rate at the inlet of pipeline

In the pipeline-system such as shown in Fig.4, the transfer functions $G_i(s) = P_s(s)/Q_i(s)$ and $Q_i(s)$ become as follows:

$$G_i(s) = \frac{P_s(s)}{Q_i(s)} = \frac{\gamma(s)l \sinh[\gamma(s)(l-x)]}{Ks \cosh \gamma(s)l} \quad (9)$$

$$Q_i(s) = \frac{Q_0}{hs^2} (1-e^{-hs}) \quad (10)$$

Fig.4 Pipeline-system

Fig.3 Input

where, $u_0 = Q_0/f$.

As Eq.(3) may be expressed as an infinite product, it can be made a Laplace inverse transform. $P_s(\tau)/\rho u_0 a$ is obtained from Eq.(3) by using the method of a Laplace inverse transform. $P_s(\tau)$ is the Laplace inverse transform of $P_s(s)$. ε_{nk} , ζ_{nk} and ω_{nk} are related to the denominator of Eq.(3)

1. Photograph
2. Oscilloscope
3. Amplifier
4. Pressure transducers
5. Inertia load
6. Displacement sensor

Fig.5 Experimental apparatus on linear decrease flow rate at the outlet of pipeline

where, Q_0 is the changing flow rate as an input. x denotes a distance from the inlet of pipeline. Otherwise the nomenclature is the same as in section 2. 1.

Obtaining $P_x(s)/\rho u_0 a$ from Eqs.(9) and (10), it coincides with Eq.(3). Therefore, $P_x(s)/\rho u_0 a$ in the pipeline-system of Fig.4 is identical with those of Figs.1 and 3.

3. Experiments and considerations on linear decrease of flow rate at the outlet of pipeline

3.1 Experimental apparatus and procedure

The experimental apparatus used in the case of a linear decrease of flow rate at the outlet of pipeline is schematically shown in Fig.5. An accumulator and a swash plate type variable displacement axial piston pump are attached at the inlet and outlet of the test pipeline. The hydraulic oil runs through the relief valve, accumulator, test pipeline, swash plate type variable displacement axial piston pump and flow control valve in that order. The piston pump is driven by a variable electronic motor, and the inertia load is attached so that the speed of pump does not change through a change in the angle of inclination of the piston pump.

In this piston pump, the theoretical delivery Q_0 is as follows:

$$Q_0 = 2\pi ANR \tan \varphi \quad (11)$$

where, R : distance from center line of cylinder block to that of piston, φ : angle of inclination of swash plate, A : sectional area of piston, z : number of pistons, N : rotational speed of pump.

When the value of φ is small, $\tan \varphi \approx \varphi$. Therefore, Eq.(11) is able to be rewritten as

$$Q_0 \approx 2\pi ANR \varphi \quad (12)$$

In this experiment, φ is linearly changed with the aid of a hydraulic cylinder in a meter-out circuit approximately, and its value changes from 0.1 to 0.3 radian. $z=7$, $A=0.994 \text{ cm}^2$, $R=1.85 \text{ cm}$. The rod moving the swash plate is moved at a constant speed with the aid of the oil hydraulic cylinder in the meter-circuit. The angle of inclination of the swash plate moved in proportion to the displacement of the rod moving the swash plate. The movement of the rod has been recorded at the displacement sensor by using the oscilloscope. An example of the records is

Fig.6 Movement of rod moving swash plate

shown in Fig.6.

The shapes of the pressure response have been recorded at the pressure transducers which were mounted in the middle and outlet of pipeline by using the oscilloscope. Examples of pressure waves for various values of τ_A are shown in Fig.7. Table 1 shows the specifications of the pipeline-system used in the experiment.

3.2 Experimental results and considerations

Figure 8 shows the comparison between the experimental results and the theoretical calculations under various line-conditions such as listed in Table 1. In this report, the calculated values of Eq(4) are called ones based on the equivalent viscous resistance.

Figures 8(a)~(f) show the shapes of the pressure response at $L=0$ and 0.5 under $\tau_A = 1, 2$ and 4 respectively in the case of comparatively small line-resistances. The pressure pulsations of high frequency and small amplitude appear in the first period of the shape of the pressure response of the experimental results. These are mainly due to the pulsations of the suction pressure of the pump and the reflecting wave at the pump. These pulsations dissipated with the lapse of time because of the line-resistance and the compressibility of oil in the pipeline. Figures 8(g) and (h) show the case of a more viscous resistance than in Figs.8(a) (f), and corresponds to Fig.8(c). In the shapes of the pressure response, the larger the line-resistance, the less the amplitude of the high frequency component becomes and the faster the damping becomes.

Fig.7 Shapes of pressure response (Experimental results)

In these cases, the pulsations of high frequency and small amplitude appear hardly.

Figures 9~11 are compared with three kinds of the calculated procedures, namely using the equivalent viscous resistance, the resistance of steady laminar flow ($R_{sk} = R_f$) and the nonviscous resistance, respectively. Figure 9 shows the influences of above mentioned viscous resistances at $\tau_k = 1, 2$ and 4 for a comparatively small line-resistance. In the first half cycle, agreement between the calculated values by the procedure of the equivalent viscous resistance and the ones by the viscous resistance of steady laminar flow is comparatively good, and becomes less good as τ becomes larger. When the viscous resistance is small, the pulsating

amplitude of the shape of the pressure response becomes small for $\tau_k > 2$ and it becomes smaller as τ_k becomes larger. For $\tau_k \geq 2$, the amplitude of the first cycle becomes $2(1-L_s)/\tau_k$ in the case of the nonviscous resistance. When the viscous resistance is very small and $\tau_k = 4$, the shape of the pressure response rises at $\tau = L_s$ and becomes an isosceles trapezoid with the length of upper base $2L_s$, that of down base $4-2L_s$ and the height $(1-L_s)/2$, but the value of the shape of the pressure response becomes zero because of the influences of the reflecting waves for $\tau \geq 4-L_s$.

Figures 10 and 11 show the influences of above mentioned viscous resistances for large line-resistance. In these figures,

$$\lambda_{nk} = r_0 \sqrt{\omega_{nk}/\nu}, \quad k=1, 2, 3, \dots$$

Table 1 Specifications of pipeline systems

Length of pipeline l m	Inner diameter of pipeline d cm	Thickness of pipe wall b mm	Density of oil ρ kg sec ² /cm ⁴	Kinematic viscosity of oil ν cm ² /sec	Modulus of longitudinal elasticity E kg/cm ²	T_s sec	Damping factor ζ_{n1}
10.8	1.60	2.8	8.81×10^{-7}	0.660	2.15×10^6	0.008 44	0.058 6
10.8	1.60	2.8	9.06×10^{-7}	2.22	2.15×10^6	0.008 50	0.075 1
11.8	0.65	2.0	8.81×10^{-7}	0.660	2.15×10^6	0.009 10	0.178
11.8	0.65	2.0	8.90×10^{-7}	1.72	2.15×10^6	0.009 15	0.379

	ζ_{n1}	$R_f T_s$	$R_{n1} T_s$	T_s s	τ_k	L_s	k
(a)	0.058 6	0.069 6	0.184	0.008 44	1	0	4
(b)	0.058 6	0.069 6	0.184	0.008 44	1	0.5	4
(c)	0.058 6	0.069 6	0.184	0.008 44	2	0	4
(d)	0.058 6	0.069 6	0.184	0.008 44	2	0.5	4
(e)	0.058 6	0.069 6	0.184	0.008 44	4	0	4
(f)	0.058 6	0.069 6	0.184	0.008 44	4	0.5	4
(g)	0.178	0.435	0.560	0.009 10	2	0	4
(h)	0.379	1.190	1.190	0.009 15	2	0	3

Fig.8 Shapes of pressure response (comparison between theoretical values and experimental ones)

In Fig.10, though $\chi_{n1} > 3.817$, as the value of χ_{n1} is in the neighbourhood of 3.817, agreement between the calculated value by the procedure of the equivalent viscous resistance and the one by the viscous resistance of steady laminar flow is comparatively good.

In the case of Fig.11, $\chi_{n1} = 3.24 < 3.817$. Thus, as $R_{n1} = R_f$, even if $R_{n1} \neq R_f$, the calculated value by the procedure of the viscous resistance of steady laminar flow has nearly agreed with the one by the equivalent viscous resistance. Therefore, when $\chi_{n1} \leq 3.817$, we may estimate the shape of the pressure response as $R_{n1} = R_f$.

According to Figs.9~11, with an increase of the line-resistance, it becomes impossible to estimate the shape of the pressure response by the calculated values using the nonviscous resistance.

Figures 12 and 13 show comparison between the calculated values for $k = 4$ and those for $k = 1$. In the calculated values for $k = 4$ on the oil hammer reported already by authors⁽¹⁾, when the line-

resistance is small, the pulsations of high frequency appear in the shape of pressure responses. But, in this report, it is seen that the pulsations of high frequency hardly appear in the calculated values or the shape of the pressure response even if line-resistance is small in the case of $\tau_k \geq 1 - L_s$ and $k = 4$. In the calculated values for $k = 1$, the discontinuity appears at $\tau = \tau_k$ because of the influences of $(1 - d(-\tau_k))f(\tau)$ in the theoretical expression. But we can make the discontinuity appear more smooth by adopting the series terms to $k = 3$ or 4.

Figure 12 shows the case of a small line-resistance. For $\tau_k < 2$, the calculated value for $k = 1$ slightly differs from that for $k = 4$ in the neighbourhood of the maximum and minimum of the initial two cycles, but agreement between these two calculated values is good in succeeding cycles due to the damping of the high frequency component. For $\tau_k \geq 2$, the flat parts do not appear in the calculated values for $k = 4$, and agreement between these values and the calculated values of $k = 1$ is good. Figure 13 shows the case of a larger line-resistance than that of Fig.12. Agreement between the calculated value for $k = 4$ and that for $k = 1$ is better as the line-resistance is larger.

Table 2 shows the reasonable numbers of the terms of the series to be taken into consideration for theoretical calculation. When $\zeta_{n1} = 0.2745$, $\chi_{n1} = 3.817$ and $R_{n1} = R_f$.

Fig.9 Influences of viscous resistance on various calculated values

Fig.10 Influences of viscous resistance on various calculated values

Fig.11 Influences of viscous resistance on various calculated values

Fig.12 Comparison of the number of the terms of series

Fig.13 Comparison of the number of the terms of series

Table 2 The number of the terms of series to be taken into consideration for calculation

conditions		The number of the terms of series
$\tau_A < 4$	$\zeta_{n1} < 0.2745$	$k = 4$
	$\zeta_{n1} \geq 0.2745$	$k = 3$
$\tau_A \geq 4$	When exact wave shapes are desired, When the maximum comes into question,	the same conditions as $\tau_A < 4$ $k = 1$

3.3 Maximum and damping property of the shape of pressure response

In this paper, we consider the case of $\tau_A \geq (1-L_s)$ and the shape of pressure response being oscillatory. As the shape of the pressure response becomes nearly non-oscillatory for $\sqrt{3}/2 \leq \zeta_{n1} < 1$, we deal with the case of $0 < \zeta_{n1} < \sqrt{3}/2$.

According to Figs. 8, 12 and 13 the virtual shapes of pressure responses are approximately represented with the calculated values for $k = 1$ in the theoretical expression. In this section, the maximum and damping property of the shape of the pressure response by using mainly the expression for $k = 1$ are described. Considering the expression for $k = 1$ from Eq.(4), it is written as

$$\frac{P_s(\tau)}{\rho u_0 a} = \frac{1}{\tau_A} (1 - d(-\tau_A)) \left[c_1 \tau + c_2 - \frac{2 \cos(\varepsilon_{n1} L_s)}{\varepsilon_{n1}^2 \sqrt{1 - \zeta_{n1}^2}} \times e^{-\zeta_{n1} \varepsilon_{n1} \tau} \left((1 - 4\zeta_{n1}^2) \sqrt{1 - \zeta_{n1}^2} \times \cos(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau) + \zeta_{n1} (3 - 4\zeta_{n1}^2) \times \sin(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau) \right) \right] \quad (13)$$

Assuming $0 < \zeta_{n1} < \sqrt{3}/2$, Eq.(13) is rewritten as the following expressions, namely Eqs.(14) and (15).

$$\text{For } \tau < \tau_A \quad F(\tau) = \frac{P_s(\tau) - (\bar{P}_1 - \bar{P}_s)}{\rho u_0 a} = \frac{1}{\tau_A} \left\{ c_1 (\tau - \tau_A) - c_2 - \frac{2 \cos(\varepsilon_{n1} L_s)}{\varepsilon_{n1}^2 \sqrt{1 - \zeta_{n1}^2}} e^{-\zeta_{n1} \varepsilon_{n1} \tau} \times \sin(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau + \phi_{n1}) \right\} \quad (14)$$

where, $\phi_{n1} = \arctan((1 - 4\zeta_{n1}^2) \sqrt{1 - \zeta_{n1}^2} / \zeta_{n1} (3 - 4\zeta_{n1}^2))$, ϕ_{n1} changes from $\pi/2$ to $-\pi/2$ as ζ_{n1} changes from zero to $\sqrt{3}/2$.

Fig.14 Shapes of the pressure response of first half cycle in nonviscous resistance

For $\tau \geq \tau_A$

$$F(\tau) = \frac{P_s(\tau) - (\bar{P}_1 - \bar{P}_s)}{\rho u_0 a} = \frac{-2 \cos(\varepsilon_{n1} L_s)}{\tau_A \varepsilon_{n1}^2 \sqrt{1 - \zeta_{n1}^2}} e^{-\zeta_{n1} \varepsilon_{n1} \tau} \times (\pm \sqrt{A_{n1}^2 + B_{n1}^2} \sin(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau + \phi_{n1} + \phi_{s1})) \quad (15)$$

where, ϕ_{n1} is identical with that of Eq.(14). $\phi_{s1} = \arctan(B_{n1}/A_{n1})$, $-\pi/2 \leq \phi_{s1} \leq \pi/2$
 $A_{n1} = 1 - e^{\zeta_{n1} \varepsilon_{n1} \tau_A} \cos(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau_A)$
 $B_{n1} = e^{\zeta_{n1} \varepsilon_{n1} \tau_A} \sin(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau_A)$

A choice of sign $\pm$ in Eq.(15) is made by the following procedures. In spite of the positive or negative sign of B_{n1} , the positive sign $+$ is chosen for $A_{n1} > 0$, and the negative one $-$ is chosen for $A_{n1} < 0$. When $A_{n1} = 0$, the choice of sign $\pm$ depends upon the positive or negative sign of B_{n1} , the positive sign $+$ is chosen for $B_{n1} > 0$, and the negative one $-$ is chosen for $B_{n1} < 0$.

In the case of $\tau < \tau_A$, putting τ which gives the maximum and minimum of the shape of the pressure response as τ_m^* , from Eq.(14) τ_m^* is a positive value which satisfies

$$c_1 + \frac{2 \cos(\varepsilon_{n1} L_s)}{\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2}} e^{-\zeta_{n1} \varepsilon_{n1} \tau_m^*} \times \sin(\varepsilon_{n1} \sqrt{1 - \zeta_{n1}^2} \tau_m^* + \phi_{n1} - \phi_{s1}) = 0 \quad (16)$$

where, $\phi_{s1} = \arctan(\sqrt{1 - \zeta_{n1}^2} / \zeta_{n1})$

In the case of $\tau \geq \tau_A$, from Eq.(15) τ_m^* becomes

$$\tau_m^* = \frac{(2m-1)}{\sqrt{1 - \zeta_{n1}^2}} \frac{2(\phi_{n1} + \phi_{s1})}{\pi \sqrt{1 - \zeta_{n1}^2}} \quad (17)$$

where, when $A_{n1} > 0$, or when $A_{n1} = 0$ and $B_{n1} > 0$, at τ_m^* the shape of the pressure response becomes maximum for an even number of m and becomes minimum for an odd number of m . When $A_{n1} < 0$ or when $A_{n1} = 0$ and $B_{n1} < 0$, at τ_m^* the shape of the pressure response becomes maximum for an odd number of m and becomes minimum for an even number of m .

When the value of ζ_{n1} is comparatively small, we can estimate the value of τ , τ_{max}^* , which gives the maximum value of the shape of the pressure response from the half cycle of it in the nonviscous resistance. Figure 14 is the half cycle of the shape of the pressure response under various values of τ_A for $\tau_A \geq 1$. From Fig.14 τ_{max}^* is deduced by the following method. Assuming that τ becomes τ_{max}^* in the middle of the upper base of the shape of the pressure response, for $1 \leq \tau_A \leq 2$

$$\tau_{max}^* = \frac{(\tau_A + L_s) + (2 - L_s)}{2} = 1 + \frac{1}{2} \tau_A \quad (18)$$

for $2 \leq \tau_A$

$$\tau_{max}^* = \frac{(2 - L_s) + (2 + L_s)}{2} = 2 \quad (19)$$

We will try to examine how Eqs.(18) and (19) are related to Eqs.(16) and (17), respectively.

In the case of $1 \leq \tau_A \leq 2$, since

Fig.15 Shape of the pressure response (comparison between theoretical values and experimental ones)

$\tau_k \leq \tau_{\max}^*$, $\tau_{\max}^*$ is obtained from Eq.(17). When $\zeta_{n1} = 0$, as $\phi_{n1} = \pi/2$, $A_{n1} = 1 - \cos(\tau_k \pi/2)$ and $B_{n1} = \sin(\tau_k \pi/2)$,

$$\begin{aligned} \tau_{\max}^* &= 2 - \frac{2}{\pi} \arctan\left(\cot \frac{\pi}{4} \tau_k\right) \\ &= 2 - \frac{2}{\pi} \left(\frac{\pi}{2} - \frac{\pi}{4} \tau_k\right) = 1 + \frac{1}{2} \tau_k \end{aligned} \quad (20)$$

Thus, Eq.(20) is identical with Eq.(18).

In the case of $2 \leq \tau_k$, since $\tau_{\max}^*$ exists in the range of $\tau \leq \tau_k$, $\tau_{\max}^*$ is obtained from Eq.(16). When $\zeta_{n1} = 0$, as $c_1 = 0$ and $\phi_{n1} = \phi_{w1} = \pi/2$, $\sin(\tau_{\max}^* \pi/2) = 0$. Therefore,

$$\tau_{\max}^* = 2 \quad (21)$$

Equation (21) is identical with Eq.(19).

Therefore, it is seen that $\tau_{\max}^*$ may be indicated as Eqs.(18) and (19) in the range of small values of ζ_{n1} .

From Eqs.(14) and (15), the nondimensional period of the shape of the pressure response, T_0 , becomes approximately as follows:

$$T_0 = 4/\sqrt{1 - \zeta_{n1}^2} \quad (22)$$

This nondimensional period T_0 is identical with that of an oil hammer.

The consecutive amplitude ratio κ and logarithmic decrement δ become as follows:

$$\kappa = \exp[\pi \zeta_{n1} (\tau_{(m+1)}^* - \tau_m^*)/2] \quad (23)$$

$$\delta = \pi \zeta_{n1} (\tau_{(m+1)}^* - \tau_m^*)/2 \quad (24)$$

According to the preceding considerations, when $\tau_k > 4$ or $\zeta_{n1} > 0.4$, the maximum of the shape of the pressure response becomes less than 0.5, and it is possible that it becomes less than 0.5 for $2 \leq \tau_k \leq 4$ even if $\zeta_{n1} \leq 0.4$.

4. Experiments and considerations on linear increase of flow rate at the inlet of pipeline

Figure 15 shows comparison between the theoretical calculated values and the experimental results. The experiment has been performed by rearranging the experimental apparatus of Fig.5. The pipeline condition is identical with that of $\zeta_{n1} = 0.0586$ in Table 1. It is seen from Fig.4 that agreement between theory and experiment is good. High frequency and small amplitude components appear in the shape of the pressure response of the experiment. Its causes and properties are similar to the case of linear decrease of flow rate at the outlet of pipe-line.

The maximum and the damping property are also similar to the case of section 3.3.

5. Conclusion

Any conclusions drawn from this study are summarized as follows.

- (1) The analytical solution for the shape of the pressure response to linear change of flow rate in oil pipeline has been obtained. The calculated values have agreed well with the experimental results.
- (2) The shapes of the pressure response are drawn with the nondimensional parameters $R_f T_e$, ζ_{n1} , τ_k and L_x .
- (3) In the analytical solutions, when $\tau_k \rightarrow 0$, it is identical with that of the oil hammer caused by closing the valve quickly.
- (4) In the case of $\tau_k \geq 1 - L_x$, even if the line-resistance ($R_f T_e$, $R_{n1} T_e$) is small, the convergence of the analytical solution is fast.
- (5) The reasonable values of the terms of the series to be taken into consideration for the theoretical calculations are listed in Table 2.
- (6) The maximum and damping property of the pressure response have been approximately indicated. The nondimensional period T_0 has become similar to that of an oil hammer.
- (7) The pressure wave rises more slowly, has a smaller amplitude and damps faster as the line-resistance is larger.
- (8) The amplitude of the pressure response becomes smaller as τ_k is larger.
- (9) In the case of $\tau_k \geq 1 - L_x$, when $\tau_k \geq 4$ or $\zeta_{n1} > 0.4$, the maximum value of the pressure response becomes less than 0.5.

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