Stress-Strain Relations of Polycrystalline Metals *

(1. Stress Fields in a Plastically Deformed Polycrystalline Aggregate)

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The expression for stress fields in a plastically deformed polycrystalline aggregate is derived by use of two methods; the one is based on the Duhamel analogy which is used by Lin and Ito, and the other is based on Eshelby's transformation problem which is used by Kröner or Budiansky and Wu. The revised formulations yield the same result for both methods and the expression by Kröner et al. is found to be correct in a sense of 'arrangement average'. The self-consistent model proposed by Hill against Kröner et al. is examined.

1. Introduction

Many investigations have been made to predict a plastic stress-strain behaviour of a polycrystalline aggregate from single crystal properties. Kröner⁽¹⁾, Budiansky and Wu⁽²⁾ proposed the self-consistent model (K.B.W. model) for an elasto-plastic material, which satisfies reasonably the static and kinematic continuity condition at the grain boundaries. This model is based on Eshelby's⁽³⁾ solution of stress field in a spherical inclusion which is embedded in an infinite elastic medium and undergoes a transformation strain. Budiansky et al. replaced the transformation strain by the plastic strain and the elastic medium by a hypothetical medium whose local strain consists of local elastic strain and macroscopic plastic strain. The stress-strain relations of F.C.C. or B.C.C. polycrystal were calculated numerically by them and Hutchinson⁽⁴⁾.

Hill⁽⁵⁾ disapproved, however, the K.B.W. model for its too artificial assumption and proposed his new self-consistent model. The numerical calculation based on this new model was followed by Hutchinson⁽⁶⁾.

On the other hand, Lin and Ito⁽⁷⁾ perceived the analogy between plastic strain distribution and equivalent body or surface force, which is similar to the Duhamel analogy in thermal stress analysis. This analogy enables us to analyze a stress field in a plastically deformed polycrystal by means of purely elastic formulation. The interference effects of neighboring slipped grains can be taken into account correctly.

The present paper, however, finds out that Lin et al. overlooked the boundary condition on the outer surface of the aggregate, so that their expression for

local stress field does not satisfy the self-consistent condition. The revised formulation yields a similar expression to the K.B.W. model except the term of interaction stress from surrounding slipped crystals. The same result is reintroduced based on the Eshelby's transformation problem without the artificial assumption of Budiansky et al. Introducing the concept of "arrangement average", the mean and the deviation of interaction stress are calculated and the K.B.W. model is found to be correct in a sense of arrangement average.

But Hill's new model is contradictory to the K.B.W. model; in other words, if we accept Hill's model, we should reject the K.B.W. model and also the present result. The reason for this contradiction is considered lastly.

2. Approach based on the Duhamel analogy

Consider a polycrystalline aggregate (denote its volume by V and its surface by A) subjected to the macroscopically homogeneous stress S_{ij} , while the local stress σ_{ij} in the individual crystal (volume v_n , surface a_n) is not homogeneous. Apply the stress increment $\dot{S}_{ij}$ which will produce the overall strain increment $\dot{\epsilon}_{ij}$ as well as the local stress and strain increments $\dot{\sigma}_{ij}$ and $\dot{\epsilon}_{ij}$.

Strain increment is divided into elastic and plastic parts as

$$\dot{\epsilon}_{ij} \equiv \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p \quad (1)$$

Stress increment is related to elastic strain increment by Hooke's law as

$$\dot{\sigma}_{ij} = \lambda \delta_{ij} \dot{\epsilon}_{kk}^e + 2\mu \dot{\epsilon}_{ij}^e \quad (2)$$

where λ and μ are Lamé's elastic constants and δ_{ij} is Kronecker's delta. The equilibrium equation within a

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grain is

$$\dot{\sigma}_{ij,j} = 0, \quad (3)$$

and the equilibrium condition on the grain boundary is

$$\dot{\sigma}_{ij}n_j = t_i \quad (4)$$

where n_i is outward normal unit vector and t_i is surface traction at the surface of the grain.

The problem is how to determine the stress field $\dot{\sigma}_{ij}$ when the plastic strain $\dot{\epsilon}_{ij}^p$ in all grains and the uniform traction $T_i = \dot{\sigma}_{ij}n_j$ on the outer surface of the aggregate are prescribed. The boundary condition on the outer surface of the aggregate can be replaced by prescribing the mean value of the local stresses.

$$\bar{\sigma}_{ij} \equiv \frac{1}{V} \int_V \dot{\sigma}_{ij} dV = \dot{S}_{ij} \quad (5)$$

This problem is converted after the manner of Lin et al.⁽⁷⁾ as follows. Equation (2) is rewritten by use of Eq.(1) as

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}^e - 2\mu \dot{\epsilon}_{ij}^p \quad (6)$$

where

$$\dot{\sigma}_{ij}^e \equiv \lambda \dot{\epsilon}_{kk} \delta_{ij} + 2\mu \dot{\epsilon}_{ij} \quad (7)$$

Equation (7) means the total strain $\dot{\epsilon}_{ij}$ and the fictitious stress $\dot{\sigma}_{ij}^e$ are determinable by purely elastic formulation. Substitution of Eq.(6) into Eqs.(3) and (4) yields

$$\dot{\sigma}_{ij,j}^e + \dot{\sigma}_{ij,j}^* = 0 \quad (8)$$

$$\dot{\sigma}_{ij}^e n_j = t_i + \dot{\sigma}_{ij}^* n_j \quad (9)$$

where

$$\dot{\sigma}_{ij}^* \equiv -2\mu \dot{\epsilon}_{ij}^{p*} \quad (10)$$

$$t_i^* \equiv 2\mu \dot{\epsilon}_{ij}^{p*} n_j \quad (11)$$

Let two grains v_p and v_q adjoin each other and denote their interface by b_{pq} . Since the traction t_i is continuous across b_{pq} , Eq.(9) gives

$$(\dot{\sigma}_{ij}^{e(p)} - \dot{\sigma}_{ij}^{e(q)})n_j = t_i^{*(p)} + t_i^{*(q)} \quad (12)$$

Equations (8) and (12) mean that $\dot{\sigma}_{ij}^e$ is caused by the two equivalent forces: the body force $\dot{\sigma}_{ij}^*$ within each grain and the surface force $t_i^{*(p)} + t_i^{*(q)}$ at each interface b_{pq} . From Eqs.(5) and (6) we see the next condition is imposed on the stress $\dot{\sigma}_{ij}^e$.

$$\bar{\sigma}_{ij}^e = \dot{S}_{ij} + 2\mu \bar{\epsilon}_{ij}^p \quad (13)$$

where inhomogeneity and anisotropy of elastic moduli are neglected and so the following is supposed, as done by Budiansky et al.⁽²⁾

$$\bar{\epsilon}_{ij}^p = \dot{E}_{ij}^p \quad (14)$$

Now we shall pursue first the particular solution $\dot{\sigma}_{ij}^*$ due to the equivalent forces in an infinite elastic medium. The displacement field u_i at the point x subjected to a point force F_i acting at the point x' is given as^{(8) (7) (8)}

$$u_i(x) = \frac{1}{4\pi\mu} \frac{F_i(x')}{r} - \frac{\lambda + \mu}{8\pi\mu(\lambda + 2\mu)} \frac{\partial^2 r}{\partial x_i \partial x_j} F_j(x') \\ = \frac{1}{16\pi\mu(1-\nu)} [(3-4\nu)\delta_{ij} + q_i q_j] \frac{F_j(x')}{r} \quad (15)$$

where

$$r^2 \equiv (x_k - x'_k)(x_k - x'_k), \quad q_i \equiv (x_i - x'_i)/r \quad (16)$$

and ν is Poisson's ratio. Corresponding strain field is

$$\epsilon_{ij}(x) = \frac{1}{16\pi\mu(1-\nu)} (q_k \delta_{ij} - 3q_i q_j q_k \\ - (1-2\nu)(q_i \delta_{kj} + q_j \delta_{ki})) \frac{F_k(x')}{r^2},$$

and stress field is

$$\sigma_{ij}(x) = \phi_{ijk}(x; x') F_k(x') \quad (17)$$

where

$$\phi_{ijk}(x; x') \equiv \frac{1}{8\pi(1-\nu)} \frac{1}{r^2} [(1-2\nu)(\delta_{ij} q_k \\ - \delta_{ki} q_j - \delta_{kj} q_i) - 3q_i q_j q_k] \quad (18)$$

Supposing that the aggregate v is embedded in the infinite elastic medium, the solution $\dot{\sigma}_{ij}^*$ due to the equivalent forces can be given by

$$\dot{\sigma}_{ij}^*(x) = \sum_m \int_{v_m} \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') dv' \\ + \sum_{p,q} \int_{b_{pq}} \phi_{ijk}(x; x') (t_k^{*(p)}(x') + t_k^{*(q)}(x')) db' \\ + \int_A \phi_{ijk}(x; x') t_k^*(x') dA' \quad (19)$$

Considering the fact that

$$\sum_{p,q} \int_{b_{pq}} \phi_{ijk}(x; x') (t_k^{*(p)}(x') + t_k^{*(q)}(x')) db' = 0,$$

the second term in the right hand of the above is rewritten as

$$\sum_{p,q} \int_{b_{pq}} \phi_{ijk}(x; x') (t_k^{*(p)}(x') + t_k^{*(q)}(x')) db' \\ = \sum_m \int_{a_m} \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') da' - \int_A \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') dA'$$

and Eq.(19) becomes

$$\dot{\sigma}_{ij}^*(x) = \sum_m \left[\int_{v_m} \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') dv' + \int_{a_m} \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') da' \right] \quad (20)$$

The volume integral in the above is divided into two parts as

$$\int_{v_m} \phi_{ijk}(x; x') \dot{\sigma}_k^*(x') dv' = -2\mu \int_{v_m} \phi_{ijk}(x; x') \frac{\partial \dot{\epsilon}_{kl}^p}{\partial x'_l} dv' \\ = 2\mu \int_{v_m} \frac{\partial \phi_{ijk}}{\partial x'_l} \dot{\epsilon}_{kl}^p dv' - 2\mu \int_{v_m} \frac{\partial}{\partial x'_l} (\phi_{ijk} \dot{\epsilon}_{kl}^p) dv'.$$

We apply the divergence theorem to the last term in the above. When the volume v_m does not contain the observation point x ,

$$\int_{v_m} \frac{\partial}{\partial x'_i} (\Phi_{ijk} \epsilon_{kl}^p) dv' = \int_{a_m} \Phi_{ijk} \epsilon_{kl}^p n_i da'.$$

On the other hand, when the volume v_m includes the point x , we should exclude an infinitely small region containing the point x from the region of volume integration, since the point x is singular. We suppose that the infinitely small region (denote its surface by c) is a sphere of infinitely small radius δ whose center is at the point x .

$$\int_{v_m} \frac{\partial}{\partial x'_i} (\Phi_{ijk} \epsilon_{kl}^p) dv' = \int_{a_m} \Phi_{ijk} \epsilon_{kl}^p n_i da' + \int_c \Phi_{ijk} \epsilon_{kl}^p n_i dc'.$$

On the surface c , we can put $r = \delta$, $q_i = n_i$ and $\epsilon_{kl}^p(x') \approx \epsilon_{kl}^p(x)$, then

$$\int_c \Phi_{ijk} \epsilon_{kl}^p n_i dc' = \frac{\epsilon_{kl}^p(x)}{8\pi(1-\nu)} \int_{\Omega} [(1-2\nu)(\delta_{ij}n_k - \delta_{ki}n_j - \delta_{kj}n_i) - 3n_i n_j n_k] n_i d\omega$$

where ω is solid angle. The integration of the product of n_i can be calculated by use of the following formulae.

$$\begin{aligned} \int_{\Omega} n_i n_j d\omega &= \frac{4\pi}{3} \delta_{ij}, \\ \int_{\Omega} n_i n_j n_k d\omega &= \frac{4\pi}{3 \times 5} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \\ \int_{\Omega} n_i n_j n_k n_l d\omega &= \frac{4\pi}{3 \times 5 \times 7} (\delta_{ij} \delta_{kl} \delta_{mn} + \delta_{ik} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{jk} \delta_{lm} + \delta_{jm} \delta_{ln} + \delta_{jn} \delta_{lm} + \delta_{il} \delta_{jk} \delta_{mn} + \delta_{jm} \delta_{ln} + \delta_{jn} \delta_{lm} + \delta_{ik} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{jk} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{jk} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{jk} \delta_{lm}) \end{aligned}$$

and so forth. (21)

Since $\epsilon_{ij}^p = \epsilon_{ji}^p$ and $\epsilon_{kk}^p = 0$,

$$\int_c \Phi_{ijk} \epsilon_{kl}^p n_i dc' = \frac{\epsilon_{kl}^p(x)}{8\pi(1-\nu)} \frac{8\pi}{15} [(1-5\nu)(\delta_{ij} \delta_{kl} - (4-5\nu)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}))] = -\beta \epsilon_{ij}^p(x) \quad (22)$$

where

$$\beta = \frac{2(4-5\nu)}{15(1-\nu)}. \quad (23)$$

Consequently Eq.(20) becomes

$$\dot{\sigma}_{ij}^*(x) = 2\mu\beta \epsilon_{ij}^p(x) + \dot{\sigma}_{ij}(x) \quad (24)$$

where

$$\dot{\sigma}_{ij}(x) = 2\mu \sum_m \int_{v_m} \Psi_{ijk}(x; x') \epsilon_{kl}^p(x') dv', \quad (25)$$

$$\Psi_{ijk}(x; x') = \frac{\partial \Phi_{ijk}}{\partial x'_i} = \frac{1}{8\pi(1-\nu)} \frac{1}{r^3} \Theta_{ijk}(q), \quad (26)$$

$$\begin{aligned} \Theta_{ijk}(q) &\equiv (1-2\nu)(\delta_{ki} \delta_{jl} + \delta_{kl} \delta_{ji} - \delta_{ij} \delta_{kl}) \\ &\quad - 3(1-2\nu)(\delta_{ij} q_k + \delta_{ki} q_j - \delta_{ij} q_k) q_l \\ &\quad + 3(\delta_{ki} q_j q_l + \delta_{ji} q_k q_l + \delta_{il} q_j q_k) - 15 q_i q_j q_k q_l. \end{aligned} \quad (27)$$

Equation (24) represents the particular solution due to the equivalent forces but does not satisfy the boundary condition on the outer surface. The solution compatible with the condition of Eq.(13) is gives as

$$\begin{aligned} \dot{\sigma}_{ij}^*(x) &= \dot{\sigma}_{ij}^*(x) - \dot{\sigma}_{ij}^* + \dot{\sigma}_{ij} + 2\mu \dot{\epsilon}_{ij}^p \\ &= \dot{\sigma}_{ij} + 2\mu \dot{\epsilon}_{ij}^p + 2\mu\beta (\epsilon_{ij}^p(x) - \dot{\epsilon}_{ij}^p) + \dot{\sigma}_{ij} - \dot{\sigma}_{ij}^*. \end{aligned} \quad (28)$$

Now we calculate the value of $\dot{\sigma}_{ij}^*$ under the assumption that the aggregate V is a sphere of radius R whose center is at the origin of the coordinates. Let the line connecting the observation point x_i and the force acting point x'_i intersect the outer surface of the aggregate at the point X_i and denote the length between the point x'_i and the point X_i by ρ . From $X_i = x'_i + \rho q_i$ and $X_i X_i = R^2$, we get

$$\rho(q) = -(q_i x'_i) + \sqrt{(q_i x'_i)^2 - (x'_i x'_i) + R^2}. \quad (29)$$

$\dot{\sigma}_{ij}^*$ is expressed by

$$\dot{\sigma}_{ij}^* = \frac{1}{V} \int_V \dot{\sigma}_{ij}(x) dV = 2\mu \sum_m \int_{v_m} \Psi_{ijk}(x; x') \epsilon_{kl}^p(x') dv' \quad (30)$$

where

$$\begin{aligned} \Psi_{ijk}(x') &\equiv \frac{1}{V} \int_V \Psi_{ijk}(x; x') dV \\ &= \frac{1}{V} \lim_{\delta \rightarrow 0} \int_{\Omega} d\omega \int_{\delta}^{\rho} \Psi_{ijk}(r^2) r^2 dr \\ &= \frac{1}{V} \frac{1}{8\pi(1-\nu)} \lim_{\delta \rightarrow 0} \int_{\Omega} \Theta_{ijk} \log \frac{\rho}{\delta} d\omega. \end{aligned}$$

Here we consider a direction q^1 and the opposite direction $-q^1$. Noting that $\Theta_{ijk}(q)$ is an even function with respect to direction cosine q and $\rho(q^1) \times \rho(-q^1) = R^2 - (x'_i x'_i)$, we can see

$$\begin{aligned} \Theta_{ijk}(q^1) \log \frac{\rho(q^1)}{\delta} + \Theta_{ijk}(-q^1) \log \frac{\rho(-q^1)}{\delta} \\ = \Theta_{ijk}(q^1) \log \frac{\rho(q^1) \rho(-q^1)}{\delta^2} \\ = \Theta_{ijk}(q^1) \log \frac{R^2 - (x'_m x'_m)}{\delta^2}. \end{aligned}$$

Therefore

$$\int_{\Omega} \Theta_{ijk} \log \frac{\rho}{\delta} d\omega = \log \frac{\sqrt{R^2 - (\mathbf{x}_n' \cdot \mathbf{x}_n')}}{\delta} \int_{\Omega} \Theta_{ijk} d\omega$$

By use of Formulae (21) we get

$$\int_{\Omega} \Theta_{ijk} d\omega = 0 \quad (31)$$

and so

$$\bar{s}_{ij} = 0 \quad (32)$$

Finally from Eqs.(6) and (28)

$$\dot{\sigma}_{ij}(x) = \dot{S}_{ij} - 2\mu(1-\beta)(\dot{\epsilon}_{ij}^*(x) - \dot{E}_{ij}^*) + \dot{s}_{ij}(x) \quad (33)$$

This expression agrees with the one of K.B.W. model except the term $\dot{s}_{ij}$.

For comparison, the expression of Lin et al. is represented with the notations used in the present paper.

$$\dot{\sigma}_{ij} = \dot{S}_{ij}^0 + \sigma_{ij}^* - 2\mu\dot{\epsilon}_{ij}^* \quad (34)$$

where $\dot{S}_{ij}^0$ is defined by them as "uniform applied aggregate stress"⁽⁷⁾ or "externally applied uniform load on the aggregate".⁽⁸⁾ If $\dot{S}_{ij}^0$ is the same as $\dot{S}_{ij}$ used in the present paper, Eq.(34) does not satisfy the self-consistent condition. In Ref.9, moreover, $\dot{\sigma}_{ij}^*$ is regarded to be equal to $\dot{s}_{ij}$

3. Formulation based on Eshelby's transformation problem

In spite of the criticism by Hill⁽⁵⁾, a similar result to K.B.W. model has been obtained in the preceding paragraph. So we consider again the same problem following Eshelby's scheme.⁽³⁾

After addition of stress increment $\dot{S}_{ij}$, release it along the same path as in the loading case. We suppose only elastic changes take place during the unloading process in all grains. The residual stress increment $\dot{\sigma}_{ij}^*$ can be represented by

$$\dot{\sigma}_{ij}^* = \dot{\sigma}_{ij} - \dot{S}_{ij} \quad (35)$$

The residual stress is considered to be elastic constraint caused by the heterogeneity of local plastic deformation, so that we can apply a purely elastic formulation.

Disintegrate the aggregate into the individual grains and suppose each grain suffers a uniform plastic strain $\dot{\epsilon}_{ij}^*$ as the "stress-free misfit strain". In the next place restore elastically the form of each grain to its original one by application of the surface traction

$$f_i = -2\mu\dot{\epsilon}_{ij}^* n_j \quad (36)$$

Rejoin all the grains and apply the surface traction $-f_i$ on the grain boundaries, so that each grain returns to the

self-equilibrated state. Denote the additional elastic field due to the surface traction $-f_i$ by $\dot{\sigma}_{ij}^*$, from which $\dot{\sigma}_{ij}^*$ can be determined as

$$\dot{\sigma}_{ij}^* = \dot{\sigma}_{ij}^* - 2\mu\dot{\epsilon}_{ij}^* \quad (37)$$

As preliminary steps for the determination of $\dot{\sigma}_{ij}^*$, the following two auxiliary problems are considered.

(1) A single crystal alone deforms plastically in an infinite homogeneous elastic medium.

The stress due to $-f_i$ is found by use of Eq.(17).

$$\dot{\sigma}_{ij}^*(x) = 2\mu\dot{\epsilon}_{kl}^* \int_{\Omega} \Phi_{ijk}(x; x') n_l da' \quad (38)$$

When the crystal is a sphere in shape and contains the point x at its center, Eq.(38) results in

$$\dot{\sigma}_{ij}^* = 2\mu\beta\dot{\epsilon}_{ij}^* \quad (39)$$

Hence

$$\dot{\sigma}_{ij}^* = 2\mu(\beta-1)\dot{\epsilon}_{ij}^* \quad (40)$$

On the other hand when the point x is located outside the crystal, $\dot{\sigma}_{ij}^*$ is identical with $\dot{\sigma}_{ij}^*$ which may be converted to the volume integral as

$$\dot{\sigma}_{ij}^* = \dot{\sigma}_{ij}^* = 2\mu\dot{\epsilon}_{kl}^* \int_{\Omega} \Psi_{ijk}(x; x') dv' \quad (41)$$

(2) An aggregate consisting of $N+1$ crystals is embedded in an infinite homogeneous elastic medium. Denote the crystal containing the observation point x by v_0 and the other surrounding crystals by $v_1, v_2, \dots, v_N$. The stress field in v_0 is given by superposing the constraint stress caused by its own plastic strain, i.e. Eq.(40), on the interaction stresses transmitted from the surrounding solid crystals, i.e. Eq.(41).

$$\begin{aligned} \dot{\sigma}_{ij}^*(x) = & 2\mu(\beta-1)\dot{\epsilon}_{ij}^*(x) \\ & + 2\mu \sum_{m=1}^N \dot{\epsilon}_{kl}^*(x_m) \int_{v_m} \Psi_{ijk}(x; x') dv' \end{aligned} \quad (42)$$

Now we return to the original problem where the outer surface of the aggregate must be traction-free, while Eq.(42) includes the elastic constraint from the outer infinite elastic medium. If we interpret the aggregate in the problem (2) to be a single crystal in the problem (1) whose plastic strain is $\dot{E}_{ij}^*$, the constraint stress is estimated in an averaged manner to be $2\mu(\beta-1)\dot{E}_{ij}^*$ from Eq.(40). Hence the residual stress is given by

$$\dot{\sigma}_{ij}^* = 2\mu(\beta-1)(\dot{\epsilon}_{ij}^* - \dot{E}_{ij}^*) + \dot{s}_{ij} \quad (43)$$

Substitution of Eq.(43) into Eq.(35) yields the same result as Eq.(33). It should be

noted that the term of $2\mu(\beta-1)\dot{\epsilon}_{ij}^p$ is introduced for a different reason from Budiansky and Wu⁽²⁾ who argued that the matrix becomes softer due to its plastic deformation.

4. Arrangement average

In this paragraph we will investigate the properties of the term $\dot{s}_{ij}$ in Eq.(33). Let the aggregate V be a sphere whose center is at the origin of the coordinates and whose radius is R . Locate the observation point at the origin and let the crystal v_0 containing the observation point be also a sphere whose center is at the origin and whose radius is R_0 . But other crystals may be arbitrary in shape. Assume the plastic strain distribution within each grain is uniform. Hereupon $\dot{s}_{ij}$ at the origin can be represented as

$$\dot{s}_{ij} = 2\mu \sum_{m=1}^N \dot{\epsilon}_{kl}^{p(m)} \int_{v_m} \Psi_{ijkl}(0; x') dv' \quad (44)$$

We can regard $\dot{s}_{ij}$ as the interaction stress transmitted from the surrounding slid crystals.

Plastic strain $\dot{\epsilon}_{ij}^{p(m)}$ in each grain depends on the orientations of all crystal axes to the macroscopic stress axis, so that $\dot{s}_{ij}$ depends on how arrangement of orientations is assigned to the surrounding crystals. Lin and Ito⁽⁷⁾ gave a definite model for the orientation arrangement; that is, the three-dimensional space of the aggregate is considered to be entirely filled with identical basic block of 64 cube-shaped crystals having different orientations. We can imagine, in fact, infinite ways of assignation of orientation arrangement.

Fix the orientation of the crystal v_0 and assign all probable ways of orientation arrangement to the surrounding crystals $v_1, v_2, \dots, v_N$; however, their positions or shapes are left as they are. The average with respect to the above operation is named "arrangement average" and denoted by $\langle \rangle$ in what follows.

The surrounding crystals may take any arbitrary orientation out of all orientations existing in the aggregate. Therefore all values of plastic strain in the aggregate may appear in the m -th crystal if it is at some distance from v_0 . Hence the arrangement average of the strain in v_m can be approximated by the volume average.

$$\langle \dot{\epsilon}_{ij}^{p(m)} \rangle = \dot{\epsilon}_{ij}^p \quad (m \neq 0) \quad (45)$$

From Eq.(44)

$$\langle \dot{s}_{ij} \rangle = 2\mu \sum_{m=1}^N \langle \dot{\epsilon}_{kl}^{p(m)} \rangle \int_{v_m} \Psi_{ijkl} dv' = 2\mu \dot{\epsilon}_{kl}^p \int_{V_1} \Psi_{ijkl} dV'$$

where $V_1 \equiv V - v_0$. Recalling Eq.(31)

we see the above integral vanishes; i.e.

$$\langle \dot{s}_{ij} \rangle = 0 \quad (46)$$

If the volume V is sufficiently large, the above result is valid anywhere in the aggregate except the neighborhood of the surface. Then Eq.(33) gives

$$\tilde{\dot{s}}_{ij} = \dot{s}_{ij} - 2\mu(1-\beta)(\dot{\epsilon}_{ij}^p - \dot{\epsilon}_{ij}^p) \quad (47)$$

Thus the expression of K.B.W. model is shown to be correct in a sense of arrangement average. If a stress-strain relation of a single crystal is linear, the averaged stress and the averaged strain are related in the same manner, which would enable us to calculate the overall stress-strain relation by use of Eq.(47) as Budiansky et al.⁽²⁾⁽⁴⁾ did. But since the relation is not linear as will be discussed in the next paragraph, we should evaluate anyhow the value of $\dot{s}_{ij}$.

As one of the ways for quantitative estimation of interaction stress, we calculate the deviation of $\dot{s}_{ij}$ under the assumption that $N \rightarrow \infty$ ($R \rightarrow \infty$). From Eqs.(45) and (46)

$$\dot{s}_{ij} = \tilde{\dot{s}}_{ij} - \dot{s}_{ij} = 2\mu \sum_{m=1}^{\infty} \Delta \dot{\epsilon}_{kl}^{p(m)} \int_{v_m} \Psi_{ijkl} dv' \quad (48)$$

where

$$\Delta \dot{\epsilon}_{kl}^{p(m)} \equiv \dot{\epsilon}_{kl}^{p(m)} - \dot{\epsilon}_{kl}^p \quad (49)$$

Denote the covariance of $\dot{s}_{ij}$ and $\dot{s}_{kl}$ by Γ_{ijkl}

$$\begin{aligned} \Gamma_{ijkl} &\equiv \langle \dot{s}_{ij} \dot{s}_{kl} \rangle \\ &= 4\mu^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\langle \Delta \dot{\epsilon}_{kl}^{p(m)} \Delta \dot{\epsilon}_{cd}^{p(n)} \rangle \int_{v_m} \Psi_{ijab} dv' \int_{v_n} \Psi_{klcd} dv' \right] \end{aligned} \quad (50)$$

Since no correlation exists between the orientations of two different crystals, we obtain

$$\langle \Delta \dot{\epsilon}_{kl}^{p(m)} \Delta \dot{\epsilon}_{cd}^{p(n)} \rangle = \langle \Delta \dot{\epsilon}_{kl}^{p(m)} \rangle \langle \Delta \dot{\epsilon}_{cd}^{p(n)} \rangle = 0 \quad (m \neq n) \quad (51)$$

On the other hand, for the case when $m=n$, the covariance of $\Delta \dot{\epsilon}_{kl}^p$ does not vanish and can be approximated, as Eq.(45), by the volume average.

$$\langle \Delta \dot{\epsilon}_{kl}^{p(m)} \Delta \dot{\epsilon}_{cd}^{p(n)} \rangle = \langle \Delta \dot{\epsilon}_{kl}^p \Delta \dot{\epsilon}_{cd}^p \rangle \quad (52)$$

Hereby

$$\Gamma_{ijkl} = 4\mu^2 (\overline{\Delta \epsilon_{ab}^p} \overline{\Delta \epsilon_{cd}^p}) \sum_{m=1}^{\infty} \int_{v_m} \int_{v_m} \Psi_{ijab}(0; x') \Psi_{klcd}(0; x'') dv' dv'' \quad (53)$$

We approximate $\Psi_{klcd}(0; x'')$ by $\Psi_{klcd}(0; x')$, which is allowed when the distance of v_m from the origin is greater enough than the size of v_m .

$$\begin{aligned} \Gamma_{ijkl} &= 4\mu^2 (\overline{\Delta \epsilon_{ab}^p} \overline{\Delta \epsilon_{cd}^p}) \bar{v}_m \int_{V_1} \Psi_{ijab} \Psi_{klcd} dV' \\ &= 4\mu^2 (\overline{\Delta \epsilon_{ab}^p} \overline{\Delta \epsilon_{cd}^p}) \bar{v}_m \frac{1}{64\pi^2(1-\nu)^2} \int_{\Lambda_c} \Theta_{ijab} \Theta_{klcd} d\omega \int_{R_0}^{\infty} \frac{dr}{r^4} \end{aligned} \quad (54)$$

Replacing $\bar{v}_m$ by $v_0 = (4\pi/3)R_0^3$ and applying the formulae (21), the integration of the above equation results in

$$\begin{aligned} \Gamma_{ijkl} &= \frac{2\mu^2}{945(1-\nu)^2} [-2(55-240\nu+168\nu^2)J_{a1} + 4(25-90\nu+63\nu^2)J_{a2} \\ &\quad - 8(10-54\nu+63\nu^2)J_{b1} + 2(50-90\nu+63\nu^2)J_{b2} \\ &\quad - 2(1+36\nu-126\nu^2)J_{c1} + 25J_{c2}] \end{aligned} \quad (55)$$

where

$$\begin{aligned} J_{a1} &\equiv (\overline{\Delta \epsilon_{ij}^p} \overline{\Delta \epsilon_{kl}^p}), \quad J_{a2} \equiv (\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{kl}^p}) + (\overline{\Delta \epsilon_{kk}^p} \overline{\Delta \epsilon_{ii}^p}) \\ J_{b1} &\equiv \bar{\partial}_{ij}(\overline{\Delta \epsilon_{kl}^p} \overline{\Delta \epsilon_{ii}^p}) + \bar{\partial}_{kl}(\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{jj}^p}) \\ J_{b2} &\equiv \bar{\partial}_{ik}(\overline{\Delta \epsilon_{jj}^p} \overline{\Delta \epsilon_{ii}^p}) + \bar{\partial}_{jk}(\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{jj}^p}) + \bar{\partial}_{il}(\overline{\Delta \epsilon_{jj}^p} \overline{\Delta \epsilon_{kk}^p}) + \bar{\partial}_{jl}(\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{kk}^p}) \\ J_{c1} &\equiv \bar{\partial}_{ij} \bar{\partial}_{kl}(\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{jj}^p}), \quad J_{c2} \equiv (\bar{\partial}_{ik} \bar{\partial}_{jl} + \bar{\partial}_{il} \bar{\partial}_{jk})(\overline{\Delta \epsilon_{ii}^p} \overline{\Delta \epsilon_{jj}^p}) \end{aligned} \quad (56)$$

Equation (55) means the effect of the interaction stress depends on the plastic strain deviation. Numerical calculations by use of the above results are left to future works.

5. Examination of Hill's self-consistent model

Against the K.B.W. model Hill⁽⁵⁾ proposed a new self-consistent model, which is summarized in the following where the same abbreviated notation is used as Hutchinson⁽⁶⁾ or Hill⁽⁵⁾.

The problem is how to determine the tensor of overall instantaneous moduli L for the polycrystalline aggregate when the tensors of instantaneous moduli L_c are prescribed for all the single crystalls.

$$\dot{\epsilon}_c = L_c \dot{\epsilon}_c, \quad \bar{\sigma} = L \bar{\epsilon} \quad (57)$$

As a preparation for this problem, we suppose the instantaneous moduli of all crystals are homogeneous and equal to L , but suppose each crystal undergoes a transformation of stress-free strain ϵ_c^T . The constraint of the surrounding matrix makes the resulting strain ϵ_c^* different from ϵ_c^T , which is written by use of Eshelby's tensor S as

$$\epsilon_c^* = S \epsilon_c^T \quad (58)$$

The associated stress in the crystal is

$$\dot{\sigma}_c^* = L(\dot{\epsilon}_c^* - \dot{\epsilon}_c^T) \quad (59)$$

Hill introduced a 'constraint' tensor L^* which is defined by

$$\dot{\sigma}_c^* = -L^* \dot{\epsilon}_c^* \quad (60)$$

Substitution of Eqs. (58) and (59) into Eq. (60) gives

$$L(S-I)\dot{\epsilon}_c^T = -L^* S \dot{\epsilon}_c^T$$

where I is unity tensor. On the premise that the above holds for all ϵ_c^T , we see the tensor L^* is related to S as

$$L^* = L(I-S)S^{-1} \quad (61)$$

Now we return to the original problem. Put

$$\dot{\sigma}_c^* = \dot{\sigma}_c - \bar{\sigma}, \quad \dot{\epsilon}_c^* = \dot{\epsilon}_c - \bar{\epsilon} \quad (62)$$

then from Eq. (60)

$$\dot{\sigma}_c - \bar{\sigma} = -L^*(\dot{\epsilon}_c - \bar{\epsilon}) \quad (63)$$

Substituting Eq. (57) into the above, we find

$$(L^* + L_c)\dot{\epsilon}_c = (L^* + L)\bar{\epsilon}$$

and so

$$\dot{\epsilon}_c = A_c \bar{\epsilon} \quad (64)$$

where

$$A_c \equiv (L^* + L_c)^{-1}(L^* + L) \quad (65)$$

Denote the average over all the crystals by $\langle \rangle$, then from Eq.(64) and $\langle \dot{\epsilon}_c \rangle = \dot{\bar{\epsilon}}$,

$$[(A_c) - I]\dot{\bar{\epsilon}} = 0 \quad (66)$$

From the above Hill derived an equation to determine the tensor L in question.

$$(A_c) = I \quad (67)$$

If we replace L^* in Eq.(63) by the overall elastic constraint tensor L^* defined for a purely elastic matrix with elastic moduli L , this model agrees with the K.B.W. model. So Hill argued that the K.B.W. model disregards the pronounced directional weakness in the constraint of an already yielded aggregate. Nevertheless, the present result obtained in the preceding paragraphs seems to support the expression of K.B.W. model. The author's viewpoints about Hill's model are as follows.

(1) In the K.B.W. model or the present study the stress-free misfit strain $\dot{\epsilon}_c^*$ is considered to be the discrepancy between the plastic strain in the crystal and the one in the surrounding matrix, i.e.

$$\dot{\epsilon}_c^* = \dot{\epsilon}_c^p - \dot{\bar{\epsilon}}^p \quad (68)$$

From Eqs.(58), (62) and (68)

$$\dot{\epsilon}_c - \dot{\bar{\epsilon}} = S(\dot{\epsilon}_c^p - \dot{\bar{\epsilon}}^p) \quad (69)$$

Supposing the elastic moduli of crystals are homogeneous and equal to L ,

$$\dot{\epsilon}_c = L^{-1}\dot{\sigma}_c + \dot{\epsilon}_c^*, \quad \dot{\bar{\epsilon}} = L^{-1}\dot{\sigma} + \dot{\bar{\epsilon}}^* \quad (70)$$

Substitution of the above into Eq.(69) gives

$$\begin{aligned} \dot{\sigma}_c - \dot{\sigma} &= -L(I-S)(\dot{\epsilon}_c^p - \dot{\bar{\epsilon}}^p) \\ &= -L(I-S)^{-1}(\dot{\epsilon}_c - \dot{\bar{\epsilon}}) \end{aligned} \quad (71)$$

This result is different from Hill's result, i.e. Eq.(61).

The two models are illustrated in Fig.1 and 2, where the lines OR and OQ represent Eq.(57), the line QR represents Eq.(63) or (71) and the length between P and R means the stress-free strain $\dot{\epsilon}_c^*$. Thus in the K.B.W. model or the present study the constraint stress is considered to be caused by the elastic compensation of the discrepancy of plastic deformation

between the crystal and the matrix. The author thinks the effect of the weakness in the constraint of the plastically deformed matrix is included in the term of $\dot{\epsilon}_c^* - \dot{\bar{\epsilon}}^*$, not in the constraint tensor L^* .

(2) In order to derive Eq.(67) from Eq.(66), the premise is necessary that Eq.(66) holds for any arbitrary strain $\dot{\bar{\epsilon}}$; in other words, (A_c) in Eq.(66) should be constant for the arbitrary directions of the vector $\dot{\bar{\epsilon}}$ in strain rate space (at least for the six independent directions). This premise is admitted for an elastic body since the stress-strain rate relation is linear, but not for a plastic body.

As Hutchinson⁽⁶⁾ mentioned, the instantaneous moduli L_c for a single crystal depend on the particular branches (i.e. set of active slip systems), which in turn depend on the prescribed value of $\dot{\epsilon}_c$ or $\dot{\sigma}_c$. When the current stress is located at a vertex of the yield surface (polyhedron) for the crystal, we must choose a set of active slip systems out of the potentially active slip systems, for which we need to prescribe the direction of the stress or strain rate vector. Equation (64) shows $\dot{\epsilon}_c$ depends on $\dot{\bar{\epsilon}}$, so that it is reasonable to say L as well as L_c depends on $\dot{\bar{\epsilon}}$. Hereupon, even though we can say from Eq.(66) that

Fig. 1 Hill's model

Fig. 2 K.B.W. model

the determinant of the matrix $(A_i - I)$ vanishes, all the elements in the matrix can not be made zero generally. The author can not accept Hill's assumption that Eq.(67) holds for a plastic body.

(3) Budiansky et al.⁽²⁾ showed that the limit yield stress for an elastic-plastic polycrystal of non-hardening F.C.C. crystals agrees precisely with the yield stress obtained by Bishop and Hill⁽¹⁰⁾ for a rigid-plastic polycrystal. But the yield stress calculated by Hutchinson⁽⁶⁾ based on Hill's new model is somewhat lower than the value by Bishop et al. This fact seems to support the K.B.W. model.

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