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Applications of Three Shell Theories for Bending Problems of Axisymmetric Shells*

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For bending problems of the general axisymmetric shell, the finite difference method has been widely used as well as the finite element method, and B. Budiansky and P.P. Radkowski have proposed a finite difference method based on Sanders' shell theory.

In this paper, two numerical methods for statical bending and free vibration problems of the general axisymmetric shell are presented. One of them is based on Flügge's shell theory, which is well used in our country, and the other is based on Mizoguchi's shell theory, the utility of which has been proved especially for various problems of cylindrical shells. The above-mentioned two methods are described by the equations of the same form as those by Budiansky and Radkowski, only the coefficients of which are different.

These three methods are applied to some statical bending and free vibration problems at the same time, and it turns out that the results by these methods almost coincide with each other.

1. Introduction

For bending problems of the general axisymmetric shell, the finite difference method has been widely used as well as the finite element method, and B. Budiansky and P.P. Radkowski⁽¹⁾ once proposed a finite difference method based on Sanders' shell theory.⁽²⁾ But at least in our country, Flügge's shell theory⁽³⁾ is more widely used. So, in this paper, a method based on Flügge's shell theory for solving the free vibration and statical bending problems of axisymmetric shells is presented. Furthermore, Mizoguchi's shell theory,⁽⁴⁾⁻⁽⁶⁾ the utility of which has been proved especially for various problems of cylindrical shells, is also applied to the above-mentioned problems.

These three methods are applied to some statical bending and free vibration problems at the same time, and their results are compared with each other.

2. Fundamental relations

In Figs.1 and 2, the notations are shown for coordinates and geometrical dimensions related with a shell element, stress resultants, stress couples, external forces acting on it, displacement components, and rotation components. Then,

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the following nondimensional quantities are defined :

$$S = Q_\xi, r = Q\rho, \omega_\xi = Q/\xi, \omega_\theta = Q/r_\theta, \lambda = h/a \quad (1)$$

where a and h are the reference length and the shell thickness, respectively. The shell thickness is assumed as to be constant throughout the shell in this paper. Also, the following relations are easily verified⁽⁷⁾:

$$\omega_\theta = \sqrt{1 - (\rho')^2}/\rho, \quad \omega_\xi = -(\gamma' + \gamma^2)/\omega_\theta \\ \omega_\theta' = \gamma(\omega_\xi + \omega_\theta), \quad \gamma' = -\omega_\xi \omega_\theta - \gamma^2, \quad \rho'/\rho = -\omega_\xi \omega_\theta \quad (2)$$

where $\gamma' = \rho'/\rho$ and $(\gamma)' = \partial(\gamma)/\partial\xi$.

Now, the equilibrium equations of the shell element $ABCD$ are obtained as follows:

$$\begin{aligned} \frac{\partial(rN_\xi)}{\partial\xi} + \frac{\partial(QN_{\theta\xi})}{\partial\theta} - N_\theta \frac{\partial r}{\partial\xi} + Q_\xi \frac{\partial r}{\partial\xi} + rX &= 0 \\ \frac{\partial(QN_\theta)}{\partial\theta} + \frac{\partial(rN_{\theta\xi})}{\partial\xi} + N_{\theta\xi} \frac{\partial r}{\partial\xi} + Q_\theta \frac{\partial r}{\partial\xi} + rY &= 0 \\ \frac{\partial(rQ_\xi)}{\partial\xi} + \frac{\partial(QQ_\theta)}{\partial\theta} - N_\theta \frac{\partial r}{\partial\xi} - N_\xi \frac{\partial r}{\partial\xi} + rZ &= 0 \\ \frac{\partial(rM_{\theta\xi})}{\partial\xi} + \frac{\partial(QM_\theta)}{\partial\theta} + M_{\theta\xi} \frac{\partial r}{\partial\xi} - rQ_\theta &= 0 \\ \frac{\partial(rM_\xi)}{\partial\xi} + \frac{\partial(QM_{\theta\xi})}{\partial\theta} - M_\theta \frac{\partial r}{\partial\xi} - rQ_\xi &= 0 \\ N_{\theta\xi} - N_{\theta\xi} + \frac{M_{\theta\xi}}{r_\xi} - \frac{M_\theta}{r_\theta} &= 0 \end{aligned} \quad (3)$$

Here, the variables defined in Figs.1 and 2 are expanded in Fourier series as follows, respectively, :

$$\begin{aligned} N_\xi &= \sigma_\xi h \sum_{n=0}^{\infty} n_\xi^{(n)} \cos n\theta, \quad N_\theta = \sigma_\theta h \sum_{n=0}^{\infty} n_\theta^{(n)} \cos n\theta \\ N_{\theta\xi} &= \sigma_\xi h \sum_{n=1}^{\infty} n_\xi^{(n)} \sin n\theta, \quad N_{\theta\xi} = \sigma_\theta h \sum_{n=1}^{\infty} n_\theta^{(n)} \sin n\theta \\ M_\xi &= \sigma_\xi (h^3/a) \sum_{n=0}^{\infty} m_\xi^{(n)} \cos n\theta, \quad M_\theta = \sigma_\theta (h^3/a) \sum_{n=0}^{\infty} m_\theta^{(n)} \cos n\theta \end{aligned}$$

$$\begin{aligned}
M_{\xi\theta} &= \sigma_0 (h^3/a) \sum_{n=1}^{\infty} m_{\xi\theta}^{(n)} \sin n\theta, \quad M_{\theta\xi} = \sigma_0 (h^3/a) \sum_{n=1}^{\infty} m_{\theta\xi}^{(n)} \sin n\theta \\
Q_{\xi} &= \sigma_0 h \sum_{n=0}^{\infty} q_{\xi}^{(n)} \cos n\theta, \quad Q_{\theta} = \sigma_0 h \sum_{n=0}^{\infty} q_{\theta}^{(n)} \sin n\theta \quad (4) \\
U &= \sigma_0 (a/E) \sum_{n=0}^{\infty} u_{\xi}^{(n)} \cos n\theta, \quad V = \sigma_0 (a/E) \sum_{n=1}^{\infty} u_{\theta}^{(n)} \sin n\theta \\
W &= \sigma_0 (a/E) \sum_{n=0}^{\infty} w^{(n)} \cos n\theta, \quad X = \sigma_0 (h/a) \sum_{n=0}^{\infty} x^{(n)} \cos n\theta \\
Y &= \sigma_0 (h/a) \sum_{n=1}^{\infty} y^{(n)} \sin n\theta, \quad Z = \sigma_0 (h/a) \sum_{n=0}^{\infty} z^{(n)} \cos n\theta
\end{aligned}$$

where σ_0 and E are the reference stress and Young's modulus, respectively.

3. Solution based on Flügge's shell theory

Neglecting the sixth equation of Eqs. (3) and assuming approximately $N_{\theta\theta} = N_{\theta\xi}$ and $M_{\xi\theta} = M_{\theta\xi}$, elimination of Q_{ξ} and Q_{θ} from Eqs. (3) with use of Eqs. (1) leads to the following three equations:

$$\begin{aligned}
& a \left[\frac{\partial(PN_{\xi})}{\partial\xi} + \frac{\partial N_{\xi\theta}}{\partial\theta} - \rho' N_{\theta} \right] \\
& + \omega_{\xi} \left[\frac{\partial(PM_{\xi})}{\partial\xi} + \frac{\partial M_{\xi\theta}}{\partial\theta} - \rho' M_{\theta} \right] + \sigma r X = 0 \\
& a \left[\frac{\partial N_{\theta}}{\partial\theta} + \frac{\partial(PN_{\xi\theta})}{\partial\xi} + \rho' N_{\xi\theta} \right] \\
& + \omega_{\theta} \left[\frac{\partial(PM_{\xi\theta})}{\partial\xi} + \frac{\partial M_{\theta}}{\partial\theta} + \rho' M_{\xi\theta} \right] + \sigma r Y = 0 \quad (5) \\
& \frac{\partial}{\partial\xi} \left[\frac{\partial(PM_{\xi})}{\partial\xi} + \frac{\partial M_{\xi\theta}}{\partial\theta} - \rho' M_{\theta} \right] \\
& + \frac{\partial}{\partial\theta} \left[\frac{\partial(PM_{\xi\theta})}{\partial\xi} + \frac{\partial M_{\theta}}{\partial\theta} + \rho' M_{\xi\theta} \right] \\
& - a \rho \omega_{\theta} N_{\theta} - a \rho \omega_{\xi} N_{\xi} + \sigma r Z = 0
\end{aligned}$$

Substitution of Eqs. (1) (2) and (4) into Eqs. (5) leads to the following nondimensional equations:

$$\begin{aligned}
& n_{\xi}^{(n)'} + \frac{n}{\rho} n_{\theta}^{(n)'} + \gamma (n_{\xi}^{(n)'} - n_{\theta}^{(n)'}) \\
& + \lambda^2 \omega_{\xi} (m_{\xi}^{(n)'} + \gamma (m_{\xi}^{(n)'} - m_{\theta}^{(n)'}) + \frac{n}{\rho} m_{\theta}^{(n)'}) + \bar{\gamma}^{(n)} = 0 \\
& n_{\xi\theta}^{(n)'} + 2\gamma n_{\theta}^{(n)'} - \frac{n}{\rho} n_{\theta}^{(n)'} \\
& + \lambda^2 \omega_{\theta} (m_{\xi\theta}^{(n)'} + 2\gamma m_{\theta}^{(n)'} - \frac{n}{\rho} m_{\theta}^{(n)'}) + \bar{\gamma}^{(n)} = 0 \quad (6) \\
& -\omega_{\xi} n_{\xi}^{(n)'} - \omega_{\theta} n_{\theta}^{(n)'} + \lambda^2 (m_{\xi}^{(n)'} + 2\gamma m_{\xi}^{(n)'} - \omega_{\xi} \omega_{\theta} m_{\theta}^{(n)'}) \\
& + (\omega_{\xi} \omega_{\theta} - \frac{n^2}{\rho^2}) m_{\theta}^{(n)'} - \gamma m_{\theta}^{(n)'} + 2\frac{n}{\rho} m_{\xi\theta}^{(n)'} + 2\gamma m_{\xi\theta}^{(n)'} + \bar{Z}^{(n)} = 0
\end{aligned}$$

The relations between the stress resultants or the stress couples and the displacements in Flügge's theory as shown in Table 1 are used in Eqs. (6), then the fundamental differential equations relating to four dependent variables u_{ξ} , u_{θ} , w and m_{ξ} , which are functions of ξ , are derived as follows:

$$\begin{aligned}
& a_1^{(n)} u_{\xi}^{(n)'} + a_2^{(n)} u_{\xi}^{(n)'} + a_3^{(n)} u_{\theta}^{(n)'} + a_4^{(n)} u_{\theta}^{(n)'} + a_5^{(n)} u_{\theta}^{(n)'} + a_6^{(n)} w^{(n)'} \\
& + a_7^{(n)} w^{(n)'} + a_8^{(n)} m_{\xi}^{(n)'} + a_9^{(n)} m_{\xi}^{(n)'} + \bar{X}^{(n)} = 0 \\
& a_{10}^{(n)} u_{\xi}^{(n)'} + a_{11}^{(n)} u_{\xi}^{(n)'} + a_{12}^{(n)} u_{\theta}^{(n)'} + a_{13}^{(n)} u_{\theta}^{(n)'} + a_{14}^{(n)} u_{\theta}^{(n)'} + a_{15}^{(n)} w^{(n)'} \\
& + a_{16}^{(n)} w^{(n)'} + a_{17}^{(n)} m_{\xi}^{(n)'} + a_{18}^{(n)} m_{\xi}^{(n)'} + \bar{Y}^{(n)} = 0 \quad (7) \\
& a_{19}^{(n)} u_{\xi}^{(n)'} + a_{20}^{(n)} u_{\xi}^{(n)'} + a_{21}^{(n)} u_{\theta}^{(n)'} + a_{22}^{(n)} u_{\theta}^{(n)'} + a_{23}^{(n)} u_{\theta}^{(n)'} + a_{24}^{(n)} w^{(n)'} + a_{25}^{(n)} w^{(n)'} \\
& + a_{26}^{(n)} w^{(n)'} + a_{27}^{(n)} m_{\xi}^{(n)'} + a_{28}^{(n)} m_{\xi}^{(n)'} + a_{29}^{(n)} m_{\xi}^{(n)'} + \bar{Z}^{(n)} = 0 \\
& a_{30}^{(n)} u_{\xi}^{(n)'} + a_{31}^{(n)} u_{\xi}^{(n)'} + a_{32}^{(n)} u_{\theta}^{(n)'} + a_{33}^{(n)} u_{\theta}^{(n)'} + a_{34}^{(n)} w^{(n)'} + a_{35}^{(n)} w^{(n)'} + a_{36}^{(n)} m_{\xi}^{(n)'} = 0
\end{aligned}$$

The coefficients $a_i^{(n)} \sim a_{36}^{(n)}$ in Eqs. (7) are shown in Table 2.

4. Solution based on Sanders' shell theory

Sanders⁽²⁾ derived the following equations from Eqs. (3) by considering the virtual work in a shell:

$$\begin{aligned}
& \frac{\partial(rN_{\xi})}{\partial\xi} + a \frac{\partial\bar{N}_{\xi\theta}}{\partial\theta} - N_{\theta} \frac{\partial r}{\partial\xi} + \frac{ar}{r_{\xi}} Q_{\xi} \\
& + \frac{a}{2} \frac{\partial}{\partial\xi} \left[\left(\frac{1}{r_{\xi}} - \frac{1}{r_{\theta}} \right) \bar{M}_{\xi\theta} \right] + \sigma r X = 0 \\
& \frac{\partial(r\bar{N}_{\xi\theta})}{\partial\xi} + a \frac{\partial N_{\theta}}{\partial\theta} + \bar{N}_{\xi\theta} \frac{\partial r}{\partial\xi} + \frac{ar}{r_{\theta}} Q_{\theta} \\
& + \frac{r}{2} \frac{\partial}{\partial\xi} \left[\left(\frac{1}{r_{\theta}} - \frac{1}{r_{\xi}} \right) \bar{M}_{\xi\theta} \right] + \sigma r Y = 0 \\
& \frac{\partial(rQ_{\xi})}{\partial\xi} + a \frac{\partial Q_{\theta}}{\partial\theta} - \frac{ar}{r_{\xi}} N_{\xi} - \frac{ar}{r_{\theta}} N_{\theta} + \sigma r Z = 0 \\
& \frac{\partial(rM_{\xi})}{\partial\xi} + a \frac{\partial\bar{M}_{\xi\theta}}{\partial\theta} - M_{\theta} \frac{\partial r}{\partial\xi} - a r Q_{\xi} = 0 \\
& \frac{\partial(r\bar{M}_{\xi\theta})}{\partial\xi} + a \frac{\partial M_{\theta}}{\partial\theta} + \bar{M}_{\xi\theta} \frac{\partial r}{\partial\xi} - a r Q_{\theta} = 0 \quad (8)
\end{aligned}$$

where $\bar{N}_{\xi\theta}$ and $\bar{M}_{\xi\theta}$ are defined as follows:

$$\bar{N}_{\xi\theta} = (N_{\xi\theta} + N_{\theta\xi})/2, \quad \bar{M}_{\xi\theta} = (M_{\xi\theta} + M_{\theta\xi})/2 \quad (9)$$

These two variables are expanded in Fourier series as

$$\bar{N}_{\xi\theta} = \sigma_0 h \sum_{n=1}^{\infty} \bar{n}_{\xi\theta}^{(n)} \sin n\theta, \quad \bar{M}_{\xi\theta} = \sigma_0 (h^3/a) \sum_{n=1}^{\infty} \bar{m}_{\xi\theta}^{(n)} \sin n\theta \quad (10)$$

Elimination of Q_{ξ} and Q_{θ} from Eqs. (8) with use of Eqs. (1), (2), (4) and (10) leads to the following nondimensional equations:

$$\begin{aligned}
& n_{\xi}^{(n)'} + \gamma (n_{\xi}^{(n)'} - n_{\theta}^{(n)'}) + \frac{n}{\rho} n_{\theta}^{(n)'} + \lambda^2 (\omega_{\xi} m_{\xi}^{(n)'} + \gamma m_{\xi}^{(n)'} - m_{\theta}^{(n)'}) \\
& + \frac{n}{2\rho} (3\omega_{\xi} - \omega_{\theta}) \bar{m}_{\xi\theta}^{(n)'} + \bar{\gamma}^{(n)} = 0 \\
& \bar{n}_{\xi\theta}^{(n)'} + 2\gamma \bar{n}_{\xi\theta}^{(n)'} - \frac{n}{\rho} n_{\theta}^{(n)'} + \lambda^2 \left(-\frac{n}{\rho} \omega_{\theta} m_{\theta}^{(n)'} + \frac{1}{2} (3\omega_{\theta} - \omega_{\xi}) \bar{m}_{\xi\theta}^{(n)'} \right. \\
& + \frac{1}{2} (3\gamma \omega_{\theta} + \gamma \omega_{\xi} - \omega_{\xi}^2) \bar{m}_{\xi\theta}^{(n)'} \left. \right) + \bar{\gamma}^{(n)} = 0 \quad (11) \\
& -\omega_{\xi} n_{\xi}^{(n)'} - \omega_{\theta} n_{\theta}^{(n)'} + \lambda^2 (m_{\xi}^{(n)'} + 2\gamma m_{\xi}^{(n)'} - \omega_{\xi} \omega_{\theta} m_{\theta}^{(n)'}) \\
& + (\omega_{\xi} \omega_{\theta} - \frac{n^2}{\rho^2}) m_{\theta}^{(n)'} - \gamma m_{\theta}^{(n)'} + 2\frac{n}{\rho} \bar{m}_{\xi\theta}^{(n)'} + 2\gamma \bar{n}_{\xi\theta}^{(n)'} + \bar{Z}^{(n)} = 0
\end{aligned}$$

The relations between the stress resultants or the stress couples and the displacements in Sanders' theory as shown in Table 1 are used in Eqs. (11), then the fundamental differential equations are obtained. They have the same forms as Eqs. (7), and their thirty-six coefficients are shown in Table 2.

5. Solution based on Mizoguchi's shell theory

Let us assume the relation $M_{\xi\theta} = M_{\theta\xi}$, then the sixth equation of Eqs. (3) is treated as

$$N_{\xi\theta} - M_{\xi\theta}/r_{\theta} = N_{\theta\xi} - M_{\theta\xi}/r_{\xi} = \bar{N} \quad (12)$$

Then the shear stress resultants are given as

$$N_{\xi\theta} = \bar{N} + M_{\xi\theta}/r_{\theta}, \quad N_{\theta\xi} = \bar{N} + M_{\theta\xi}/r_{\xi} \quad (13)$$

Substituting Eqs.(13) into the first to fifth equations of Eqs.(3), and eliminating $\hat{Q}_\xi$ and $\hat{Q}_\theta$ from these equations with use of Eqs.(1), we obtain the following three equations :

$$\begin{aligned} & \alpha \rho \frac{\partial N_\xi}{\partial \xi} + \alpha \rho' (N_\xi - N_\theta) + \alpha \frac{\partial \bar{N}}{\partial \theta} + \omega_\xi \frac{\partial M_{\xi\theta}}{\partial \theta} \\ & + \omega_\xi \left(\rho \frac{\partial M_\xi}{\partial \xi} + \rho' M_\xi + \frac{M_{\xi\theta}}{\partial \theta} - \rho' M_\theta \right) + \alpha^2 \rho \chi = 0 \\ & \alpha \rho \frac{\partial \bar{N}}{\partial \xi} + 2\alpha \rho' \bar{N} + (\rho \omega_\theta + \rho' \omega_\theta + \rho' \omega_\xi) M_{\xi\theta} + \omega_\theta \rho \frac{\partial M_{\xi\theta}}{\partial \xi} \\ & + \alpha \frac{\partial N_\theta}{\partial \theta} + \omega_\theta \left(\rho \frac{\partial M_\theta}{\partial \xi} + 2\rho' M_{\xi\theta} + \frac{\partial M_\theta}{\partial \theta} \right) - \alpha^2 \rho \gamma = 0 \\ & \rho \left(\frac{\partial M_\xi}{\partial \xi} + \gamma \frac{\partial M_\xi}{\partial \xi} + \gamma M_\xi + \frac{\partial M_{\xi\theta}}{\partial \theta} - \frac{\gamma}{\rho} \frac{\partial M_{\xi\theta}}{\partial \theta} - \gamma \frac{\partial M_\theta}{\partial \xi} \right. \\ & \left. - \gamma' M_\theta \right) + \rho' \left(\frac{\partial M_\xi}{\partial \xi} + \gamma M_\xi + \frac{\partial M_{\xi\theta}}{\partial \theta} - \gamma M_\theta \right) + \left(\frac{\partial M_{\xi\theta}}{\partial \xi} \right. \\ & \left. + 2\gamma \frac{\partial M_{\xi\theta}}{\partial \theta} + \frac{\partial M_\theta}{\partial \theta} \right) - \alpha \rho \left(\omega_\xi N_\xi + \omega_\theta N_\theta \right) + \alpha^2 \rho Z = 0 \end{aligned} \quad (14)$$

Here, the variable $\bar{N}$ is expanded in Fourier series as

$$\bar{N} = \sigma_0 h \sum_{n=1}^{\infty} \bar{n}^{(n)} \sin n\theta \quad (15)$$

By using Eqs.(1) (2) (4) and (15), the non-dimensional forms of Eqs.(14) are obtained as follows :

$$\begin{aligned} & n_\xi^{(n)} + \gamma (n_\xi^{(n)} - n_\theta^{(n)}) + \frac{n}{\rho} \bar{n}^{(n)} + \lambda^2 \omega_\xi (m_\xi^{(n)} + \gamma m_\xi^{(n)} \\ & + 2 \frac{n}{\rho} m_{\xi\theta}^{(n)} - \gamma m_\theta^{(n)}) + \bar{x}^{(n)} = 0 \\ & \bar{n}^{(n)} + 2\gamma \bar{n}^{(n)} - \frac{n}{\rho} n_\theta^{(n)} + \lambda^2 \left(2\gamma (\omega_\xi + \omega_\theta) m_{\xi\theta}^{(n)} \right. \\ & \left. + 2\omega_\theta m_{\xi\theta}^{(n)} - \omega_\theta \frac{n}{\rho} m_\theta^{(n)} \right) + \bar{y}^{(n)} = 0 \\ & -\omega_\xi n_\xi^{(n)} - \omega_\theta n_\theta^{(n)} + \lambda^2 \left(m_\xi^{(n)} + 2\gamma m_\xi^{(n)} - \omega_\xi \omega_\theta m_\xi^{(n)} - \gamma m_\theta^{(n)} \right. \\ & \left. + (\omega_\xi \omega_\theta - \frac{n^2}{\rho^2}) m_\theta^{(n)} + 2 \frac{n}{\rho} m_{\xi\theta}^{(n)} + 2\gamma \frac{n}{\rho} m_{\xi\theta}^{(n)} \right) + \bar{z}^{(n)} = 0 \end{aligned} \quad (16)$$

In Mizoguchi's shell theory, the relations between the stress resultants or the stress couples and the strains are given by the following relations :

$$\begin{aligned} N_\xi &= \frac{12D}{h^2} (\epsilon_\xi + \nu \epsilon_\theta), \quad N_\theta = \frac{12D}{h^2} (\epsilon_\theta + \nu \epsilon_\xi) \\ \bar{N} &= 6(1-\nu) \frac{D}{h^2} \epsilon_{\xi\theta}, \quad M_\xi = D(\kappa_\xi + \nu \kappa_\theta) \\ M_\theta &= D(\kappa_\theta + \nu \kappa_\xi), \quad M_{\xi\theta} = (1-\nu) D \kappa_{\xi\theta} \end{aligned} \quad (17)$$

where $D = E h^3 / 12(1-\nu^2)$. Further, the relations between the strains and the displacements are expressed in the following relations :

$$\begin{aligned} \epsilon_\xi &= \frac{\partial U}{\partial \xi} + \frac{W}{r_\xi}, \quad \epsilon_\theta = \frac{\partial V}{r \partial \theta} + \frac{\gamma}{\alpha} U + \frac{W}{r_\theta} \\ \epsilon_{\xi\theta} &= \frac{\partial U}{r \partial \theta} - \frac{\gamma}{\alpha} V + \frac{\partial V}{\partial \xi}, \quad \kappa_\xi = \frac{\partial}{\partial \xi} \left(\frac{U}{r_\xi} \right) - \frac{\partial^2 W}{\alpha^2 \partial \xi^2} \\ \kappa_\theta &= \frac{\gamma}{\alpha r_\xi} U + \frac{\partial}{r \partial \theta} \left(\frac{V}{r_\theta} \right) - \frac{\partial^2 W}{\alpha^2 \partial \theta^2} - \frac{\gamma}{\alpha^2} \frac{\partial W}{\partial \xi} \\ \kappa_{\xi\theta} &= \frac{\partial U}{r_\xi \partial \theta} - \frac{\gamma}{\alpha r_\xi} V + \frac{\partial}{\partial \xi} \left(\frac{V}{r_\theta} \right) - \frac{\partial^2 W}{\alpha^2 \partial \xi \partial \theta} + \frac{\gamma}{\alpha} \frac{\partial W}{\partial \theta} \end{aligned} \quad (18)$$

From Eqs.(4) (15) (17) and (18), the relations between the stress resultants or the stress couples and the displacements are obtained as shown in Table 1. Substitution of these relations into Eqs.(16) leads to the fundamental differential equa-

tions. They have the same forms as Eqs. (7), and their thirty-six coefficients are shown in Table 2.

6. Boundary conditions

6.1 Boundary conditions in Flügge's and Mizoguchi's theories

The stress resultants and the stress couples which are to be considered at the boundaries are N_ξ , $\hat{N}_{\xi\theta}$, $\hat{Q}_\xi$ and M_ξ as shown in Fig.3. $\hat{N}_{\xi\theta}$ and $\hat{Q}_\xi$ are the effective shearing stress resultant and the effective transverse shearing stress resultant, respectively. They are given by the following relation⁽¹⁾⁽⁶⁾:

$$\begin{aligned} \hat{N}_{\xi\theta} &= N_{\xi\theta} + \frac{M_{\xi\theta}}{r_\theta} \\ \hat{Q}_\xi &= Q_\xi + \frac{\partial M_{\xi\theta}}{r \partial \theta} \end{aligned} \quad (19)$$

They are expanded in Fourier series as

$$\begin{aligned} \hat{N}_{\xi\theta} &= \sigma_0 h \sum_{n=1}^{\infty} \hat{n}_{\xi\theta}^{(n)} \sin n\theta \\ \hat{Q}_\xi &= \sigma_0 h \sum_{n=1}^{\infty} \hat{q}_\xi^{(n)} \cos n\theta \end{aligned} \quad (20)$$

Substitution of Eqs.(4), (20) and the fifth equation of Eqs.(3) into Eqs.(19) leads to the following equations :

$$\begin{aligned} \hat{n}_{\xi\theta}^{(n)} &= n_{\xi\theta}^{(n)} + \lambda^2 \omega_\theta m_{\xi\theta}^{(n)}, \\ \hat{q}_\xi^{(n)} &= \lambda^2 \left(m_\xi^{(n)} + \gamma (m_\xi^{(n)} - m_\theta^{(n)}) + 2 \frac{n}{\rho} m_{\xi\theta}^{(n)} \right) \end{aligned} \quad (21)$$

From Eqs.(21) and the relations between the stress resultants or the stress couples and the displacements due to Flügge's theory in Table 1, the following relations are derived :

$$\begin{aligned} n_\xi^{(n)} &= b_1^{(n)} u_\xi^{(n)} + b_2^{(n)} u_\theta^{(n)} + b_3^{(n)} u_\theta^{(n)} + b_4^{(n)} w^{(n)} \\ \hat{n}_{\xi\theta}^{(n)} &= b_5^{(n)} u_\xi^{(n)} + b_6^{(n)} u_\theta^{(n)} + b_7^{(n)} u_\theta^{(n)} + b_8^{(n)} w^{(n)} + b_9^{(n)} w^{(n)} \\ \hat{q}_\xi^{(n)} &= b_{10}^{(n)} u_\xi^{(n)} + b_{11}^{(n)} u_\theta^{(n)} + b_{12}^{(n)} u_\theta^{(n)} + b_{13}^{(n)} w^{(n)} + b_{14}^{(n)} w^{(n)} \\ &+ b_{15}^{(n)} m_\xi^{(n)} + b_{16}^{(n)} m_\xi^{(n)} \end{aligned} \quad (22)$$

The coefficients $b_1^{(n)} \sim b_{16}^{(n)}$ are shown in Table 3.

In quite the same way, Eqs.(22) are derived for Mizoguchi's theory. The coefficients $b_1^{(n)} \sim b_{16}^{(n)}$ for this case are shown in Table 3.

6.2 Boundary conditions in Sanders' theory
According to Sanders' shell theory⁽²⁾, the virtual work at a boundary is given as

$$N_\xi U + \hat{N}_{\xi\theta} V + \hat{Q}_\xi W + M_\xi \Phi_\xi \quad (23)$$

where $\hat{N}_{\xi\theta}$ and $\hat{Q}_\xi$ are given as

$$\hat{N}_{\xi\theta} = N_{\xi\theta} + \frac{1}{2} \left(\frac{3}{r_\theta} - \frac{1}{r_\xi} \right) M_{\xi\theta}, \quad \hat{Q}_\xi = Q_\xi + \frac{\partial M_{\xi\theta}}{r \partial \theta} \quad (24)$$

Use of Eqs.(1) (10) (20) and the fourth equation of Eqs.(8) in Eqs.(24) leads to

$$\begin{aligned} \hat{n}_{\xi\theta}^{(n)} &= \bar{n}_{\xi\theta}^{(n)} - \frac{\lambda^2}{2} (\omega_\xi - 3\omega_\theta) \bar{m}_{\xi\theta}^{(n)} \\ \hat{q}_\xi^{(n)} &= \lambda^2 \left(m_\xi^{(n)} + \gamma (m_\xi^{(n)} - m_\theta^{(n)}) + 2 \frac{n}{\rho} \bar{m}_{\xi\theta}^{(n)} \right) \end{aligned} \quad (25)$$

From Eqs.(25) and the relations in Table 1 attributed to Sanders' Eqs.(22) are obtained in this case, too. The coefficients $b_i^{(n)} \sim b_{16}^{(n)}$ are shown in Table 3.

6.3 Typical examples of boundary conditions

The boundary conditions which are used frequently in practical cases are shown in the following :

Free edge : $\eta_{\xi}^{(n)} = \hat{\eta}_{\xi\theta}^{(n)} = \hat{\eta}_{\xi\xi}^{(n)} = m_{\xi}^{(n)} = 0$

Fixed edge : $u_{\xi}^{(n)} = u_{\theta}^{(n)} = w^{(n)} = w^{(n)'} = 0$

Simply supported edge (movable) :

$$\eta_{\xi}^{(n)} = u_{\theta}^{(n)} = w^{(n)} = m_{\xi}^{(n)} = 0$$

Simply supported edge (immovable) :

$$u_{\xi}^{(n)} = u_{\theta}^{(n)} = w^{(n)} = m_{\xi}^{(n)} = 0$$

For the case of the pole as the vertex of a spherical shell, the following relations⁽⁷⁾ are used :

$$n = 0 : u_{\xi} = u_{\theta} = w' = m_{\xi}' = 0$$

$$n = 1 : u_{\xi} + u_{\theta} = u_{\theta}' = w = m_{\xi} = 0$$

$$n \geq 2 : u_{\xi}^{(n)} = u_{\theta}^{(n)} = w^{(n)} = w^{(n)'} = 0$$

7. Fundamental equations for free vibration problems

Since statical bending problems have been considered in the previous chapters, the method for free vibration problems is treated in this chapter. The following vibrational inertia forces should be considered as external forces :

$$X = -\rho^* h \frac{\partial^2 u}{\partial t^2}, \quad Y = -\rho^* h \frac{\partial^2 v}{\partial t^2}, \quad Z = -\rho^* h \frac{\partial^2 w}{\partial t^2} \quad (26)$$

where ρ^* is the mass density, and t is the time. The following nondimensional expressions are defined as Eqs.(4) (9) and (15) for the case of statical bending problems :

$$N_{\xi} = \sigma_0 h \eta_{\xi}^{(n)} \cos n\theta \cos \Omega \tau, \quad N_{\theta} = \sigma_0 h \eta_{\theta}^{(n)} \cos n\theta \cos \Omega \tau$$

$$N_{\xi\theta} = \sigma_0 h \eta_{\xi\theta}^{(n)} \sin n\theta \cos \Omega \tau, \quad \bar{N}_{\xi\theta} = \sigma_0 h \bar{\eta}_{\xi\theta}^{(n)} \sin n\theta \cos \Omega \tau$$

$$\bar{N} = \sigma_0 h \bar{\eta}^{(n)} \sin n\theta \cos \Omega \tau,$$

$$M_{\xi} = \sigma_0 (h^3/a) m_{\xi}^{(n)} \cos n\theta \cos \Omega \tau$$

$$M_{\theta} = \sigma_0 (h^3/a) m_{\theta}^{(n)} \cos n\theta \cos \Omega \tau,$$

$$M_{\xi\theta} = \sigma_0 (h^3/a) m_{\xi\theta}^{(n)} \sin n\theta \cos \Omega \tau$$

$$\bar{M}_{\xi\theta} = \sigma_0 (h^3/a) \bar{m}_{\xi\theta}^{(n)} \sin n\theta \cos \Omega \tau, \quad (27)$$

$$Q_{\xi} = \sigma_0 h q_{\xi}^{(n)} \cos n\theta \cos \Omega \tau$$

$$Q_{\theta} = \sigma_0 h q_{\theta}^{(n)} \sin n\theta \cos \Omega \tau,$$

$$U = \sigma_0 (a/E) u_{\xi}^{(n)} \cos n\theta \cos \Omega \tau$$

$$V = \sigma_0 (a/E) u_{\theta}^{(n)} \sin n\theta \cos \Omega \tau,$$

$$W = \sigma_0 (a/E) w^{(n)} \cos n\theta \cos \Omega \tau$$

where Ω and τ are the nondimensional frequency and time, respectively, defined as

$$\Omega = 2\pi a f \sqrt{\rho^*/E} \quad \tau = (t/a) \sqrt{E/\rho^*} \quad (28)$$

where f is the free frequency.

In free vibration problems, the Fourier index n expresses half the number of the nodal lines parallel to the axis of symmetry. Applying Eqs.(3) and (8) to Eqs.(26) and (27), and modifying equations

in the same way as in chapters 3, 4, and 5, the fundamental differential equations are derived. They have the same forms as Eqs.(7), and the coefficients $a_i^{(n)} \sim a_{16}^{(n)}$ are given by Table 2 as in the cases of statical bending problems. But in this case, the terms of external forces are given by

$$\bar{X}^{(n)} = \Omega^2 u_{\xi}^{(n)}, \quad \bar{Y}^{(n)} = \Omega^2 u_{\theta}^{(n)}, \quad \bar{Z}^{(n)} = \Omega^2 w^{(n)} \quad (29)$$

8. Numerical examples

The finite difference method is used in this research for solving the fundamental differential equations (7), but the details of its procedures are omitted here for brevity. They may be found in references (8) and (9). In the following, numerical results for statical bending or free vibration problems of some axisymmetric shells are stated. The materials of these shells are assumed as mild steel, i.e.

$$\rho^* = 7.9 \times 10^{-6} \text{ kg sec}^2/\text{mm}^4$$

$$E = 2.1 \times 10^4 \text{ kg/mm}^2$$

$$\nu = 0.3$$

8.1 Statical bending problem

We consider a truncated conical shell, in which the upper edge is free and the lower edge is fixed, being subjected to a lateral wind pressure. For simplification, its pressure distribution is assumed as

$$X = 0, \quad Y = 0, \quad Z = \sigma_0 (h/a) p_0 \cos \theta$$

where p_0 is the nondimensional reference pressure and assumed as $p_0 = a/h$. In numerical calculations, the number of the difference points is 151, and the mean radius R_0 is taken as the reference length a . The distributions of the stress resultants and the stress couples, which are obtained by the above-mentioned three methods, are shown in Figs.5 and 6 for the cases of $a/h = 30$ and 150, respectively.

The stresses on shell surfaces are calculated by the following equations :

$$(\sigma_{\xi}/\sigma_0)_{out} = n_{\xi} + 6\lambda m_{\xi}, \quad (\sigma_{\xi}/\sigma_0)_{in} = n_{\xi} - 6\lambda m_{\xi}$$

$$(\sigma_{\theta}/\sigma_0)_{out} = n_{\theta} + 6\lambda m_{\theta}, \quad (\sigma_{\theta}/\sigma_0)_{in} = n_{\theta} - 6\lambda m_{\theta}$$

where $()_{in}$ and $()_{out}$ are the nondimensional stresses on the inner and outer surfaces, respectively. Table 4 and 5 indicate the surface stresses at the lower edge.

8.2 Free vibration problems

The free vibration problems of three kinds of shells are solved, and the results are checked by experiments.

8.2.1 Cylindrical shell

A cylindrical shell with both edges fixed as shown in Fig.7 is considered. The radius R is taken as the reference length a , and the number of the difference points is 101. The free frequencies calculated for some values of the Fourier index n and the number of the nodal line C are shown in Table 6 together with the corresponding experimental data. It is

found that the numerical results obtained by the three methods and the experimental data almost coincide with each other except for the case of $n = 5$ and $C = 0$.

8.2.2 Spherical shell

A spherical shell with the lower edge fixed as shown in Fig.8 is considered. The radius R is taken as the reference length a , and the number of the difference points is 151. The free frequencies calculated are shown in Table 7. The experimental value in Table 7 is only for $n = 1$ and $C = 0$, since the frequencies for other values of n or C are so high that they can not be measured by use of the authors' measuring apparatus.

8.2.3 Model of pressure vessel

A model of pressure vessel as shown in Fig.9 is considered. Its lower edge is fixed, and the radius of the cylindrical part is taken as the reference length a . The difference length in each part of the model is assumed to be almost equal, and the number of the difference points is $51+51+151+15+51 = 319$. The theoretical values of free frequencies are shown in Table 8 together with the experimental data. It is found from this table that the theoretical values and the experimental ones almost coincide except for the case of $n = 6$ and $C = 1$.

9. Considerations

The accuracy of solution by the finite difference method is affected by the number of the difference points and of the effective figures used in numerical calculations. In this research, this fact is always taken into consideration.

Comparing the coefficients of fundamental differential equations (7) for the above-mentioned three methods, it is found that the differences between them occur almost in the terms containing $(h/a)^2$. In the above static bending problem, the differences between the stress values for the three methods are found to be very small both for $a/h = 30$ and 150 as shown in Tables 4 and 5. Also in the three free vibration problems treated above, the differences between the free frequencies by the three methods are very small as shown in Tables 6, 7 and 8, although they become a little larger for the case of a shell of rather complex shape (the model of pressure vessel).

The experimental data of free frequency almost coincide with the corresponding calculated values except in two cases, i.e. the cases of $n = 5$, $C = 0$ for the cylindrical shell and $n = 6$, $C = 1$ for the model of pressure vessel. These two cases are the lowest frequency of shells, respectively, and we could observe remarkably large amplitudes and some random vibrational phenomena in the experiment. The authors suppose that nonlinear vibration phenomena are occurring in these cases.⁽¹⁰⁾ Therefore, the methods discussed in this research can not be applied to such cases.

10. Conclusion

Following the method based on Sanders' shell theory, two methods based on Flügge's and Mizoguchi's shell theories are proposed in this research for solving the static bending and free vibration problems of the general axisymmetric shell. And it is found that the differences between the results obtained by these three methods are considerably small for the usual shells with small values of h/a .

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Table 1. Relations of stress resultants stress couples and displacements.

(F: Flügge, M: Mizoguchi, S: Sanders) $b = \frac{1}{(1-\nu^2)}$, $d = \frac{b}{12}$

$n_{\xi}^{(n)}$	F·M·S	$b[u_{\xi}^{(n)} + \nu r u_{\xi}^{(n)} + \nu \frac{n}{\rho} u_{\theta}^{(n)} + (\omega_{\xi} + \nu \omega_{\theta}) W^{(n)}]$
$n_{\theta}^{(n)}$	F·M·S	$b[\nu u_{\xi}^{(n)} + r u_{\xi}^{(n)} + \frac{n}{\rho} u_{\theta}^{(n)} + (\omega_{\theta} + \nu \omega_{\xi}) W^{(n)}]$
$n_{\xi\theta}^{(n)}$	F	$b \frac{(1-\nu)}{2} (u_{\theta}^{(n)} - r u_{\xi}^{(n)} - \frac{n}{\rho} u_{\xi}^{(n)})$
	M	$b \frac{(1-\nu)}{2} (u_{\theta}^{(n)} - r u_{\xi}^{(n)} - \frac{n}{\rho} u_{\xi}^{(n)}) + \lambda^2 \omega_{\theta} m_{\xi\theta}^{(n)}$
$\bar{n}_{\xi\theta}^{(n)}$	S	$b \frac{(1-\nu)}{2} (u_{\theta}^{(n)} - r u_{\xi}^{(n)} - \frac{n}{\rho} u_{\xi}^{(n)})$
$n_{\theta\xi}^{(n)}$	F	$n_{\xi\theta}^{(n)}$
	M	$b \frac{(1-\nu)}{2} (u_{\theta}^{(n)} - r u_{\xi}^{(n)} - \frac{n}{\rho} u_{\xi}^{(n)}) + \lambda^2 \omega_{\xi} m_{\theta\xi}^{(n)}$
$m_{\xi}^{(n)}$	F	$d(-W^{(n)}) - \nu r W^{(n)} + \nu \frac{n^2}{\rho^2} W^{(n)}$
	M·S	$d[\omega_{\xi} u_{\xi}^{(n)} + (\omega_{\xi}' + \nu r \omega_{\xi}) u_{\xi}^{(n)} + \nu \omega_{\theta} \frac{n}{\rho} u_{\theta}^{(n)} - W^{(n)'} - \nu r W^{(n)'} + \nu \frac{n^2}{\rho^2} W^{(n)'}]$
$m_{\theta}^{(n)}$	F	$\nu m_{\xi}^{(n)} + d(1-\nu^2)(\frac{n^2}{\rho^2} W^{(n)} - r W^{(n)})$
	M·S	$\nu m_{\xi}^{(n)} + d(1-\nu^2)(r \omega_{\xi} u_{\xi}^{(n)} + \omega_{\theta} \frac{n}{\rho} u_{\theta}^{(n)} - r W^{(n)} + \frac{n^2}{\rho^2} W^{(n)})$

$m_{\xi 0}^{(n)}$	F	$d(1-\nu)\left(\frac{n}{\rho}W^{(n)} - \frac{n}{\rho}rW^{(n)}\right)$	$a_{12}^{(n)}$	F	$b\frac{(1-\nu)}{2}$
$m_{\xi 0}^{(n)}$	M	$d(1-\nu)\left(-\omega_{\xi}\frac{n}{\rho}u_{\xi}^{(n)} + \omega_{\theta}u_{\theta}^{(n)} - r\omega_{\theta}u_{\theta}^{(n)} + \frac{n}{\rho}W^{(n)} - r\frac{n}{\rho}W^{(n)}\right)$	$a_{12}^{(n)}$	M	$b\frac{(1-\nu)}{2} + 2\lambda^2d(1-\nu)\omega_{\theta}^2$
$\bar{m}_{\xi 0}^{(n)}$	S	$d\frac{(1-\nu)}{2}\left[(\omega_{\theta}-3\omega_{\xi})\frac{n}{2\rho}u_{\xi}^{(n)} + \frac{(3\omega_{\theta}-\omega_{\xi})}{2}u_{\theta}^{(n)} + \frac{(\omega_{\xi}-3\omega_{\theta})}{2}r u_{\theta}^{(n)} + 2\frac{n}{\rho}W^{(n)} - 2\frac{n}{\rho}rW^{(n)}\right]$	$a_{12}^{(n)}$	S	$b\frac{(1-\nu)}{2} + \lambda^2d\frac{1-\nu}{8}(3\omega_{\theta}-\omega_{\xi})^2$
$m_{\theta \xi}^{(n)}$	F.M	$m_{\xi 0}^{(n)}$	$a_{13}^{(n)}$	F	$b\frac{(1-\nu)}{2}r$
$q_{\xi}^{(n)}$	F.M	$\lambda^2\left[m_{\xi}^{(n)} + r(m_{\xi}^{(n)} - m_{\theta}^{(n)}) + \frac{n}{\rho}m_{\xi 0}^{(n)}\right]$	$a_{13}^{(n)}$	M	$b\frac{(1-\nu)}{2}r + 2\lambda^2d(1-\nu)r\omega_{\theta}(2\omega_{\xi}-\omega_{\theta})$
$q_{\xi}^{(n)}$	S	$\lambda^2\left[m_{\xi}^{(n)} + r(m_{\xi}^{(n)} - m_{\theta}^{(n)}) + \frac{n}{\rho}\bar{m}_{\xi 0}^{(n)}\right]$	$a_{13}^{(n)}$	S	$b\frac{(1-\nu)}{2}r + \lambda^2d\frac{(1-\nu)}{8}(3\omega_{\theta}-\omega_{\xi})[r(5\omega_{\xi}-3\omega_{\theta})-2\omega_{\xi}^2]$
$q_{\theta}^{(n)}$	F.M	$\lambda^2\left[m_{\theta}^{(n)} + 2r\bar{m}_{\theta 0}^{(n)} - \frac{n}{\rho}m_{\theta}^{(n)}\right]$	$a_{14}^{(n)}$	F	$b\frac{(1-\nu)}{2}(\omega_{\xi}\omega_{\theta}-r^2) - b\frac{n^2}{\rho^2}$
$q_{\theta}^{(n)}$	S	$\lambda^2\left[\bar{m}_{\theta}^{(n)} + 2r\bar{m}_{\theta 0}^{(n)} - \frac{n}{\rho}m_{\theta}^{(n)}\right]$	$a_{14}^{(n)}$	M	$b\frac{(1-\nu)}{2}(\omega_{\xi}\omega_{\theta}-r^2) - b\frac{n^2}{\rho^2} + \lambda^2d(1-\nu)(2r\omega_{\theta}^2 + 2\omega_{\xi}\omega_{\theta}^3 - 4r^2\omega_{\xi}\omega_{\theta} - \omega_{\theta}^2\frac{n^2}{\rho^2})$
			$a_{14}^{(n)}$	S	$b\frac{(1-\nu)}{2}(\omega_{\xi}\omega_{\theta}-r^2) - b\frac{n^2}{\rho^2} + \lambda^2d(1-\nu)\left[\frac{\omega_{\xi}\omega_{\theta}}{8}(3\omega_{\theta}-\omega_{\xi})^2 - (1+\nu)\omega_{\theta}^2\frac{n^2}{\rho^2} + \frac{r}{8}(3\omega_{\theta}-\omega_{\xi})(2\omega_{\xi}^2-5r\omega_{\xi}+3r\omega_{\theta})\right]$

Table 2. Coefficients of fundamental equation.

(F: Flügge, M: Mizoguchi, S: Sanders)

$a_1^{(n)}$	F.M.S	b	$a_{15}^{(n)}$	F	$\lambda^2d(1-\nu)\omega_{\theta}\frac{n}{\rho}$
$a_2^{(n)}$	F.M.S	br	$a_{15}^{(n)}$	M	$2\lambda^2d(1-\nu)\omega_{\theta}\frac{n}{\rho}$
$a_3^{(n)}$	F	$-\nu b\omega_{\xi}\omega_{\theta} - br^2 - b\frac{(1-\nu)}{2}\frac{n^2}{\rho^2}$	$a_{15}^{(n)}$	S	$\lambda^2d(1-\nu)(3\omega_{\theta}-\omega_{\xi})\frac{n}{2\rho}$
$a_3^{(n)}$	M	$-\nu b\omega_{\xi}\omega_{\theta} - br^2 - b\frac{(1-\nu)}{2}\frac{n^2}{\rho^2} - \lambda^2d(1-\nu)\left[2\frac{n^2}{\rho^2} + (1+\nu)r^2\right]\omega_{\xi}^2$	$a_{16}^{(n)}$	F	$\lambda^2d(1-\nu)r\omega_{\theta}\frac{n}{\rho}$
$a_3^{(n)}$	S	$-\nu b\omega_{\xi}\omega_{\theta} - br^2 - b\frac{(1-\nu)}{2}\frac{n^2}{\rho^2} - \lambda^2d(1-\nu)\left[(1+\nu)r^2\omega_{\xi}^2 + \frac{(3\omega_{\xi}-\omega_{\theta})^2n^2}{8\rho^2}\right]$	$a_{16}^{(n)}$	M	$\lambda^2d(1-\nu)r(2\omega_{\xi}-\omega_{\theta}+\nu\omega_{\theta})\frac{n}{\rho}$
$a_4^{(n)}$	F	$b(1+\nu)\frac{n}{2\rho}$	$a_{16}^{(n)}$	S	$\lambda^2d(1-\nu)\left[2(1+\nu)r\omega_{\theta}-\omega_{\xi}^2+3r(\omega_{\xi}-\omega_{\theta})\right]\frac{n}{2\rho}$
$a_4^{(n)}$	M	$b(1+\nu)\frac{n}{2\rho} + 2\lambda^2d(1-\nu)\omega_{\xi}\omega_{\theta}\frac{n}{\rho}$	$a_{17}^{(n)}$	F	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho} + \lambda^2d(1-\nu)\omega_{\theta}\frac{n}{\rho}\left[\omega_{\xi}\omega_{\theta} - (1+\nu)\frac{n^2}{\rho^2}\right]$
$a_4^{(n)}$	S	$b(1+\nu)\frac{n}{2\rho} + \lambda^2d(1-\nu)(3\omega_{\xi}-\omega_{\theta})(3\omega_{\theta}-\omega_{\xi})\frac{n}{8\rho}$	$a_{17}^{(n)}$	M	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho} + \lambda^2d(1-\nu)\frac{n}{\rho} \times [-2r^2\omega_{\xi} + 2r^2\omega_{\theta} + 2\omega_{\xi}\omega_{\theta}^2 - (1+\nu)\omega_{\theta}\frac{n^2}{\rho^2}]$
$a_5^{(n)}$	F	$-b(3-\nu)r\frac{n}{2\rho}$	$a_{17}^{(n)}$	S	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho} + \lambda^2d(1-\nu)\frac{n}{2\rho} \times [r\omega_{\xi}^2 - 2r^2\omega_{\xi} - 2(1+\nu)\omega_{\theta}\frac{n^2}{\rho^2} + (3\omega_{\theta}-\omega_{\xi})(r^2+\omega_{\xi}\omega_{\theta})]$
$a_5^{(n)}$	M	$-b(3-\nu)r\frac{n}{2\rho} - \lambda^2d(1-\nu)(3+\nu)r\omega_{\xi}\omega_{\theta}\frac{n}{\rho}$	$a_{18}^{(n)}$	F.M.S	$-\lambda^2\nu\omega_{\theta}\frac{n}{\rho}$
$a_5^{(n)}$	S	$-b(3-\nu)r\frac{n}{2\rho} - \lambda^2d(1-\nu)r\frac{n}{\rho} \times \left[\frac{(3\omega_{\xi}-\omega_{\theta})(3\omega_{\theta}-\omega_{\xi})}{8} + (1+\nu)\omega_{\xi}\omega_{\theta}\right]$	$a_{19}^{(n)}$	F	$-b(\omega_{\xi}+\nu\omega_{\theta})$
$a_6^{(n)}$	F	$b(\omega_{\xi}+\nu\omega_{\theta}) + \lambda^2d(1-\nu)\left[\frac{n^2}{\rho^2} + (1+\nu)r^2\right]\omega_{\xi}$	$a_{19}^{(n)}$	M	$-b(\omega_{\xi}+\nu\omega_{\theta}) - \lambda^2d(1-\nu)\omega_{\xi}\left[(1+\nu)r^2 + 2\frac{n^2}{\rho^2}\right]$
$a_6^{(n)}$	M	$b(\omega_{\xi}+\nu\omega_{\theta}) + \lambda^2d(1-\nu)\left[2\frac{n^2}{\rho^2} + (1+\nu)r^2\right]\omega_{\xi}$	$a_{19}^{(n)}$	S	$-b(\omega_{\xi}+\nu\omega_{\theta}) - \lambda^2d(1-\nu)\left[(1+\nu)r^2\omega_{\xi} + (3\omega_{\xi}-\omega_{\theta})\frac{n^2}{2\rho^2}\right]$
$a_6^{(n)}$	S	$b(\omega_{\xi}+\nu\omega_{\theta}) + \lambda^2d(1-\nu)\left[(1+\nu)r^2\omega_{\xi} + (3\omega_{\xi}-\omega_{\theta})\frac{n^2}{2\rho^2}\right]$	$a_{20}^{(n)}$	F	$-br(\omega_{\theta}+\nu\omega_{\xi})$
$a_7^{(n)}$	F	$b(\omega_{\xi}^2+r\omega_{\xi}-r\omega_{\theta}) - \lambda^2d(1-\nu)(2+\nu)r\omega_{\xi}\frac{n^2}{\rho^2}$	$a_{20}^{(n)}$	M	$-br(\omega_{\theta}+\nu\omega_{\xi}) + \lambda^2d(1-\nu) \times [(1+\nu)r(2\omega_{\xi}^2\omega_{\theta}+r\omega_{\xi}^2-r\omega_{\xi}\omega_{\theta}\frac{n^2}{\rho^2}) - 2\omega_{\xi}^2\frac{n^2}{\rho^2}]$
$a_7^{(n)}$	M	$b(\omega_{\xi}^2+r\omega_{\xi}-r\omega_{\theta}) - \lambda^2d(1-\nu)(3+\nu)r\omega_{\xi}\frac{n^2}{\rho^2}$	$a_{20}^{(n)}$	S	$-br(\omega_{\theta}+\nu\omega_{\xi}) + \lambda^2d(1-\nu)\left[(1+\nu)r(2\omega_{\xi}^2\omega_{\theta} + r^2\omega_{\xi}^2 - r\omega_{\xi}^2 - \omega_{\xi}\omega_{\theta}\frac{n^2}{\rho^2}) + (r\omega_{\xi}^2 - r\omega_{\theta} - 3\omega_{\xi})\frac{n^2}{2\rho^2}\right]$
$a_7^{(n)}$	S	$b(\omega_{\xi}^2+r\omega_{\xi}-r\omega_{\theta}) - \lambda^2d(1-\nu)r\frac{n^2}{\rho^2} \times \left[\frac{(3\omega_{\xi}-\omega_{\theta})}{2} + (1+\nu)\omega_{\xi}\right]$	$a_{21}^{(n)}$	F	0
$a_8^{(n)}$	F.M.S	$\lambda^2\omega_{\xi}$	$a_{21}^{(n)}$	M	$2\lambda^2d(1-\nu)\omega_{\theta}\frac{n}{\rho}$
$a_9^{(n)}$	F.M.S	$\lambda^2(1-\nu)r\omega_{\xi}$	$a_{21}^{(n)}$	S	$\lambda^2d(1-\nu)(3\omega_{\theta}-\omega_{\xi})\frac{n}{2\rho}$
$a_{10}^{(n)}$	F	$-b(1+\nu)\frac{n}{2\rho}$	$a_{22}^{(n)}$	F	0
$a_{10}^{(n)}$	M	$-b(1+\nu)\frac{n}{2\rho} - 2\lambda^2d(1-\nu)\omega_{\xi}\omega_{\theta}\frac{n}{\rho}$	$a_{22}^{(n)}$	M	$\lambda^2d(1-\nu)r[2\omega_{\xi} - (3+\nu)\omega_{\theta}]\frac{n}{\rho}$
$a_{10}^{(n)}$	S	$-b(1+\nu)\frac{n}{2\rho} - \lambda^2d(1-\nu)(3\omega_{\xi}-\omega_{\theta})(3\omega_{\theta}-\omega_{\xi})\frac{n}{8\rho}$	$a_{22}^{(n)}$	S	$\lambda^2d(1-\nu)[3r\omega_{\xi}-r\omega_{\theta}(5+2\nu)-\omega_{\xi}^2]\frac{n}{2\rho}$
$a_{11}^{(n)}$	F	$-b(3-\nu)r\frac{n}{2\rho}$	$a_{23}^{(n)}$	F	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho}$
$a_{11}^{(n)}$	M	$-b(3-\nu)r\frac{n}{2\rho} - \lambda^2d(1-\nu)\frac{n}{\rho} \times (2r\omega_{\xi}^2 + 2\omega_{\xi}^2\omega_{\theta} + r\omega_{\xi}\omega_{\theta} + \nu r\omega_{\xi}\omega_{\theta})$	$a_{23}^{(n)}$	M	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho} + \lambda^2d(1-\nu)\left[(3+\nu)\omega_{\xi}\omega_{\theta}^2 + (4+2\nu)r^2\omega_{\theta} - (3+\nu)r^2\omega_{\xi} - (1+\nu)\omega_{\theta}\frac{n^2}{\rho^2}\right]\frac{n}{\rho}$
$a_{11}^{(n)}$	S	$-b(3-\nu)r\frac{n}{2\rho} - \lambda^2d(1-\nu)\frac{n}{\rho}\left[(1+\nu)r\omega_{\xi}\omega_{\theta} - \frac{r}{8}(6\omega_{\xi}\omega_{\theta} - 7\omega_{\xi}^2 - 3\omega_{\theta}^2) + \frac{\omega_{\xi}^2}{4}(5\omega_{\theta} - 3\omega_{\xi})\right]$	$a_{23}^{(n)}$	S	$-b(\omega_{\theta}+\nu\omega_{\xi})\frac{n}{\rho} + \lambda^2d(1-\nu)\left[2(1+\nu)(\omega_{\xi}\omega_{\theta}^2 - r^2\omega_{\xi} + 2r^2\omega_{\theta} - \omega_{\theta}\frac{n^2}{\rho^2}) + r\omega_{\xi}^2 + 3r^2(\omega_{\theta}-\omega_{\xi})\right]\frac{n}{2\rho}$
			$a_{24}^{(n)}$	F.M.S	$\lambda^2d(1-\nu)\left[(1+\nu)r^2 + 2\frac{n^2}{\rho^2}\right]$
			$a_{25}^{(n)}$	F.M.S	$-\lambda^2d(1-\nu)r\left[(1+\nu)(2\omega_{\xi}\omega_{\theta}+r^2) + 2\frac{n^2}{\rho^2}\right]$

$a_{26}^{(n)}$	F.M.S	$-b(\omega_\varphi^2 + 2\gamma\omega_\varphi\omega_\theta + \omega_\theta^2) + \lambda^2 d(1-\nu) \times [(3+\nu)\omega_\varphi\omega_\theta + 2(2+\nu)r^2 - (1+\nu)\frac{n^2}{\rho^2}] \frac{n^2}{\rho^2}$
$a_{27}^{(n)}$	F.M.S	λ^2
$a_{28}^{(n)}$	F.M.S	$\lambda^2(2-\nu)r$
$a_{29}^{(n)}$	F.M.S	$-\lambda^2[(1-\nu)\omega_\varphi\omega_\theta + \nu\frac{n^2}{\rho^2}]$
$a_{30}^{(n)}$	F	0
	M.S	$d\omega_\varphi$
$a_{31}^{(n)}$	F	0
	M.S	$d(\omega_\varphi' + \nu r\omega_\varphi)$
$a_{32}^{(n)}$	F	0
	M.S	$d\nu\omega_\theta \frac{n}{\rho}$
$a_{33}^{(n)}$	F.M.S	$-d$
$a_{34}^{(n)}$	F.M.S	$-d\nu r$
$a_{35}^{(n)}$	F.M.S	$d\nu \frac{n^2}{\rho^2}$
$a_{36}^{(n)}$	F.M.S	-1

Table 3. Coefficients of equation of boundary condition

(F: Flügge, M: Mizoguchi, S: Sanders)

$b_1^{(n)}$	F.M.S	b
$b_2^{(n)}$	F.M.S	$b\nu r$
$b_3^{(n)}$	F.M.S	$b\nu \frac{n}{\rho}$
$b_4^{(n)}$	F.M.S	$b(\omega_\varphi + \nu\omega_\theta)$
$b_5^{(n)}$	F	$-b(1-\nu)\frac{n}{2\rho}$
	M	$-b(1-\nu)\frac{n}{2\rho} - 2\lambda^2 d(1-\nu)\omega_\varphi\omega_\theta \frac{n}{\rho}$
	S	$-b(1-\nu)\frac{n}{2\rho} - d\lambda^2(1-\nu)(3\omega_\varphi - \omega_\theta)(3\omega_\varphi - \omega_\theta) \frac{n}{8\rho}$
$b_6^{(n)}$	F	$b\frac{(1-\nu)}{2}$
	M	$b\frac{(1-\nu)}{2} + 2\lambda^2 d(1-\nu)\omega_\theta^2$
	S	$b\frac{(1-\nu)}{2} + \lambda^2 d\frac{(1-\nu)}{8}(3\omega_\theta - \omega_\varphi)^2$
$b_7^{(n)}$	F.M.S	$-r b_6^{(n)}$
$b_8^{(n)}$	F	$\lambda^2 d(1-\nu)\omega_\theta \frac{n}{\rho}$
	M	$2\lambda^2 d(1-\nu)\omega_\theta \frac{n}{\rho}$
	S	$\lambda^2 d(1-\nu)(3\omega_\theta - \omega_\varphi) \frac{n}{2\rho}$
$b_9^{(n)}$	F.M.S	$-r b_8^{(n)}$
$b_{10}^{(n)}$	F	0
	M	$-\lambda^2 d(1-\nu)[(1+\nu)r^2\omega_\varphi + 2\omega_\varphi \frac{n^2}{\rho^2}]$
	S	$-\lambda^2 d(1-\nu)[(1+\nu)r^2\omega_\varphi + (3\omega_\varphi - \omega_\theta) \frac{n^2}{2\rho^2}]$
$b_{11}^{(n)}$	F	0
	M	$2\lambda^2 d(1-\nu)\omega_\theta \frac{n}{\rho}$
	S	$\lambda^2 d(1-\nu)(3\omega_\theta - \omega_\varphi) \frac{n}{2\rho}$
$b_{12}^{(n)}$	F	0
	M	$-\lambda^2 d(1-\nu)(3+\nu)r\omega_\theta \frac{n}{\rho}$
	S	$-\lambda^2 d(1-\nu)r[3\omega_\theta - \omega_\varphi + 2(1+\nu)\omega_\theta] \frac{n}{2\rho}$
$b_{13}^{(n)}$	F.M.S	$\lambda^2 d(1-\nu)[(1+\nu)r^2 + 2\frac{n^2}{\rho^2}]$
$b_{14}^{(n)}$	F.M.S	$-\lambda^2 d(1-\nu)(3+\nu)r \frac{n^2}{\rho^2}$
$b_{15}^{(n)}$	F.M.S	λ^2
$b_{16}^{(n)}$	F.M.S	$\lambda^2(1-\nu)r$

Table 4. Stresses at lower edge

(Conical shell with $a/h = 30$)

$(\frac{\sigma_\varphi}{\sigma_\theta})_{out}$	Flügge	0.18818×10^6
	Mizoguchi	0.18823×10^6
	Sanders	0.18818×10^6
$(\frac{\sigma_\varphi}{\sigma_\theta})_{in}$	Flügge	-0.77214×10^5
	Mizoguchi	-0.77184×10^5
	Sanders	-0.77195×10^5
$(\frac{\sigma_\theta}{\sigma_\varphi})_{out}$	Flügge	0.56455×10^5
	Mizoguchi	0.56471×10^5
	Sanders	0.56454×10^5
$(\frac{\sigma_\theta}{\sigma_\varphi})_{in}$	Flügge	-0.23164×10^5
	Mizoguchi	-0.23155×10^5
	Sanders	-0.23158×10^5

Table 5. Stresses at lower edge

(Conical shell with $a/h = 150$)

$(\frac{\sigma_\varphi}{\sigma_\theta})_{out}$	Flügge	0.93503×10^6
	Mizoguchi	0.93498×10^6
	Sanders	0.93497×10^6
$(\frac{\sigma_\varphi}{\sigma_\theta})_{in}$	Flügge	-0.39142×10^6
	Mizoguchi	-0.39143×10^6
	Sanders	-0.39144×10^6
$(\frac{\sigma_\theta}{\sigma_\varphi})_{out}$	Flügge	0.28050×10^6
	Mizoguchi	0.28049×10^6
	Sanders	0.28049×10^6
$(\frac{\sigma_\theta}{\sigma_\varphi})_{in}$	Flügge	-0.11742×10^6
	Mizoguchi	-0.11743×10^6
	Sanders	-0.11743×10^6

Table 6. Free frequencies Ω of cylindrical shell

c	n	4	5	6	7	8	9	10
0	Flügge	0.092221	0.074712	0.075560	0.089258	0.11088	0.13784	0.16898
	Mizoguchi	0.091972	0.074230	0.074885	0.088470	0.11006	0.13701	0.16829
	Sanders	0.091978	0.074237	0.074885	0.088473	0.11009	0.13710	0.16814
	Experiment	0.104	0.0432	0.0790	0.0930	0.115	0.146	0.1772
1	Flügge	0.20290	0.15422	0.12905	0.12288	0.13178	0.15156	0.17882
	Mizoguchi	0.20275	0.15392	0.12857	0.12221	0.13103	0.15073	0.17797
	Sanders	0.20275	0.15390	0.12859	0.12223	0.13105	0.15074	0.17796
	Experiment	—	0.133	0.122	0.117	0.153	0.158	0.1843
2	Flügge	0.32970	0.25450	0.20697	0.18095	0.17311	0.18055	0.19964
	Mizoguchi	0.32955	0.25429	0.20661	0.18042	0.17248	0.17977	0.19882
	Sanders	0.32955	0.25431	0.20665	0.18044	0.17249	0.17975	0.19881
	Experiment	—	—	—	—	—	—	0.2018

Table 7. Free frequencies Ω of spherical shell

c	n	0	1	2
0	Flügge	—	0.13495	—
	Mizoguchi	—	0.13494	—
	Sanders	—	0.13495	—
	Experiment	—	0.0985	—
1	Flügge	0.37661	0.69607	0.77779
	Mizoguchi	0.37660	0.69606	0.77775
	Sanders	0.37660	0.69606	0.77778
	Experiment	—	—	—
2	Flügge	0.81673	—	0.90125
	Mizoguchi	0.81672	—	0.90125
	Sanders	0.81672	—	0.90125
	Experiment	—	—	—

Table 8. Free frequencies Ω of model of pressure vessel

c	n	5	6	7	8
1	Flügge	0.09694	0.08656	0.09333	0.1114
	Mizoguchi	0.09695	0.08596	0.09258	0.1104
	Sanders	0.09694	0.08595	0.09245	0.1093
	Experiment	0.127	0.0435	0.0950	0.119

Fig.1 Coordinates and surface geometry

Fig.2 Notations (a) Stress resultants (b) Stress couples (c) Displacements and rotations (d) External forces

Fig.3 Stress resultants and stress couples at boundary

Fig. 4 Conical shell with lower edge fixed

Fig. 6 Stress resultants and stress couples due to lateral wind pressure (conical shell with $a/h = 150$)

Fig. 7 Cylindrical shell with both edges fixed

Fig. 5 Stress resultants and stress couples due to lateral wind pressure (conical shell with $a/h = 30$)

Fig. 8 Spherical shell with lower edge fixed

Fig. 9 Model of pressure vessel with lower edge fixed