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Analysis of an Infinite Elastic Thick Plate Subjected to Loads Symmetrical to Axis of Revolution*

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We make a stress analysis on the above problem by means of a couple of stress functions with subsequent Fourier-Bessel expansion of the given loads on the boundary, and we propose a new method for a numerical calculation of stress components.

First, stress functions are obtained for a general loading condition. Next, stress functions and stress components are obtained for a typical example, and then we perform a numerical calculation of stress components.

In the course of our calculation, we expand the part of hyperbolic functions of integrand contained within the stress functions expressed in infinite integrals by Maclaurin law. Thus, integrating by terms, we obtain each stress component. Then, we solve the simultaneous linear algebraic equations with four unknowns so as to satisfy the boundary conditions at two given points on the boundary.

The result will show that we are able to obtain an approximate solution in finite terms sufficiently accurate for practical use.

1. Introduction

Stress analysis of an infinite elastic thick plate subjected to loads symmetrical to axis of revolution, one of the most fundamental three-dimensional problems in the theory of elasticity, has been made by means of Michell's function and Hankel transformation⁽¹⁾. But a lot of time is required for numerical calculation, for the expressions for stress components take the form of infinite integral. In recent years, due to the development of an electronic computer, the calculations of stress components in a comparatively short time are made possible by way of numerical integration method. In this case, however, though approximate values of the stress components are obtained with high precision in the neighbourhood of middle plane, it still takes a lot of time to calculate, for convergence becomes slow as the calculation approaches the boundary surface.

Here, in this paper, we would like to make a proposal of the way of numerical calculation for stress components in a thick plate subjected to loads symmetrical to axis of revolution, using a similar way to the one we used in a thick plate subjected to loads symmetrical to axis of revolution and to middle plane⁽²⁾. The proposal is as follows. After representing

the part of hyperbolic functions of the integrand of stress functions expressed in infinite integral in terms of exponential functions, we expand the denominator by Maclaurin law and integrate by terms. And then we get each stress component, of which approximate values are obtained in finite terms with sufficient accuracy for practical use by solving the simultaneous linear algebraic equations with four unknowns so as to satisfy the boundary conditions at two points on the boundary surfaces.

In the case of integrating by terms, we utilize integral formulas of Hankel-Lipschitz and its enlargements consisting of the product of two Bessel functions, and we express them approximately in finite terms.

2. Stress functions

As for the stress function of this problem, we apply the method in which two stress functions are used, as the problem can be solved by fixing only variables which are in the direction of rotating axis of cylindrical coordinates⁽³⁾.

Taking the cylindrical coordinates (Fig. 1) with r in the direction of radius

Fig. 1 Stress components in cylindrical coordinates

* Received 27th December, 1972.

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and z in the axis of revolution, in which the stress component in the latter is denoted by σ_z , the stress component in the former, by σ_r , the shearing stress component acting in both directions, by τ , and the normal stress in the direction of circumference, by σ_t , the stress function F is to satisfy the following equation:

$$\nabla^2 \nabla^2 F = 0 \quad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (2)$$

Each stress component is indicated as follows:

$$\left. \begin{aligned} \sigma_z &= \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} \\ \sigma_r &= -\frac{\partial^2}{\partial r^2} (F + 2\phi_2) - \frac{2}{m} \frac{1}{r} \frac{\partial \phi_2}{\partial r} \\ \sigma_t &= -\frac{1}{r} \frac{\partial}{\partial r} (F + 2\phi_2) - \frac{2}{m} \frac{\partial^2 \phi_2}{\partial r^2} \\ \tau &= -\frac{\partial^2 F}{\partial r \partial z} \end{aligned} \right\} \quad (3)$$

in which m is Poisson's number, and the stress function ϕ_2 must satisfy the following equation:

$$\nabla^2 F = 2 \frac{\partial^2 \phi_2}{\partial z^2} \quad (4)$$

Now, we consider the case where vertical loads are acting symmetrically about the axis z at the upper and lower boundary surfaces of an infinite elastic thick plate with a height $2h$. Among the solutions which will satisfy Eq. (1) we select F as follows:

$$F = \sum_{n=1}^{\infty} (A_n \cosh \alpha_n z + B_n \sinh \alpha_n z + C_n z \cosh \alpha_n z + D_n z \sinh \alpha_n z) J_0(\alpha_n r) \quad (5)$$

where A_n , B_n , C_n and D_n are arbitrary constants determined by the boundary conditions, α_n indicates the positive root of number n in $J_0(\alpha a) = 0$, and a denotes the

width of expansion. We expand the normal load $-p_1(r)$ acting on $z = +h$; $-p_2(r)$, $z = -h$ to Fourier-Bessel series of zero order. Setting

$$\left. \begin{aligned} p_1(r) &= \sum_{n=1}^{\infty} A_n' J_0(\alpha_n r) \\ p_2(r) &= \sum_{n=1}^{\infty} B_n' J_0(\alpha_n r) \end{aligned} \right\} \quad (6)$$

A_n' and B_n' can be represented by the following expressions:

$$\left. \begin{aligned} A_n' &= \frac{2}{a^2 J_1^2(\alpha_n a)} \int_0^a r p_1(r) J_0(\alpha_n r) dr \\ B_n' &= \frac{2}{a^2 J_1^2(\alpha_n a)} \int_0^a r p_2(r) J_0(\alpha_n r) dr \end{aligned} \right\} \quad (7)$$

Now, making an asymptotic expansion of Bessel function for a sufficiently large α ,

$$J_0(\alpha a) = \sqrt{\frac{2}{\pi \alpha a}} \cos\left(\alpha a - \frac{\pi}{4}\right) = 0$$

hence

$$\alpha_n a - \pi/4 = (2n-1)\pi/2 \quad n=1, 2, 3, \dots \quad (8)$$

from which

$$J_1(\alpha_n a) = \sqrt{\frac{2}{\pi \alpha_n a}} \sin\left(\alpha_n a - \frac{\pi}{4}\right) = (-1)^{n+1} \sqrt{\frac{2}{\pi \alpha_n a}}$$

hence

$$\alpha_n a J_1^2(\alpha_n a) = 2/\pi$$

Consequently, A_n' and B_n' are given as follow:

$$A_n' = \frac{\pi \alpha_n}{a} f(p_1), \quad B_n' = \frac{\pi \alpha_n}{a} f(p_2) \quad (9)$$

provided that

$$\left. \begin{aligned} f(p_1) &= \int_0^a r p_1(r) J_0(\alpha_n r) dr \\ f(p_2) &= \int_0^a r p_2(r) J_0(\alpha_n r) dr \end{aligned} \right\} \quad (10)$$

The boundary conditions are $(\sigma_z)_{z=\pm h} = -p_i(r)$, $(\sigma_z)_{z=\pm h} = -p_i(r)$ and $(\tau)_{z=\pm h} = 0$. Substituting Eq. (5) into Eq. (3) with an application of the above boundary conditions to it, the constants A_n , B_n , C_n and D_n are determined, with which Eq. (5) is represented as follows:

$$F = \sum_{n=1}^{\infty} \frac{\pi}{a \alpha_n} \left[\frac{(\sinh \alpha_n h + \alpha_n h \cosh \alpha_n h) \cosh \alpha_n z - \alpha_n z \sinh \alpha_n h \sinh \alpha_n z}{\sinh 2\alpha_n h + 2\alpha_n h} (f(p_1) + f(p_2)) \right. \\ \left. + \frac{(\cosh \alpha_n h + \alpha_n h \sinh \alpha_n h) \sinh \alpha_n z - \alpha_n z \cosh \alpha_n h \cosh \alpha_n z}{\sinh 2\alpha_n h - 2\alpha_n h} (f(p_1) - f(p_2)) \right] J_0(\alpha_n r) \quad (11)$$

From Eq. (8), we multiply each side of Eq. (11) by $\Delta n = a \Delta \alpha_n / \pi (=1)$ and change the form of summation into that of integral. Putting $z/h = \eta$, $r/h = \rho$ and $\alpha_n h = \lambda$, we get

$$F = \int_0^{\infty} \left[\frac{(\sinh \lambda + \lambda \cosh \lambda) \cosh \eta \lambda - \eta \lambda \sinh \lambda \sinh \eta \lambda}{\lambda (\sinh 2\lambda + 2\lambda)} (f(p_1) + f(p_2)) \right. \\ \left. + \frac{(\cosh \lambda + \lambda \sinh \lambda) \sinh \eta \lambda - \eta \lambda \cosh \lambda \cosh \eta \lambda}{\lambda (\sinh 2\lambda - 2\lambda)} (f(p_1) - f(p_2)) \right] J_0(\rho \lambda) d\lambda \quad (12)$$

From Eq. (4), ϕ_2 becomes as follows. There, the arbitrary functions of η only derived from integration become zero in order to satisfy the boundary conditions.

$$\phi_2 = - \int_0^{\infty} \left[\frac{\sinh \lambda \cosh \eta \lambda}{\lambda (\sinh 2\lambda + 2\lambda)} (f(p_1) + f(p_2)) + \frac{\cosh \lambda \sinh \eta \lambda}{\lambda (\sinh 2\lambda - 2\lambda)} (f(p_1) - f(p_2)) \right] J_0(\rho \lambda) d\lambda \quad (13)$$

From Eqs. (3), (12) and (13), each stress component can be obtained.

3. Uniformly distributed load over circular area supported by circular linear load

As an example we treat the case shown in Fig. 2. After obtaining the stress functions and stress components, we perform numerical calculations for each stress component.

3.1 Derivation of stress functions

The boundary conditions on the surface $z = +h$ are $p_1(r) = p_1$ for the interval $0 \leq r \leq c$ and $p_1(r) = 0$ for other intervals; those on

Fig. 2 A thick plate subjected to loads symmetrical to axis of revolution

the surface $z=-h$ are $p_1(r)=p_2$ for the interval $d \leq r \leq d+\Delta d$ and $p_2(r)=0$ for other intervals. Then setting $\pi c^2 p_1 = 2\pi d \Delta d p_2 = P$ and making $\Delta d \rightarrow 0$, we get from Eq. (10)

$$f(p_1) = \frac{Ph}{\pi c \lambda} J_1\left(\frac{c}{h} \lambda\right), \quad f(p_2) = \frac{P}{2\pi} J_0\left(\frac{d}{h} \lambda\right) \dots (14)$$

Substituting these values into Eqs. (12) and (13),

$$F = \frac{P}{\pi} \int_0^\infty \left[\frac{(\sinh \lambda + \lambda \cosh \lambda) \cosh \eta \lambda - \eta \lambda \sinh \lambda \sinh \eta \lambda}{\lambda (\sinh 2 \lambda + 2 \lambda)} \left\{ \frac{h}{c \lambda} J_1\left(\frac{c}{h} \lambda\right) + \frac{1}{2} J_0\left(\frac{d}{h} \lambda\right) \right\} \right. \\ \left. + \frac{(\cosh \lambda + \lambda \sinh \lambda) \sinh \eta \lambda - \eta \lambda \cosh \lambda \cosh \eta \lambda}{\lambda (\sinh 2 \lambda - 2 \lambda)} \left\{ \frac{h}{c \lambda} J_1\left(\frac{c}{h} \lambda\right) - \frac{1}{2} J_0\left(\frac{d}{h} \lambda\right) \right\} \right] J_0(\rho \lambda) d\lambda \dots (15)$$

$$\phi_z = -\frac{P}{\pi} \int_0^\infty \left[\frac{\sinh \lambda \cosh \eta \lambda}{\lambda (\sinh 2 \lambda + 2 \lambda)} \left\{ \frac{h}{c \lambda} J_1\left(\frac{c}{h} \lambda\right) + \frac{1}{2} J_0\left(\frac{d}{h} \lambda\right) \right\} \right. \\ \left. - \frac{\cosh \lambda \sinh \eta \lambda}{\lambda (\sinh 2 \lambda - 2 \lambda)} \left\{ \frac{h}{c \lambda} J_1\left(\frac{c}{h} \lambda\right) - \frac{1}{2} J_0\left(\frac{d}{h} \lambda\right) \right\} \right] J_0(\rho \lambda) d\lambda \dots (16)$$

From Eq. (3) we can obtain stress components and they coincide with the solution obtained by Hankel transformation. However, in this paper, we use a different method shown in the next article.

3.2 Derivation of stress components

The stress functions expressed by Eqs. (12) and (13) or (15) and (16), and stress components derived from them include the same part of hyperbolic functions as that of the strip in two-dimensional problems. Several methods for numerical calculations for these stress components have been proposed. They are a numerical integration method⁽⁴⁾, a method of substituting the part of hyperbolic functions of integrand with an approximate equation⁽¹⁾, a method of integrating by terms after representing the part of hyperbolic functions in terms of exponential functions and expanding the denominator by Maclaurin law⁽⁵⁾, and so on.

If we perform a numerical integration of each stress component by means of an electronic computer, we can get approximate values with quick convergence and high accuracy in the neighbourhood of axis r , while as we draw near the boundary surfaces, the values converge slower, and it requires a lot of hours. Therefore, we resort to the third method. After representing the part of hyperbolic functions of Eqs. (12) and (13) or (15) and (16) in terms of exponential functions and expanding the denominator by Maclaurin law, we get

$$2 \left\{ \frac{(\sinh \lambda + \lambda \cosh \lambda) \cosh \eta \lambda - \eta \lambda \sinh \lambda \sinh \eta \lambda}{\lambda (\sinh 2 \lambda + 2 \lambda)} \right. \\ = \{1/\lambda + (1-\eta)\} X_1 e^{-\lambda(1-\eta)} + \{1/\lambda + (1+\eta)\} X_1 e^{-\lambda(1+\eta)} \\ - \{1/\lambda + (1-\eta)\} X_2 + \{1/\lambda - (1+\eta)\} X_1 \} e^{-\lambda(3-\eta)} - \{1/\lambda + (1+\eta)\} X_2 + \{1/\lambda - (1-\eta)\} X_1 \} e^{-\lambda(3+\eta)} \\ + \{1/\lambda + (1-\eta)\} X_3 + \{1/\lambda - (1+\eta)\} X_2 \} e^{-\lambda(5-\eta)} + \{1/\lambda + (1+\eta)\} X_3 + \{1/\lambda - (1-\eta)\} X_2 \} e^{-\lambda(5+\eta)} \\ + \dots \dots \dots (17)$$

$$2 \left\{ \frac{(\cosh \lambda + \lambda \sinh \lambda) \sinh \eta \lambda - \eta \lambda \cosh \lambda \cosh \eta \lambda}{\lambda (\sinh 2 \lambda - 2 \lambda)} \right. \\ = \{1/\lambda + (1-\eta)\} X_1 e^{-\lambda(1-\eta)} - \{1/\lambda + (1+\eta)\} X_1 e^{-\lambda(1+\eta)} \\ + \{1/\lambda + (1-\eta)\} X_2 + \{1/\lambda - (1+\eta)\} X_1 \} e^{-\lambda(3-\eta)} - \{1/\lambda + (1+\eta)\} X_2 + \{1/\lambda - (1-\eta)\} X_1 \} e^{-\lambda(3+\eta)} \\ + \{1/\lambda + (1-\eta)\} X_3 + \{1/\lambda - (1+\eta)\} X_2 \} e^{-\lambda(5-\eta)} - \{1/\lambda + (1+\eta)\} X_3 + \{1/\lambda - (1-\eta)\} X_2 \} e^{-\lambda(5+\eta)} \\ + \dots \dots \dots (18)$$

$$\frac{2 \sinh \lambda \cosh \eta \lambda}{\lambda (\sinh 2 \lambda + 2 \lambda)} = X_1 (1/\lambda) e^{-\lambda(1-\eta)} + X_1 (1/\lambda) e^{-\lambda(1+\eta)} \\ - (X_1 + X_2) (1/\lambda) e^{-\lambda(3-\eta)} - (X_1 + X_2) (1/\lambda) e^{-\lambda(3+\eta)} \\ + (X_2 + X_3) (1/\lambda) e^{-\lambda(5-\eta)} + (X_2 + X_3) (1/\lambda) e^{-\lambda(5+\eta)} - \dots \dots \dots (19)$$

$$\frac{2 \cosh \lambda \sinh \eta \lambda}{\lambda (\sinh 2 \lambda - 2 \lambda)} = X_1 (1/\lambda) e^{-\lambda(1-\eta)} - X_1 (1/\lambda) e^{-\lambda(1+\eta)} \\ + (X_1 + X_2) (1/\lambda) e^{-\lambda(3-\eta)} - (X_1 + X_2) (1/\lambda) e^{-\lambda(3+\eta)} \\ + (X_2 + X_3) (1/\lambda) e^{-\lambda(5-\eta)} - (X_2 + X_3) (1/\lambda) e^{-\lambda(5+\eta)} + \dots \dots \dots (20)$$

where

$$X_1 = 1 \quad X_2 = 2\lambda^2, \quad X_3 = 1 + (2\lambda)^2 2^2, \dots$$

Now, setting

$${}_n I_\nu(n, a, r) = \int_0^\infty \lambda^n e^{-a\lambda} J_\nu\left(\frac{r}{h} \lambda\right) J_\nu(\rho \lambda) d\lambda \dots (21)$$

Eqs. (15) and (16) can be represented as follows:

$$\frac{F}{P/\pi} = \left(\frac{h}{c} A_1 + \frac{1}{2} A_2\right) - \left(\frac{1}{2} A_3 + \frac{h}{c} A_4\right) + \left(\frac{h}{c} A_5 + \frac{1}{2} A_6\right) - \dots \dots \dots (22)$$

$$\left. \begin{aligned} A_1 &= {}_1 I_0(-2, 1-\eta, c) + (1-\eta) {}_1 I_0(-1, 1-\eta, c) \\ A_2 &= {}_0 I_0(-1, 1+\eta, d) + (1+\eta) {}_0 I_0(0, 1+\eta, d) \\ A_3 &= {}_0 I_0(-1, 3-\eta, d) + (3-\eta) {}_0 I_0(0, 3-\eta, d) + 2^2 (1-\eta) {}_0 I_0(1, 3-\eta, d) \\ A_4 &= {}_1 I_0(-2, 3+\eta, c) + (3+\eta) {}_1 I_0(-1, 3+\eta, c) + 2^2 (1+\eta) {}_1 I_0(0, 3+\eta, c) \\ A_5 &= {}_1 I_0(-2, 5-\eta, c) + (5-\eta) {}_1 I_0(-1, 5-\eta, c) \\ &\quad + 2^2 (3-\eta) {}_1 I_0(0, 5-\eta, c) + 2^4 (1-\eta) {}_1 I_0(1, 5-\eta, c) \\ A_6 &= {}_0 I_0(-1, 5+\eta, d) + (5+\eta) {}_0 I_0(0, 5+\eta, d) \\ &\quad + 2^2 (3+\eta) {}_0 I_0(1, 5+\eta, d) + 2^4 (1+\eta) {}_0 I_0(2, 5+\eta, d) \end{aligned} \right\} \dots \dots \dots (22)'$$

$$\frac{\phi_z}{P/\pi} = - \left\{ \left(\frac{h}{c} B_1 + \frac{1}{2} B_2\right) - \left(\frac{1}{2} B_3 + \frac{h}{c} B_4\right) + \left(\frac{h}{c} B_5 + \frac{1}{2} B_6\right) - \dots \dots \dots \right\} \dots (23)$$

$$\left. \begin{aligned} B_1 &= I_0(-2, 1-\eta, c), \quad B_2 = I_0(-1, 1+\eta, d) \\ B_3 &= I_0(-1, 3-\eta, d) + 2^2 I_0(0, 3-\eta, d) \\ B_4 &= I_0(-2, 3+\eta, c) + 2^2 I_0(-1, 3+\eta, c) \\ B_5 &= I_0(-2, 5-\eta, c) + 2^2 I_0(-1, 5-\eta, c) + 2^4 I_0(0, 5-\eta, c) \\ B_6 &= I_0(-1, 5+\eta, d) + 2^2 I_0(0, 5+\eta, d) + 2^4 I_0(1, 5+\eta, d) \end{aligned} \right\} \dots\dots\dots (23)'$$

In the above equations $A_1, A_2, \dots, A_6$, satisfy Eq. (1) independently, and A_1 and $B_1, \dots, A_6$ and B_6 , satisfy Eq. (4). Stress components obtained from Eqs. (22), (22)', (23) and (23)' by using Eq. (3) are shown as follows. Putting $P/\pi h^2 = K$,

$$\frac{\sigma_z}{K} = -\left\{ \left(\frac{h}{c} D_1 + \frac{1}{2} D_2 \right) - \left(\frac{1}{2} D_3 + \frac{h}{c} D_4 \right) + \left(\frac{h}{c} D_5 + \frac{1}{2} D_6 \right) - \dots \right\} \dots\dots\dots (24)$$

$$\left. \begin{aligned} D_1 &= I_0(0, 1-\eta, c) + (1-\eta) I_0(1, 1-\eta, c) \\ D_2 &= I_0(1, 1+\eta, d) + (1+\eta) I_0(2, 1+\eta, d) \\ D_3 &= I_0(1, 3-\eta, d) + (3-\eta) I_0(2, 3-\eta, d) + 2^2 (1-\eta) I_0(3, 3-\eta, d) \\ D_4 &= I_0(0, 3+\eta, c) + (3+\eta) I_0(1, 3+\eta, c) + 2^2 (1+\eta) I_0(2, 3+\eta, c) \\ D_5 &= I_0(0, 5-\eta, c) + (5-\eta) I_0(1, 5-\eta, c) \\ &\quad + 2^2 (3-\eta) I_0(2, 5-\eta, c) + 2^4 (1-\eta) I_0(3, 5-\eta, c) \\ D_6 &= I_0(1, 5+\eta, d) + (5+\eta) I_0(2, 5+\eta, d) \\ &\quad + 2^2 (3+\eta) I_0(3, 5+\eta, d) + 2^4 (1+\eta) I_0(4, 5+\eta, d) \end{aligned} \right\} \dots\dots\dots (24)'$$

$$\frac{\tau}{K} = \left(\frac{h}{c} E_1 - \frac{1}{2} E_2 \right) + \left(\frac{1}{2} E_3 - \frac{h}{c} E_4 \right) + \left(\frac{h}{c} E_5 - \frac{1}{2} E_6 \right) + \dots\dots\dots (25)$$

$$\left. \begin{aligned} E_1 &= (1-\eta) I_1(1, 1-\eta, c), \quad E_2 = (1+\eta) I_1(2, 1+\eta, d) \\ E_3 &= (1+\eta) I_1(2, 3-\eta, d) - 2^2 (1-\eta) I_1(3, 3-\eta, d) \\ E_4 &= (1-\eta) I_1(1, 3+\eta, c) - 2^2 (1+\eta) I_1(2, 3+\eta, c) \\ E_5 &= (1-\eta) I_1(1, 5-\eta, c) - 2^2 (1+\eta) I_1(2, 5-\eta, c) + 2^4 (1-\eta) I_1(3, 5-\eta, c) \\ E_6 &= (1+\eta) I_1(2, 5+\eta, d) - 2^2 (1-\eta) I_1(3, 5+\eta, d) + 2^4 (1+\eta) I_1(4, 5+\eta, d) \end{aligned} \right\} \dots\dots\dots (25)'$$

In Eqs. (24) and (24)', $-hD_1/c$ means σ_z/K of a semi-infinite body of the case subjected to a uniformly distributed load over circular area on the surface $z=+h$; $-D_2/2$ means that of the case subjected to a circular linear load on the surface $z=-h$. If we take up to D_3 , the boundary condition on the surface $z=+h$ is perfectly fulfilled. If we take D_4 instead of D_3 , the boundary condition on the surface $z=-h$ is perfectly satisfied. Similarly, if we take up to D_5 , the boundary condition on the surface $z=+h$ is fulfilled. If we take D_6 instead of D_5 , the boundary condition on the surface $z=-h$ is perfectly satisfied. The same is true for τ/K of Eqs. (25) and (25)'. Stress components σ_r and σ_t are represented as follows:

$$\frac{\sigma_r + \sigma_t}{K} = -\left\{ \left(\frac{h}{c} F_1 + \frac{1}{2} F_2 \right) - \left(\frac{1}{2} F_3 + \frac{h}{c} F_4 \right) + \left(\frac{h}{c} F_5 + \frac{1}{2} F_6 \right) - \dots \right\} \\ - \frac{2}{m} \left\{ \left(\frac{h}{c} G_1 + \frac{1}{2} G_2 \right) - \left(\frac{1}{2} G_3 + \frac{h}{c} G_4 \right) + \left(\frac{h}{c} G_5 + \frac{1}{2} G_6 \right) - \dots \right\} \dots\dots\dots (26)$$

$$\left. \begin{aligned} F_1 &= I_0(0, 1-\eta, c) - (1-\eta) I_0(1, 1-\eta, c) \\ F_2 &= I_0(1, 1+\eta, d) - (1+\eta) I_0(2, 1+\eta, d) \\ F_3 &= I_0(1, 3-\eta, d) + (5+\eta) I_0(2, 3-\eta, d) - 2^2 (1-\eta) I_0(3, 3-\eta, d) \\ F_4 &= I_0(0, 3+\eta, c) + (5-\eta) I_0(1, 3+\eta, c) - 2^2 (1+\eta) I_0(2, 3+\eta, c) \\ F_5 &= I_0(0, 5-\eta, c) + (3+\eta) I_0(1, 5-\eta, c) \\ &\quad + 2^2 (5+\eta) I_0(2, 5-\eta, c) - 2^4 (1-\eta) I_0(3, 5-\eta, c) \\ F_6 &= I_0(1, 5+\eta, d) + (3-\eta) I_0(2, 5+\eta, d) \\ &\quad + 2^2 (5-\eta) I_0(3, 5+\eta, d) - 2^4 (1+\eta) I_0(4, 5+\eta, d) \end{aligned} \right\} \dots\dots\dots (26)'$$

$$\left. \begin{aligned} G_1 &= I_0(0, 1-\eta, c), \quad G_2 = I_0(1, 1+\eta, d) \\ G_3 &= I_0(1, 3-\eta, d) + 2^2 I_0(2, 3-\eta, d) \\ G_4 &= I_0(0, 3+\eta, c) + 2^2 I_0(1, 3+\eta, c) \\ G_5 &= I_0(0, 5-\eta, c) + 2^2 I_0(1, 5-\eta, c) + 2^4 I_0(2, 5-\eta, c) \\ G_6 &= I_0(1, 5+\eta, d) + 2^2 I_0(2, 5+\eta, d) + 2^4 I_0(3, 5+\eta, d) \end{aligned} \right\}$$

From Eqs. (26) and (26)', representing Eq. (26) as

$$(\sigma_r + \sigma_t)/K = -f_1(I_0, I_0) - 2f_2(I_0, I_0)/m \dots\dots\dots (26)''$$

we obtain

$$(\sigma_r - \sigma_t)/K = f_1(I_1, I_1) - 2f_2(I_1, I_1)/m \dots\dots\dots (27)$$

As we mentioned above, when a thick plate is subjected to loads symmetrical to axis of revolution, we obtained Eqs. (24) ~ (27) for each stress component following

the similar method to that we used in a thick plate subjected to loads symmetrical to axis of revolution and to middle plane. These equations show that the latter can be treated as a special case of the former.

3.3 Enlargements of integral formulas of Hankel-Lipschitz

Each stress component, mentioned in the previous article, consists of the infinite integrals defined by Eq. (21). As for the calculating method of these infi-

nite integrals, there is suggested a method⁽¹⁾ of integration by terms for the function $J_\mu(\gamma\lambda/h)$ represented by series. In this paper, however, we will use the integral formulas of Hankel-Lipschitz and its enlargements:

$$\begin{aligned} {}_0I_0(0, a, r) &= \int_0^\infty e^{-a\lambda} J_0\left(\frac{r}{h}\lambda\right) J_0(\rho\lambda) d\lambda \\ &= \frac{1}{\pi\sqrt{b\rho}} Q_{-1/2}\left(\frac{1}{x}\right) \\ &= \frac{1}{\pi\sqrt{b\rho}} \frac{\sqrt{\pi}}{(2x^{-1})^{1/2}} \frac{\Gamma(1/2)}{\Gamma(1)} F\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right) \end{aligned} \quad (28)$$

where Q_n denotes Legendre's function of the second kind, and $F()$, hypergeometric series; with abbreviation of

$$r/h = b, \quad g^2 = a^2 + b^2 + \rho^2, \quad x = 2b\rho/g^2 \quad (28)'$$

Representing ${}_0I_0(0, a, \gamma)$ by series,

$${}_0I_0(0, a, r) = \frac{1}{g} \sum_{n=0}^{\infty} f(n) \frac{x^{2n}}{(n!)^2} \quad (29)$$

where

$$f(n) = \frac{1}{4} \frac{5}{4} \dots \left(\frac{1}{4} + n - 1\right) \frac{3}{4} \frac{7}{4} \dots \left(\frac{3}{4} + n - 1\right) \quad (29)'$$

for $n=0$, $\{f(n)\}_{n=0}^{\infty} = 1$. Here we shall take terms as many as $n=15$, and approximate terms for $n \geq 16$ with geometrical series:

$$\begin{aligned} {}_0I_0(0, a, r) &= \frac{1}{g} \left(1 + \frac{1 \cdot 3}{4^2} x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2} x^4 + \dots\right) \\ &\approx (1 + 0.18750 x^2 + 0.10254 x^4 + \dots \\ &\quad + 0.014819 x^{30} (1 - \beta x^2)) / g \quad (29)'' \end{aligned}$$

The ratio of coefficient of $n=15$ to that of $n=16$ is $3843/4096$; and is the case $n \rightarrow \infty$, the ratio is 1. As for β , therefore, we use 0.96912 as the mean value of these ratios, according to the equation $(1+3843/4096)/2 = 0.96912$.

Partially differentiating each side of Eq. (29) with respect to $(-a)$,

$${}_0I_0(1, a, r) = \frac{a}{g^2} \sum_{n=0}^{\infty} f(n) \frac{4n+1}{(n!)^2} x^{2n} \quad (30)$$

Using the same approximate method as ${}_0I_0(0, a, \gamma)$, we get

$$\begin{aligned} {}_0I_0(1, a, r) &\approx (1 + 0.93750 x^2 + 0.92285 x^4 + \dots \\ &\quad + 0.90395 x^{30} (1 - \beta x^2)) / g^3 \quad (30)' \end{aligned}$$

$${}_0I_0(2, a, r) = \frac{ab}{g^2} \sum_{n=0}^{\infty} f(n) \frac{(4n+1)(4n+3)}{(n!)^2} \left(1 - \frac{2n}{4n+3} \frac{g^2}{b^2}\right) x^{2n} \quad (32)$$

$${}_0I_0(3, a, r) = \frac{b}{g^2} \sum_{n=0}^{\infty} f(n) \frac{(4n+1)(4n+3)(4n+5)}{(n!)^2} x^{2(n-1)} \left\{ x^2 u^2 - \frac{2n}{4n+5} y^2 u^2 + \frac{2n}{(4n+3)(4n+5)} y^2 - \frac{x^2}{4n+5} \right\} \quad (33)$$

where $y^2 = (2\rho)^2/g^2$ and $u^2 = a^2/g^2$.

Next, integrating each side of ${}_0I_0(1, a, \gamma)$ with respect to a , from 0 to a , we can obtain ${}_0I_0(0, a, \gamma)$. Putting $p^2 = b^2 + \rho^2$, $q^2 = b^2 - \rho^2$, $v = p^2/g^2$ and $w^2 = (2b\rho/p^2)^2$,

$$\begin{aligned} {}_0I_0(0, a, r) &= f(b) - \frac{ab}{p^2 g} \left\{ g {}_0I_0(0, a, r) + \frac{2q^2 \rho^2}{p^4} \sum_{n=0}^{\infty} \frac{1}{2} \frac{3}{4} \dots \frac{2n+1}{2n+2} \right. \\ &\quad \left. \times \left(1 + \sum_{m=0}^{2n} \frac{1}{2} \frac{3}{4} \dots \frac{2m+1}{2m+2} v^{m+1}\right) w^{2n} \right\} \quad (34) \end{aligned}$$

where

$$f(b) = 0 \dots \rho > 0, \quad = 1/2b \dots \rho = b, \quad = 1/b \dots \rho < b$$

Hence,

$$\begin{aligned} {}_0I_0(0, a, r) &\approx f(b) - \frac{ab}{p^2 g} \left[g {}_0I_0(0, a, r) + \frac{q^2 \rho^2}{p^4} \{s_0 + 0.75 s_1 w^2 + 0.625 s_2 w^4 + \dots\right. \\ &\quad \left. + 0.28893 s_{14} w^{28} + 0.27990 s_{15} w^{30} (1 - \beta w^2)\} \right] \quad (34)' \end{aligned}$$

where $s_0 = 1 + 0.5v$, $s_1 = s_0 + 0.375v^2 + 0.3125v^3, \dots$, and $s_{14} = s_{14} + 0.10258v^{30} + 0.10092v^{31}$. When the value a is large, $\beta \rightarrow 1$ for $n \rightarrow \infty$ and the ratio of coefficient of $n=15$ to that of $n=16$ is $33/34$; and as for the value β , we use the mean value of these ratios 0.98529 , according to the equation $(1+33/34)/2 = 0.98529$. When the value a is small, we will take the next value considering the fact that β becomes nearly equal to 1:

$$\begin{aligned} \beta &= 0.98529 \dots \dots \dots a \geq 0.5 \\ &= 1 - 0.020797 \sqrt{a} \dots \dots \dots a \leq 0.5 \end{aligned}$$

We use $\beta = (1+4095/4096)/2 = 0.99988 \approx 1$.

Next, partially differentiating each side of ${}_0I_0(0, a, \gamma)$ with respect to $(-b)$, we get ${}_0I_0(1, a, \gamma)$:

$${}_0I_0(1, a, r) = \frac{b}{g^2} \sum_{n=0}^{\infty} f(n) \frac{4n+1}{(n!)^2} \left(1 - \frac{2n}{4n+1} \frac{g^2}{b^2}\right) x^{2n} \quad (31)$$

Hence,

$$\begin{aligned} {}_0I_0(1, a, r) &\approx b {}_0I_0(1, a, r) / a - 4\rho^2 b \{0.37500 \\ &\quad + 0.41016 x^2 + \dots \\ &\quad + 0.44417 x^4 + 0.44827 x^{28} (1 - x^2)\} / g^5 \end{aligned} \quad (31)'$$

where $0.44827 = 0.90395(0.49180 + 0.5)/2$, in which the value 0.5 is the limiting value of $2n/(4n+1)$ for $n \rightarrow \infty$; the value 0.49180 is the value of $2n/(4n+1)$ for $n=15$; the value 0.90395 is the value of $f(n)(4n+1)/(n!)^2$ for $n=16$. For $x \leq 1$, convergence of series in Eq. (31)' is slow, and so comparing the values of this approximate equation with those obtained from numerical integral for $b=\rho=1$ by changing a , we get Table 1. From the table, the errors of ${}_0I_0(1, a, \gamma)$, for $a=0.1$ and 0.2 , are $0.0108(20.1\%)$ and $-0.0002(-0.2\%)$ respectively.

Partially differentiating each side of ${}_0I_0(1, a, \gamma)$ with respect to $(-a)$ successively, we obtain ${}_0I_0(2, a, \gamma)$ and ${}_0I_0(3, a, \gamma)$. Since those formulas are applied in the ranges of $a \geq 2$ and $a \geq 4$ respectively, the convergence of series are so rapid that they can be approximated in finite terms with sufficient accuracy:

Table 1 Values of ${}_0I_0(1, a, \gamma)$ ($b=\rho=1$)

a	Approximate values	Numerical integration	Errors	Errors %
0.1	0.6456	0.5374	0.1082	20.1
0.2	0.4244	0.4252	-0.0008	-0.2
0.4	0.3090	0.3100	-0.0010	-0.3
0.6	0.2406	0.2406	0.0000	0.0
0.8	0.1911	0.1911	0.0000	0.0
1.0	0.1536	0.1536	0.0000	0.0

Comparing these approximate values with those of numerical integrals, we get Table 2 and Table 3. Table 2 shows the case in which we change a for $b=1$ and $\rho=1.2$, and Table 3, the case in which we change ρ for $b=1$ and $a=0.1$. Next,

$$I_1(0, a, \gamma) = \int_0^\infty e^{-\lambda} J_1\left(\frac{\gamma}{h} \lambda\right) J_1(\rho \lambda) d\lambda = \frac{1}{\pi \sqrt{b\rho}} Q_{1/2}\left(\frac{1}{x}\right) \\ = \frac{1}{\pi \sqrt{b\rho}} \frac{\sqrt{\pi}}{(2x^{-1})^{3/2}} \frac{\Gamma(3/2)}{\Gamma(2)} F\left(\frac{3}{4}, \frac{5}{4}; 2; x^2\right) \quad (35)$$

where we shall use the symbols in Eq.(28)'. We put

$$f'(n) = \frac{3}{4} \frac{7}{4} \dots \left(\frac{3}{4} + n - 1\right) \frac{5}{4} \frac{9}{4} \dots \left(\frac{5}{4} + n - 1\right) \quad (35)'$$

and define $\{f'(n)\}_{n=0}^\infty = 1$, then representing Eq.(35) by series,

$$I_1(0, a, \gamma) = \frac{b\rho}{2g^3} \sum_{n=0}^\infty \frac{f'(n)}{n!(n+1)!} x^{2n} \\ = b\rho \{1 + 0.46875 x^2 + 0.30762 x^4 + \dots \\ + 0.056497 x^{10} / (1 - \beta x^2)\} / 2g^3 \quad (35)''$$

As for β , we use $\beta = (1 + 4095/4352) / 2 = 0.97047$. Putting $b=\rho=1$, the errors of $a, I_1(0, a, \gamma)$, for $a=0.1$ and 0.2 , are -0.0012 (-1.56%) and 0.0017 (1.54%) respectively.

Partially differentiating each side of $I_1(0, a, \gamma)$ with respect to $(-a)$,

$$I_1(1, a, \gamma) = \frac{ab\rho}{2g^3} \sum_{n=0}^\infty \frac{f'(n)}{n!(n+1)!} \frac{4n+3}{x^{2n}} \\ = 3ab\rho \{1 + 1.0938 x^2 + 1.1279 x^4 + \dots \\ + 1.1864 x^{10} / (1 - x^2)\} / 2g^3 \quad (36)$$

where we use the value $\beta = (1 + 4355/4352) / 2 \approx 1$. When $b=\rho=1$, the errors of $a, I_1(1, a, \gamma)$, for $a=0.1$ and 0.2 , are -0.0023 (-0.74%) and -0.0010 (-0.32%) respectively.

Repeating the partial differentiation with respect to $(-a)$, we get $I_1(2, a, \gamma)$ and $I_1(3, a, \gamma)$. Since the ranges of application of those formulas are $a \geq 2$ and $a \geq 4$ respectively, the convergences of series is so rapid that they can be approximated in finite terms. Putting $u^2 = a^2/g^2$,

$$I_1(2, a, \gamma) = \frac{b\rho}{2g^3} \sum_{n=0}^\infty \frac{f'(n)}{n!(n+1)!} \frac{(4n+3)(4n+5)}{x^{2n}} \\ \times \left(u^2 - \frac{1}{4n+5}\right) x^{2n} \quad (37)$$

Table 2 Values of $I_0(0, a, \gamma)$ ($b=1, \rho=1.2$)

a	Approximate values	Numerical integration	Errors	Errors %
0.1	0.0872	0.0965	-0.0093	-9.6
0.2	0.1485	0.1544	-0.0059	-3.8
0.5	0.1892	0.1856	0.0036	1.9
1.0	0.1486	0.1433	0.0053	3.7
2.0	0.0801	0.0747	0.0054	7.2
4.0	0.0319	0.0265	0.0054	20
6.0	0.0182	0.0128	0.0054	42

Table 3 Values of $I_0(0, a, \gamma)$ ($b=1, a=0.1$)

ρ	Approximate values	Numerical integration	Errors	Errors %
0.4	0.8868	0.8868	0.0000	0.00
0.8	0.7912	0.7880	0.0032	0.41
1.0	0.4310	0.4303	0.0007	0.2
1.2	0.0872	0.0965	-0.0093	-9.6
1.6	0.0208	0.0209	-0.0001	0.5
2.0	0.0086	0.0086	0.0000	0.00

$$I_1(3, a, \gamma) = \frac{ab\rho}{2g^3} \sum_{n=0}^\infty \frac{f'(n)}{n!(n+1)!} \frac{(4n+3)(4n+5)(4n+7)}{x^{2n}} \\ \times \left(u^2 - \frac{3}{4n+7}\right) x^{2n} \quad (38)$$

Next, integrating each side of Eq.(35)'' with respect to a , from 0 to a , we obtain $I_1(-1, a, \gamma)$:

$$I_1(-1, a, \gamma) = f(b, \rho) - \frac{ab\rho}{2g^2} \left\{ 1 + \sum_{n=1}^\infty \frac{1}{n!} \frac{3}{4} \frac{2n-1}{2n+2} \right. \\ \times \left. \left(1 + \sum_{m=0}^{2n-1} \frac{1}{2} \frac{3}{4} \dots \frac{2m+1}{2m+2} \frac{1}{v^{n+1}} \right) w^{2n} \right\} \quad (39)$$

where

$$f(b, \rho) = \rho/2b \dots \rho < b, = b/2\rho \dots \rho > b$$

and p^2, v and w^2 are the same as those defined by Eq.(34). After all,

$$I_1(-1, a, \gamma) = f(b, \rho) - \frac{ab\rho}{2g^2} \{ t_0 + 0.25 t_1 w^2 \\ + 0.125 t_2 w^4 + \dots + 0.009963 t_{15} w^{28} \\ + 0.009029 t_{15} w^{30} / (1 - \beta w^2) \} \quad (39)'$$

where $t_0=1$, $t_1=t_0+0.5v+0.375v^2, \dots$, and $t_{15}=t_{14}+0.10432v^{14}+0.10258v^{16}$. As for β , we will take the next values from the same idea for $I_0(0, a, \gamma)$ in Eq.(34)':

$$\beta = (1 + 31.34) / 2 = 0.95588 \dots a \geq 0.5 \\ = 1 - 0.062391\sqrt{a} \dots a \leq 0.5$$

The error in this case takes the maximum value for $b=\rho=1$. Then, comparing the approximate values in Eq.(39)' with the values of numerical integrals by changing the value a , we get Table 4. Table 5 shows the case in which we change ρ for $a=0.1$.

As for $I_2(n, a, \gamma)$, from the recurrence formula of Bessel functions,

$$I_2(n, a, \gamma) = -I_0(n, a, \gamma) + 2I_1((n-1), a, \gamma) / \rho \quad (40)$$

As we mentioned above, among the terms shown in Eqs.(24)~(27), we have got the approximate equations of $\mu I_\nu(n, a, \gamma)$ which are caused by uniformly distributed load over circular area on the boundary surface $z=h$. Since those which are caused by circular linear load on $z=-h$ are the same as in the case subjected to loads symmetrical to axis of revolution and to middle plane in the previous paper⁽²⁾, in this

Table 4 Values of $I_1(-1, a, \gamma)$ ($b=\rho=1$)

a	Approximate values	Numerical integration	Errors	Errors %
0.1	0.3974	0.3922	0.0052	1.3
0.2	0.3359	0.3282	0.0077	2.3
0.5	0.2324	0.2110	0.0214	10.1
1.0	0.1372	0.1161	0.0211	18.2
2.0	0.0678	0.0468	0.0210	44.8
4.0	0.0354	0.0143	0.0211	148
6.0	0.0277	0.0067	0.0210	313

Table 5 Values of $I_1(-1, a, \gamma)$ ($b=1, a=0.1$)

ρ	Approximate values	Numerical integration	Errors	Errors %
0.4	0.1788	0.1788	0.0000	0.00
0.8	0.3423	0.3441	-0.0018	-0.52
1.0	0.3974	0.3922	0.0052	1.3
1.2	0.3627	0.3654	-0.0027	-0.74
1.6	0.2892	0.2892	0.0000	0.00
2.0	0.2361	0.2361	0.0000	0.00

paper we shall omit them.

3.4 An example of numerical calculation

As an example, we make a numerical calculation for the case $b=\gamma/h=c/h=d/h=1.0$.

As we stated in the article 3.2, σ_z/K and τ/K will satisfy the boundary conditions successively with an increase in number of terms. But it is known that the convergence is so slow that it is not good for practical use⁽⁷⁾. In the previous paper⁽²⁾, we got approximate solutions with sufficient accuracy for practical use by observing the fact that these conditions of convergence change in the mode of damped oscillation. In this paper, however, we perform the following manipulation so that we may obtain a better approximate solution.

As we stated in the article 3.2, since $A_1 \sim A_6$ independently satisfy Eq.(1), any linear combination of individual terms is also a solution of Eq.(1). And A_1 and B_1 , A_2 and B_2 ,... respectively satisfy Eq.(4). Therefore, we try to express Eqs.(24)~(27) in the finite terms as follows:

$$\sigma_z/K \approx -[(D_1+D_2/2)-(k_1D_3+k_4D_4) + (k_5D_5+k_6D_6)] \quad (41)$$

$$\tau/K \approx (E_1-E_2/2)+(k_1E_3-k_4E_4) + (k_5E_5-k_6E_6) \quad (42)$$

$$(\sigma_r+\sigma_t)/K \approx -[(F_1+F_2/2)-(k_1F_3+k_4F_4) + (k_5F_5+k_6F_6)] - 2[(G_1+G_2/2)-(k_1G_3+k_4G_4) + (k_5G_5+k_6G_6)]/m \quad (43)$$

Representing Eq.(43) as

$$(\sigma_r+\sigma_t)/K \approx k_7[-f_1(I_0,0I_0)-2f_2(I_0,0I_0)/m] \quad (43')$$

$(\sigma_r-\sigma_t)/K$ is

$$(\sigma_r-\sigma_t)/K \approx k_8[f_1(I_2,0I_2)-2f_2(I_2,0I_2)/m] \quad (44)$$

In analyzing the stresses with a loading condition similar to punching an elastic thick plate, in the previous paper⁽²⁾, we determined the constants by observing the fact that the boundary conditions are satisfied in the mode of damped oscillation. In this example, considering in the same way as in the previous paper, σ_z/K is, for instance,

$$\begin{aligned} \sigma_z/K \approx & -[(D_1+D_2/2)+((D_1+D_2/2)-(D_3/2+D_4) \\ & + (D_5+D_6/2))] / 2 + [(D_1+D_2/2)-(D_3/2+D_4)] / 2 \\ = & -[(D_1+D_2/2)-0.75(D_3/2+D_4) \\ & + 0.25(D_5+D_6/2)] \end{aligned}$$

that is,

$$2k_3=k_4=0.75, \quad k_5=2k_6=0.25 \quad (45)$$

In this paper, in order to obtain a still better approximate solution, we shall determine the values $k_3 \sim k_6$ by solving the simultaneous linear algebraic equations with four unknowns in such a way that $\sigma_z/K = \tau/K = 0$ for $\eta = \pm 1.0$, $\rho = x$. Tables 6 and 7 show the values of σ_z/K and τ/K at the boundary surfaces in the case of changing x from 1.2 to 2.0 at an interval 0.2 in comparison with those in the case of using the constants in Eq. (45). The exact boundary conditions are, as for σ_z/K , -1.0 for $\rho < 1.0$, -0.5 for $\rho = 1.0$, zero for $\rho > 1.0$ respectively when $\eta = +1.0$, and ∞ for $\rho = 1.0$, zero for the other points respectively when $\eta = -1.0$; as for τ/K , zero for all ρ when $\eta = \pm 1.0$. It is evident from the tables that when we use the coefficients k_n in the case of Eq.(45), the boundary con-

Table 6 Stresses σ_z/K and τ/K on the boundary, $\eta = +1.0$

ρ		0.0	0.4	0.8	1.2	1.4	1.6	1.8	2.0
σ_z/K	Eq.(45)	-0.9977	-0.9984	-1.0026	0.0031	0.0018	0.0017	0.0021	0.0025
	$x=1.2$	-0.9995	-0.9999	-1.0019	0.0000	-0.0009	-0.0013	-0.0014	-0.0015
	1.4	-0.9972	-0.9979	-1.0015	0.0017	0.0000	-0.0008	-0.0010	-0.0012
	1.6	-0.9953	-0.9962	-1.0007	0.0031	0.0010	0.0000	-0.0004	-0.0006
	1.8	-0.9947	-0.9956	-1.0002	0.0036	0.0015	0.0004	0.0000	-0.0003
	2.0	-0.9943	-0.9951	-0.9997	0.0040	0.0019	0.0008	0.0004	0.0000
τ/K	Eq.(45)	0.0000	0.0001	-0.0008	-0.0021	-0.0025	-0.0026	-0.0024	-0.0021
	$x=1.2$	0.0000	0.0004	0.0003	0.0000	-0.0002	-0.0003	-0.0004	-0.0004
	1.4	0.0000	0.0007	0.0008	0.0003	0.0000	-0.0002	-0.0003	-0.0004
	1.6	0.0000	0.0010	0.0011	0.0005	0.0002	0.0000	-0.0001	-0.0002
	1.8	0.0000	0.0010	0.0012	0.0006	0.0003	0.0001	0.0000	-0.0001
	2.0	0.0000	0.0010	0.0012	0.0007	0.0004	0.0002	0.0001	0.0000

Table 7 Stresses σ_z/K and τ/K on the boundary, $\eta = -1.0$

ρ		0.0	0.4	0.8	1.2	1.4	1.6	1.8	2.0
σ_z/K	Eq.(45)	-0.0263	-0.0234	-0.0131	-0.0101	-0.0045	-0.0007	0.0016	0.0030
	$x=1.2$	0.0028	0.0031	0.0085	0.0000	0.0026	0.0033	0.0027	0.0019
	1.4	-0.0001	0.0002	0.0045	-0.0021	0.0000	0.0007	0.0004	-0.0002
	1.6	-0.0011	-0.0008	0.0030	-0.0026	-0.0007	0.0000	-0.0002	-0.0006
	1.8	-0.0017	-0.0013	0.0027	-0.0026	-0.0006	0.0001	0.0000	-0.0003
	2.0	-0.0023	-0.0018	0.0024	-0.0026	-0.0005	0.0004	0.0003	0.0000
τ/K	Eq.(45)	0.0000	0.0065	0.0105	0.0110	0.0101	0.0088	0.0072	0.0056
	$x=1.2$	0.0000	0.0006	0.0006	0.0000	-0.0004	-0.0009	-0.0012	-0.0014
	1.4	0.0000	0.0005	0.0007	0.0003	0.0000	-0.0003	-0.0005	-0.0006
	1.6	0.0000	0.0006	0.0008	0.0005	0.0002	0.0000	-0.0002	-0.0003
	1.8	0.0000	0.0007	0.0010	0.0008	0.0005	0.0002	0.0000	-0.0002
	2.0	0.0000	0.0009	0.0014	0.0011	0.0008	0.0005	0.0002	0.0000

Table 8 Errors of σ_z/K on the axis r

ρ	Approximate values	Numerical integration	Errors	Errors %
0.0	-0.6169	-0.6181	0.0012	-0.19
0.4	-0.6133	-0.6144	0.0011	-0.18
0.8	-0.5422	-0.5467	0.0045	-0.82
1.0	-0.4534	-0.4540	0.0006	-0.13
1.2	-0.3330	-0.3301	-0.0029	0.88
1.6	-0.1120	-0.1117	-0.0003	0.27
2.0	-0.0171	-0.0162	-0.0009	5.6

Fig. 3 Distribution of σ_z/K Fig. 4 Distribution of τ/K

ditions for $\eta=+1.0$ are fairly well fulfilled, but for $\eta=-1.0$, there exist noticeable errors. On the other hand, if we determine the values k_n by solving the simultaneous linear algebraic equations by changing the value x , the errors produced between σ_z/K , τ/K on the boundaries and the boundary conditions change with x . As the boundary condition for $\eta=-1.0$, $\rho=0.0$ is almost satisfied in the case of $x=1.4$, we shall adopt the values k_n in this case:

$$\begin{aligned} k_1 &= 0.43876, & k_2 &= 0.93611, \\ k_3 &= 0.55493, & k_4 &= 0.15625 \end{aligned} \quad (46)$$

In Tables 6 and 7, the errors of σ_z/K are more or less large for $\rho=0.8$ and 1.2, and those errors depend on the errors of $I_0(0, a, \gamma)$ represented by Eq.(34). In the case of $x=1.4$, if we remove the errors by using Table 2 and Eq.(46), we get the fol-

Fig. 5 Distribution of σ_r/K ($m=10/3$) Fig. 6 Distribution of σ_t/K ($m=10/3$)Table 9 Stresses σ_z/K and τ/K on the boundaries, $\eta=\pm 1.0$ ($b=2.0$)

ρ		0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8
$\frac{\sigma_z}{K}$	$\eta=+1.0$	-0.2746	-0.2729	-0.2683	-0.2616	-0.2545	-0.1257	0.0000	-0.0004
	$\eta=-1.0$	-0.0154	-0.0139	-0.0099	-0.0049	0.0012	0.0015	0.0000	0.0003
$\frac{\tau}{K}$	$\eta=+1.0$	0.0000	-0.0038	-0.0067	-0.0076	-0.0061	-0.0030	0.0000	0.0016
	$\eta=-1.0$	0.0000	0.0031	0.0050	0.0052	0.0038	0.0018	0.0000	-0.0010

lowing corrected values: -0.9992 for $\rho=0.8$ and -0.0004 for $\rho=1.2$ when $\eta=+1.0$, 0.0008 for $\rho=0.8$ and 0.0012 for $\rho=1.2$ when $\eta=-1.0$. Table 8 shows the approximate values of σ_z/K on the axis r in the case of using k_n in Eq.(46) in comparison with the values of numerical integration. In obtaining the values of numerical integration, we used the equation obtained by substituting Eq.(15) in Eq.(3), in which we put $\eta=0$ and $c/h=d/h=1.0$:

$$(\sigma_z/K)_{r=0} = \int_0^\infty \frac{\sinh \lambda + \lambda \cosh \lambda}{\sinh 2\lambda + 2\lambda} \times \left\{ J_1(\lambda) + \frac{\lambda}{2} J_0(\lambda) \right\} J_0(\rho\lambda) d\lambda \dots (47)$$

and we used Simpson's formula. Similar to Table 6 and Table 7, Table 8 shows that the errors caused by $I_0(0, a, \gamma)$ are more or less large for $\rho=0.8$ and 1.2. Removing these errors, the values of σ_z/K take the values -0.5459 for $\rho=0.8$ and -0.3297 for $\rho=1.2$, and errors between the values above-mentioned and those of numerical integration are 0.0008(-0.15%) and 0.0004(-0.12%) respectively.

Figures 3~6 show each stress component obtained by using Eqs. (24)'~(26)', Eqs.(41)~(44) and Eq.(46). In Eqs.(43) and (44) we use 10/3 as Poisson's number m .

There are two kinds of errors of numerical calculation caused by approximate equations: namely, errors of $\mu I_\nu(n, a, \gamma)$ and those which are introduced by the fact that the boundary conditions are not completely satisfied. Considering σ_z/K , we find that the former reaches its maximum value in the neighbourhood of $\rho=1.0$, due to $I_0(0, a, \gamma)$. For $\rho=0.8$, the error is about 0.004 and for $\rho=1.2$, the error is about 0.003. In the latter case, as shown in Table 6, the maximum value of errors is about 0.003 for $\rho=0$, when $\eta=+1.0$, and even for the inside of the boundary surfaces, there are some errors indeed, but they don't seem to exceed the above-mentioned maximum value.

Next, investigating the state where the boundary conditions are fulfilled in the case when $b=\gamma/h=c/h=d/h=2.0$, we obtain Table 9. In this case, we choose the point, $\rho=2.4$, at which the boundary conditions are completely satisfied. The exact boundary conditions are, as for σ_z/K , -0.25 for $\rho<2.0$, -0.125 for $\rho=2.0$ and zero for $\rho>2.0$ respectively when $\eta=+1.0$, and $-\infty$ for $\rho=2.0$ and zero for the other points respectively when $\eta=-1.0$; as for τ/K , zero for all ρ when $\eta=\pm 1.0$. The maximum value of errors is considerably large, about 0.025.

4. Conclusions

We made a stress analysis of an infinite elastic thick plate subjected to loads symmetrical to axis of revolution, which is a similar method to that we used for a thick plate subjected to loads symmetrical to axis of revolution and to middle plane as stated in previous paper.

Namely, after representing the part of hyperbolic function of the integrand of stress functions expressed in infinite integral in terms of exponential functions, we expanded the denominator by Maclaurin law and integrated by terms. In this way we obtained each stress component and showed that the case in the previous paper can be treated as a special case in this paper.

Next, utilizing the fact that each term integrated by terms becomes a stress function, and therefore, any linear combination of terms is also a stress function, and solving the simultaneous linear algebraic equations with four unknowns so as to satisfy the boundary conditions at two points on the boundary surfaces, we obtained an approximate solution with higher accuracy than that of the case in the previous paper. This method has such a merit as to save the calculation hours to a great extent compared with that of numerical integration but with inevitable errors more or less. As shown in Tables 2~5, $I_0(0, a, \gamma)$, $I_1(-1, a, \gamma)$ and $\rho I_1(0, a, \gamma)$ have errors more or less, and therefore, we need to re-examine them.

In the next place, as for $b=2.0$, we examined the state in which the boundary conditions were satisfied by using the same method as for $b=1.0$. Table 9 shows that the errors in this case are larger than those in the case $b=1.0$. We, therefore, need to consider some better methods. For instance, we can suggest the following methods. After solving the simultaneous linear algebraic equations with four unknowns so as to satisfy the boundary conditions at the points $\rho=1.0$, 2.0 and 3.0 respectively, we calculate the stress components partially. In this paper, as for the stress function F , we took up to $A_4 \sim A_6$ of Eq.(22)' and performed numerical calculations. We can calculate the stress components with higher accuracy if we take up to $A_4 \sim A_{10}$, for the conditions satisfying the boundary conditions increase in number.

We wish to thank Professor Hideo Miki for his advice in the preparation of this work.

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