Flexural Vibrations in an Infinite Thick Plate with a Circular Hole to Dynamical Loads at the Hole*

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This paper deals with a problem of flexural wave propagation in an infinite elastic plate with a circular hole subjected to dynamical loads at the hole. In the approach presented herein, the effects of rotatory inertia and thickness-shear in the plate are considered and the Fourier transform method is applied to this problem. As a numerical example, we treat a case when the motion of the load is sinusoidal. When the thickness of the plate is small compared with a wave length, the results obtained coincide with the ones by the classical plate theory. The differences in the results between improved plate theory and the classical one are also discussed.

1. Introduction

The two dimensional elastic wave propagation problems in an infinite elastic medium with a hole subjected to dynamical loads at the hole have been studied by many investigators^{(1)~(6)}. However the cases of the flexural wave propagation in a plate for such kind of problems have not been fully investigated yet, though such plates are widely used in machines and structures. Since it is very difficult to analyze these problems exactly by using the three dimensional theory of elasticity, the classical plate theory based on the treatments of Poisson and Kirchhoff has been used. But, when the plate is excited by high speed loads, we can not clarify the behaviors of such a plate enough by the classical plate theory because the effects of thickness-shear and rotatory inertia in a plate become large. However, the effects of the thickness of a plate in this kind of problems have not been examined yet.

This paper deals with the problems of the flexural wave propagation in an infinite elastic plate with a hole subjected to some dynamical loads at the hole by using the improved plate theory (thick plate theory) in which the effects of thickness-shear and rotatory inertia are considered⁽⁴⁾ and the dynamical behaviors of the plate are examined. The Fourier transform method which was developed in the two dimensional elastic wave problems by Eringen⁽²⁾ is used in this paper.

As a numerical example, we consider the case of a plate excited by sinusoidal forced motions, and the effects of circular frequency of the load and the ratio of radius of the hole to the thickness of the plate on the bending moment at the hole are discussed. The same problem is also calculated by

the classical plate theory and the results obtained are compared with those by the improved plate theory.

2. Equations of motion

An infinite elastic plate with a circular hole subjected to the dynamical shearing force, bending moment and twisting moment at the hole is shown in Fig. (1) in which the x, y axes are spaced on the middle plane of the plate and the axis normal to the plate surface. Let us choose the center O of the hole as the origin of the coordinates r, θ and z as in Fig. 1. The differential equations of motion for the plate in which the thickness-shear and rotatory inertia are considered are⁽⁴⁾

$$\left. \begin{aligned} \frac{\partial}{\partial x} \left\{ \nabla^2 \varphi - \left(R \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} + \frac{1}{S} \right) \varphi - \frac{1}{S} w \right\} + \frac{1-\nu}{2} \frac{\partial}{\partial y} \left\{ \nabla^2 - \frac{2}{1-\nu} \right. \\ \times \left(\frac{1}{S} + R \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} \right) \Big\} H = 0 \\ \frac{\partial}{\partial y} \left\{ \nabla^2 \varphi - \left(R \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} + \frac{1}{S} \right) \varphi - \frac{1}{S} w \right\} - \frac{1-\nu}{2} \frac{\partial}{\partial x} \left\{ \nabla^2 - \frac{2}{1-\nu} \right. \\ \times \left(\frac{1}{S} + R \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} \right) \Big\} H = 0 \\ \nabla^2 (\varphi + w) - S \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} w = 0 \end{aligned} \right\} \dots (1)$$

Fig. 1. Infinite plate with a hole subjected to dynamical loads at the hole.

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where

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad R = \frac{h^2}{12} \\ S &= \frac{D}{\kappa^2 G h}, \quad D = \frac{G h^3}{6(1-\nu)} \end{aligned}$$

and $w(r, \theta, t)$ is the displacement of the plate in the z direction, $\phi(r, \theta, t), H(r, \theta, t)$ are the wave potentials, h is the thickness of the plate, ν Poisson's ratio, ρ the mass density, D the flexural rigidity of the plate, G the shear modulus of elasticity, κ the shear coefficient and t the time,

If we assume $f(t)$ is a known original function, its Fourier transform is defined as

$$\bar{f}(\tau) = \int_{-\infty}^{\infty} f(t) e^{i\tau t} dt \quad (2)$$

where τ is the subsidiary variable. When this transform is applied to Eqs.(1) and $\bar{\phi}$ is eliminated by using new potentials $\eta_j (j=1, 2, 3)$ as

$$\eta_1 + \eta_2 = \bar{w}, \quad \eta_3 = \bar{H}$$

the transformed expressions of Eqs.(1) with respect to η_j can be obtained. If we write these with respect to polar coordinates r, θ , we have

$$(\nabla^2 + \alpha_j^2) \eta_j = 0 \quad j=1, 2, 3 \quad (3)$$

where

$$\left. \begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\ \alpha_1^2 &= \frac{\pi^2 \tau^2}{24 \kappa^2 \omega_0^2} \left[\frac{1}{S} + \frac{1}{R} \pm \left\{ \frac{1}{R^2} + \frac{1}{S^2} + \frac{2}{RS} \left(\frac{24 \kappa^2 \omega_0^2}{\pi^2 \tau^2} - 1 \right) \right\}^{1/2} \right] \\ \alpha_2^2 &= \frac{\pi^2}{h^2} \left(\frac{\tau^2}{\omega_0^2} - \frac{12 \kappa^2}{\pi^2} \right) \\ \omega_0 &= \frac{\pi c}{h}, \quad c = \sqrt{\frac{G}{\rho}} \end{aligned} \right\} \quad (4)$$

the transformed expressions of the bending moments M_r, M_θ , twisting moment $M_{r\theta}$, and shearing forces Q_r, Q_θ are denoted by using Eq.(2) as

$$\left. \begin{aligned} \bar{M}_r &= D \left\{ \frac{\partial \psi_r}{\partial r} + \frac{\nu}{r} \left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta} \right) \right\} \\ \bar{M}_\theta &= D \left\{ \frac{1}{r} \left(\frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right) + \nu \frac{\partial \psi_r}{\partial r} \right\} \\ \bar{M}_{r\theta} &= \frac{1}{2} (1-\nu) D \left\{ \frac{1}{r} \left(\frac{\partial \psi_r}{\partial \theta} - \psi_\theta \right) + \frac{\partial \psi_\theta}{\partial r} \right\} \\ \bar{Q}_r &= \kappa^2 G h \left(\psi_r + \frac{\partial \bar{w}}{\partial r} \right) \\ \bar{Q}_\theta &= \kappa^2 G h \left(\psi_\theta + \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} \right) \end{aligned} \right\} \quad (5)$$

where

$$\left. \begin{aligned} \psi_r &= (\sigma_1 - 1) \frac{\partial \eta_1}{\partial r} + (\sigma_2 - 1) \frac{\partial \eta_2}{\partial r} + \frac{1}{r} \frac{\partial \eta_3}{\partial \theta} \\ \psi_\theta &= \frac{(\sigma_1 - 1)}{r} \frac{\partial \eta_1}{\partial \theta} + \frac{(\sigma_2 - 1)}{r} \frac{\partial \eta_2}{\partial \theta} - \frac{\partial \eta_3}{\partial r} \\ \sigma_1 &= \frac{2}{1-\nu} \left(\frac{\alpha_2}{\alpha_3} \right)^2, \quad \sigma_2 = \frac{2}{1-\nu} \left(\frac{\alpha_1}{\alpha_3} \right)^2 \end{aligned} \right\} \quad (6)$$

we can find six wave solutions from Eq.(3). The solutions in this problem are obtained by selecting the waves which propagate into the plate from the hole and they are given by

$$\eta_j = \sum_{n=-\infty}^{\infty} A_{jn} H_n^{(1)}(\alpha_j r) e^{in\theta} \quad j=1, 2, 3 \quad (7)$$

in which $H_n^{(1)}(\alpha_j r)$ is a Hankel function of the first kind of order n and $A_{jn} (j=1, 2, 3)$ are the arbitrary constants to be determined from the boundary conditions. Now we consider the case of a plate excited by the dynamical bending moment $m_r = m_r(\theta, t)$, the shearing force $q_r = q_r(\theta, t)$, and the twisting moment $m_{r\theta} = m_{r\theta}(\theta, t)$ per unit length along the hole. These loads can be expanded into Fourier series and the boundary conditions at $r=a$ are expressed as follows:

$$\left. \begin{aligned} (\bar{M}_r)_{r=a} &= \sum_{n=-\infty}^{\infty} \bar{m}_r e^{in\theta} \\ (\bar{Q}_r)_{r=a} &= \sum_{n=-\infty}^{\infty} \bar{q}_r e^{in\theta} \\ (\bar{M}_{r\theta})_{r=a} &= \sum_{n=-\infty}^{\infty} \bar{m}_{r\theta} e^{in\theta} \end{aligned} \right\} \quad (8)$$

where

$$\left. \begin{aligned} \bar{m}_r &= \frac{1}{2\pi} \int_0^{2\pi} m_r(\theta, \tau) e^{-in\theta} d\theta \\ \bar{q}_r &= \frac{1}{2\pi} \int_0^{2\pi} q_r(\theta, \tau) e^{-in\theta} d\theta \\ \bar{m}_{r\theta} &= \frac{1}{2\pi} \int_0^{2\pi} m_{r\theta}(\theta, \tau) e^{-in\theta} d\theta \end{aligned} \right\} \quad (9)$$

If we substitute Eq.(7) into Eqs.(6) and (5) and use Eqs.(8), the unknown constants A_{jn} are obtained as

$$\left. \begin{aligned} A_{1n}(\tau) &= \left| \begin{array}{ccc} \frac{\bar{m}_r}{D\alpha_1^2} & E_{2n}^* & E_{3n}^* \\ \frac{\bar{q}_r}{\kappa^2 G h \alpha_1} & F_{2n}^* & F_{3n}^* \\ \frac{\bar{m}_{r\theta}}{(1-\nu) D \alpha_1^2} & G_{2n}^* & G_{3n}^* \end{array} \right| / \Delta_n \\ A_{2n}(\tau) &= \left| \begin{array}{ccc} E_{1n}^* & \frac{\bar{m}_r}{D\alpha_1^2} & E_{3n}^* \\ F_{1n}^* & \frac{\bar{q}_r}{\kappa^2 G h \alpha_1} & F_{3n}^* \\ G_{1n}^* & \frac{\bar{m}_{r\theta}}{(1-\nu) D \alpha_1^2} & G_{3n}^* \end{array} \right| / \Delta_n \\ A_{3n}(\tau) &= \left| \begin{array}{ccc} E_{1n}^* & E_{2n}^* & \frac{\bar{m}_r}{D\alpha_1^2} \\ F_{1n}^* & F_{2n}^* & \frac{\bar{q}_r}{\kappa^2 G h \alpha_1} \\ G_{1n}^* & G_{2n}^* & \frac{\bar{m}_{r\theta}}{(1-\nu) D \alpha_1^2} \end{array} \right| / \Delta_n \end{aligned} \right\}$$

$$A_n = \begin{pmatrix} E_{1n}^* & E_{2n}^* & E_{3n}^* \\ F_{1n}^* & F_{2n}^* & F_{3n}^* \\ G_{1n}^* & G_{2n}^* & G_{3n}^* \end{pmatrix} \quad \dots\dots\dots (10)$$

and

where

$$\left. \begin{aligned} E_{jn}^* &= \varepsilon_j(\tau)^2 (E_{jn})_{r=a} \\ F_{jn}^* &= \varepsilon_j(\tau) (F_{jn})_{r=a} \\ G_{jn}^* &= \varepsilon_j(\tau)^2 (G_{jn})_{r=a} \\ \varepsilon_j(\tau) &= \alpha_j(\tau) / \alpha_1(\tau) \end{aligned} \right\} \dots\dots\dots (11)$$

$$\left. \begin{aligned} E_{jn}(\tau) &= (\sigma_j - 1) \left\{ \frac{n(n+1)(1-\nu)}{\delta_j^2} - 1 \right\} H_n^{(1)}(\delta_j) - \frac{1-\nu}{\delta_j} H_{n-1}^{(1)}(\delta_j) \quad j=1, 2 \\ E_{3n}(\tau) &= in(1-\nu) \left\{ -\frac{(n+1)}{\delta_3^2} H_n^{(1)}(\delta_3) + \frac{1}{\delta_3} H_{n-1}^{(1)}(\delta_3) \right\} \\ F_{jn}(\tau) &= -\sigma_j \left\{ \frac{n}{\delta_j} H_n^{(1)}(\delta_j) - H_{n-1}^{(1)}(\delta_j) \right\} \quad j=1, 2, \quad F_{3n}(\tau) = \frac{in}{\delta_3} H_n^{(1)}(\delta_3) \\ G_{jn}(\tau) &= in(\sigma_j - 1) \left\{ -\frac{(n+1)}{\delta_j^2} H_n^{(1)}(\delta_j) + \frac{1}{\delta_j} H_{n-1}^{(1)}(\delta_j) \right\} \quad j=1, 2 \\ G_{3n}(\tau) &= \left\{ -\frac{n(n+1)}{\delta_3^2} + \frac{1}{2} \right\} H_n^{(1)}(\delta_3) + \frac{1}{\delta_3} H_{n-1}^{(1)}(\delta_3), \quad \delta_j = \alpha_j(\tau)r \end{aligned} \right\} \dots\dots\dots (12)$$

Substituting Eqs.(10) ~ (12) into Eqs.(7),(6),(5) and using Fourier inverse transform formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) e^{-i\tau t} d\tau \quad \dots\dots\dots (13)$$

The bending moment and shearing force quantities can be obtained as follows:

$$\left. \begin{aligned} M_r &= D \sum_{n=-\infty}^{\infty} \sum_{j=1}^3 \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_j(\tau)^2 E_{jn}(\tau) A_{jn}(\tau) e^{in\theta} e^{-i\tau t} d\tau \\ M_\theta &= D \sum_{n=-\infty}^{\infty} \sum_{j=1}^3 \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_j(\tau)^2 M_{jn}(\tau) A_{jn}(\tau) e^{in\theta} e^{-i\tau t} d\tau \\ Q_r &= \kappa^2 Gh \sum_{n=-\infty}^{\infty} \sum_{j=1}^3 \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_j(\tau) F_{jn}(\tau) A_{jn}(\tau) e^{in\theta} e^{-i\tau t} d\tau \\ Q_\theta &= \kappa^2 Gh \sum_{n=-\infty}^{\infty} \sum_{j=1}^3 \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_j(\tau) L_{jn}(\tau) A_{jn}(\tau) e^{in\theta} e^{-i\tau t} d\tau \end{aligned} \right\} \dots\dots\dots (14)$$

where

$$\left. \begin{aligned} M_{jn}(\tau) &= (\sigma_j - 1) \left\{ \frac{-n(n+1)(1-\nu)}{\delta_j^2} - \nu \right\} H_n^{(1)}(\delta_j) + \frac{1-\nu}{\delta_j} H_{n-1}^{(1)}(\delta_j) \quad j=1, 2 \\ M_{3n}(\tau) &= \frac{in(1-\nu)}{\delta_3} \left\{ \frac{(n+1)}{\delta_3} H_n^{(1)}(\delta_3) - H_{n-1}^{(1)}(\delta_3) \right\} \\ L_{jn}(\tau) &= \frac{\sigma_j}{\delta_j} in H_n^{(1)}(\delta_j) \quad j=1, 2, \quad L_{3n}(\tau) = \frac{n}{\delta_3} H_n^{(1)}(\delta_3) - H_{n-1}^{(1)}(\delta_3) \end{aligned} \right\} \dots\dots\dots (15)$$

3. Vibrations of an infinite plate subjected to some dynamical shearing forces at the hole

3.1. Distributed loads

Let us consider the case of a dynamical shearing force $q_r(\theta, t)$ which is distributed at N sections of the hole with the intensity $q_m(t)$ ($m=1, 2, \dots, N$) as shown in Fig.2. We bring the end of the first load q_1 to the x axis and take other loads $q_2, q_3, \dots, q_N$ counter clockwise along the hole as in the figure. The angle between x axis and a line B_2O is denoted by γ_1 and the angles between x axis and A_2O , x axis and B_2O are denoted by β_1 , γ_2 respectively. Other angles γ_m , β_m are taken in the same way. Then $q_r(\theta, t)$ can be expressed as

$$q_r(\theta, t) = \begin{cases} q_m(t) & \beta_{m-1} < \theta < \gamma_m \\ 0 & \gamma_m < \theta < \beta_m \end{cases} \quad m=1, 2, \dots, N \quad \dots\dots\dots (16)$$

In Eqs.(16), β_0 is zero and β_N is 2π . Since we consider that the load acting at

Fig.2. Infinite plate subjected to dynamical shearing forces along the hole

the hole is the shearing force only, thus

$$\tilde{m}_r = \tilde{m}_{r,\theta} = 0$$

Substituting Eqs.(16) into Eqs.(9), we have

$$\tilde{q}_r = \begin{cases} \frac{1}{2\pi} \sum_{n=1}^N \tilde{q}_n(\tau) (\gamma_n - \beta_{n-1}) & n=0 \\ -\frac{1}{2\pi i n} \sum_{n=1}^N \tilde{q}_n(\tau) (e^{-i n \gamma_n} - e^{-i n \beta_{n-1}}) & n \neq 0 \end{cases}$$

$$\tilde{m}_r = \tilde{m}_{r,\theta} = 0 \quad \dots\dots\dots(17)$$

The bending moment and shearing forces are obtained from Eqs.(17),(10) and (14). When the intensities of the loads are the same and phase differences do not exist among these loads, we can write

$$q_m(t) = q_0(t)$$

Then the bending moment and shearing forces are represented in the form

$$\left. \begin{aligned} M_r &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{q_0(\tau)}{\alpha_1(\tau)} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \alpha_j^*(\tau)^2 E_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ M_\theta &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{q_0(\tau)}{\alpha_1(\tau)} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \alpha_j^*(\tau)^2 M_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ Q_r &= \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(\tau) \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \varepsilon_j(\tau) F_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ Q_\theta &= \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(\tau) \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \varepsilon_j(\tau) L_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \end{aligned} \right\} \dots\dots\dots(18)$$

where

$$\left. \begin{aligned} C_{10} &= \frac{1}{2\pi} \sum_{n=1}^N (\gamma_n - \beta_{n-1}), \quad C_{1n} = \frac{-1}{2\pi i n} \sum_{n=1}^N (e^{-i n \gamma_n} - e^{-i n \beta_{n-1}}) \quad n \neq 0 \\ \alpha_j^* &= \alpha_j h, \quad A_{1n}^* = (E_{1n}^* G_{1n}^* - E_{1n}^* G_{1n}^*) / \Delta_n \\ A_{2n}^* &= (E_{1n}^* G_{1n}^* - E_{1n}^* G_{1n}^*) / \Delta_n, \quad A_{3n}^* = (E_{2n}^* G_{1n}^* - E_{1n}^* G_{2n}^*) / \Delta_n \end{aligned} \right\} \dots\dots\dots(19)$$

3.2. Concentrated loads

We consider the case of the concentrated loads $P_m (m=1,2,\dots,N)$ acting at the hole, values of which are the same. If we assume that there is no phase differences among these loads, the load can be given by

$$P_m(t) = P_0(t)$$

and

$$\tilde{P}_0(\tau) = \lim_{\gamma_n \rightarrow \beta_{n-1}} q_0(\tau) (\gamma_n - \beta_{n-1}) a \quad \dots\dots\dots(20)$$

If we use Eqs.(17) and Eq.(20), we have

$$\tilde{q}_r = \frac{\tilde{P}_0(\tau)}{a} C_{1n} \quad \dots\dots\dots(21)$$

where

$$C_{1n} = \sum_{n=1}^N \frac{1}{2\pi} e^{-i n \beta_{n-1}} \quad \dots\dots\dots(22)$$

The bending moment and shearing force quantities are calculated in a similar way to the case of distributed loads, and they are given as

$$\left. \begin{aligned} M_r &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{\tilde{P}_0(\tau)}{\alpha_1(\tau)a} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \alpha_j^*(\tau)^2 E_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ M_\theta &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{\tilde{P}_0(\tau)}{\alpha_1(\tau)a} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \alpha_j^*(\tau)^2 M_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ Q_r &= \frac{1}{2\pi a} \int_{-\infty}^{\infty} \tilde{P}_0(\tau) \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \varepsilon_j(\tau) F_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \\ Q_\theta &= \frac{1}{2\pi a} \int_{-\infty}^{\infty} \tilde{P}_0(\tau) \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^3 \varepsilon_j(\tau) L_{jn}(\tau) A_{jn}^*(\tau) e^{i n \theta} e^{-i \tau} d\tau \end{aligned} \right\} \dots\dots\dots(23)$$

3.3. Axially symmetric load

In the previous analysis, the nonaxially symmetric cases were treated. The results for the case of the axially symmetric load can be obtained from the previous results. In the case of the axially symmetric load, the deformation of the plate is independent of the angle θ , and we can find the solution by taking the terms of $n=0$ only in Eqs.(18) and (19). The bending moments and shearing forces in this case are

$$\left. \begin{aligned} M_r &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{\bar{q}_0(\tau)}{\alpha_1(\tau)} \sum_{j=1}^2 \alpha_j^*(\tau)^2 E_{j0}(\tau) A_{j0}^*(\tau) e^{-i\tau} d\tau \\ M_\theta &= \frac{1}{12\pi(1-\nu)\kappa^2} \int_{-\infty}^{\infty} \frac{\bar{q}_0(\tau)}{\alpha_1(\tau)} \sum_{j=1}^2 \alpha_j^*(\tau)^2 M_{j0}(\tau) A_{j0}^*(\tau) e^{-i\tau} d\tau \\ Q_r &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{q}_0(\tau) \sum_{j=1}^2 \epsilon_j(\tau) F_{j0}(\tau) A_{j0}^*(\tau) e^{-i\tau} d\tau \end{aligned} \right\} \dots\dots\dots (24)$$

where

$$\left. \begin{aligned} A_{10}^*(\tau) &= -\epsilon_2^2(\sigma_2-1) \left\{ \frac{(1-\nu)}{\delta_2^*} H_1^{(1)}(\delta_2^*) - H_0^{(1)}(\delta_2^*) \right\} / \Delta_0 \\ A_{20}^*(\tau) &= (\sigma_1-1) \left\{ \frac{(1-\nu)}{\delta_1^*} H_1^{(1)}(\delta_1^*) - H_0^{(1)}(\delta_1^*) \right\} / \Delta_0 \\ \Delta_0(\tau) &= \epsilon_2[(1-\nu)(\sigma_2-\sigma_1)H_1^{(1)}(\delta_1^*)H_1^{(1)}(\delta_2^*)/\delta_1^* \\ &\quad + \sigma_2(\sigma_1-1)H_0^{(1)}(\delta_1^*)H_1^{(1)}(\delta_2^*) - \epsilon_2\sigma_1(\sigma_2-1)H_0^{(1)}(\delta_2^*)H_1^{(1)}(\delta_1^*)] \\ E_{j0}(\tau) &= (\sigma_j-1) \left\{ \frac{(1-\nu)}{\delta_j} H_1^{(1)}(\delta_j) - H_0^{(1)}(\delta_j) \right\} \\ M_{j0}(\tau) &= -(\sigma_j-1) \left\{ \nu H_0^{(1)}(\delta_j) + \frac{(1-\nu)}{\delta_j} H_1^{(1)}(\delta_j) \right\}, \quad F_{j0}(\tau) = -\sigma_j H_1^{(1)}(\delta_j), \quad \delta_j^* = \alpha_j a \end{aligned} \right\} \dots\dots\dots (25)$$

4. Numerical examples

In numerical examples, we assume that the motion of the loads is sinusoidal. The cases of distributed loads, concentrated loads and axially symmetric load are calculated.

4.1. Distributed sinusoidal loads

We consider a plate subjected to sinusoidal loads at the hole as shown in Fig.2, the frequencies and amplitude of which are the same. The load $q_0(t)$ at the hole is

$$q_0(t) = p_0 e^{-i\omega t} \dots\dots\dots (26)$$

and the Fourier transform of Eq.(26) is

$$\bar{q}_0(\tau) = 2\pi p_0 \delta(\tau - \omega) \dots\dots\dots (27)$$

in which p_0 is the amplitude, ω the circular frequency of the load and $\delta(\tau - \omega)$ Dirac's delta function. Substituting Eq.(27) into Eqs.(18) and taking inverse transform, the bending moment and shearing force quantities can be obtained in the dimensionless forms as

$$\left. \begin{aligned} \frac{M_r}{ap_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^2 \alpha_j^*(\omega)^2 E_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{M_\theta}{ap_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^2 \alpha_j^*(\omega)^2 M_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{Q_r}{p_0} &= \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^2 \epsilon_j(\omega) F_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{Q_\theta}{p_0} &= \sum_{n=-\infty}^{\infty} C_{1n} \sum_{j=1}^2 \epsilon_j(\omega) L_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \end{aligned} \right\} \dots\dots\dots (28)$$

4.2. Concentrated sinusoidal loads

When a plate is excited by the concentrated sinusoidal loads of the same amplitude, the load P_0 at the hole is expressed in the form

$$P_0(t) = Q_0 e^{-i\omega t}$$

thus the Fourier transform of $P_0(t)$ is

$$\bar{P}_0(\tau) = 2\pi Q_0 \delta(\tau - \omega) \dots\dots\dots (29)$$

From Eqs.(23) and Eq.(29), we have

$$\left. \begin{aligned} \frac{M_r}{Q_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{n=-\infty}^{\infty} C_{2n} \sum_{j=1}^3 \alpha_j^*(\omega)^2 E_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{M_\theta}{Q_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{n=-\infty}^{\infty} C_{2n} \sum_{j=1}^3 \alpha_j^*(\omega)^2 M_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{aQ_r}{Q_0} &= \sum_{n=-\infty}^{\infty} C_{2n} \sum_{j=1}^3 \varepsilon_j(\omega) F_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \\ \frac{aQ_\theta}{Q_0} &= \sum_{n=-\infty}^{\infty} C_{2n} \sum_{j=1}^3 \varepsilon_j(\omega) L_{jn}(\omega) A_{jn}^*(\omega) e^{in\theta} e^{-i\omega t} \end{aligned} \right\} \dots\dots\dots(30)$$

4.3. Axially symmetric sinusoidal load

We consider that a sinusoidal load acts on the whole range of a hole and its intensity per unit length along the hole is shown in Eq.(26). Substituting Eq.(27) into Eqs.(24) and taking inverse transform, we can obtain

$$\left. \begin{aligned} \frac{M_r}{ap_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{j=1}^2 \alpha_j^*(\omega)^2 E_{j0}(\omega) A_{j0}^*(\omega) e^{-i\omega t} \\ \frac{M_\theta}{ap_0} &= \frac{1}{6(1-\nu)\kappa^2} \frac{1}{\alpha_1(\omega)a} \sum_{j=1}^2 \alpha_j^*(\omega)^2 M_{j0}(\omega) A_{j0}^*(\omega) e^{-i\omega t} \\ \frac{Q_r}{p_0} &= \sum_{j=1}^2 \varepsilon_j(\omega) F_{j0}(\omega) A_{j0}^*(\omega) e^{-i\omega t} \end{aligned} \right\} \dots\dots\dots(31)$$

When the wave length and the radius of the hole are large compared with the thickness of the plate, the effects of thickness-shear and rotatory inertia in the plate can be neglected. Putting R and S as zero which give the effects of thickness-shear and rotatory inertia in Eqs.(28), (30) and (31), we have the result of classical plate theory (thin plate theory). The wave numbers in this case are

$$\alpha_{1,2}^2 = \pm \omega \left(\frac{\rho h}{D} \right)^{1/2}, \quad \alpha_3^2 = \frac{-12\kappa^2}{h^2} \dots\dots\dots(32)$$

Substituting Eqs.(32) into Eqs.(31) and putting $\alpha_j \rightarrow 0$, the result of a thin plate subjected to an axially symmetric statical load at the hole is obtained. If we calculate the bending moment and shearing force in this way, the bending moment M_θ becomes infinite and the shearing force Q_r has the values

$$Q_r = \frac{ap_0}{r} \dots\dots\dots(33)$$

The case of a simply supported thin circular plate with a hole subjected to the axially symmetric statical load has been discussed already. The result in this paper coincides with the one in reference (5) if only the effects of outer boundary conditions are removed.

Now we show the results of numerical calculations and discuss the dynamical behaviors of the plate in the previous three cases 4.1, 4.2 and 4.3 by taking $\nu=0.3$, $\kappa=\pi^2/12$ (4). The same case is calculated by the classical plate theory and the differences between the results of improved plate theory and classical one are examined. Fig.3 shows

Fig.3. Relations between bending moments at point A ($\theta=15^\circ$) and frequencies in the case of one distributed load acting along the hole ($\gamma_1=\pi/6, \beta_1=2\pi, \nu=0.3$)

Fig.4. Relations between bending moments at point A ($\theta=15^\circ$) and frequencies in the case of three distributed loads acting along the hole with same intervals ($\gamma_1=\pi/6, \gamma_2=5\pi/6, \gamma_3=3\pi/2, \beta_1=2\pi/3, \beta_2=4\pi/3, \beta_3=2\pi, \nu=0.3$)

the maximum bending moment $|M_\theta/a\rho_0|_{r=a}$ in one cycle of Vibrations of the plate at the point A($\theta=15^\circ$) versus ω/ω_0 for the case of one distributed sinusoidal load acting from 0 to 30 degrees along the hole, which is calculated from Eqs.(28). The case of three distributed loads acting along the hole with same intervals is shown in Fig.4. In the Figures, the solid line shows the values for the case of improved plate theory and the dashed one for the classical plate theory. Fig.5 illustrates the relations between $M_\theta/a\rho_0|_{r=a}$ and θ for the case of one distributed load. From Figs.3 and 4, it is noted that, when a/h is large and ω/ω_0 is small, the differences between the result by the improved plate theory and that by the classical one are small. Fig.6 shows the relations between $|M_\theta/Q_0|_{r=a}$ and θ for the case of one concentrated sinusoidal load acting at the point $\theta=0$, which are obtained from Eqs.(30). The bending moment $|M_\theta/a\rho_0|_{r=a}$ versus ω/ω_0 for the axially symmetric load is calculated and shown in Fig.7 by using Eqs.(31). In the figure, the dashed line

denotes the result by the classical plate theory. comparing the both results of improved theory and classical one, the above discussion is also applicable to this case. From Figs.3~7, we can find that the difference between the result of improved plate theory and classical one is small in general for the case of axially symmetric load compared with that for others. The important factor in this aspect is that the effects of in-plane twisting will vanish in the axially symmetric problems⁽⁶⁾. from these figures, it is seen that the classical plate theory may be used in this kind of problem when a/h is above 3, ω/ω_0 less than 1 for the axially symmetric problem and a/h is above 7, ω/ω_0 less than 1 for the other ones. Fig.8 shows the relations between $|M_\theta/a\rho_0|$

Fig.5. Relations between bending moments at the hole and θ in the case of one distributed load acting along the hole ($\gamma_1=\pi/6, \beta_1=2\pi, \nu=0.3$)

Fig.6. Relations between bending moments at the hole and θ in the case of one concentrated load acting at the hole ($\beta_0=0, \beta_1=2\pi, \nu=0.3$)

Fig.7. Relations between bending moments at the hole and frequencies in the case of axially symmetric load acting along the hole ($\nu=0.3$)

Fig.8. Relations between bending moments and the distances from the hole in the case of axially symmetric load acting along the hole ($\omega/\omega_0=0.1, \nu=0.3$)

and r/a for the case of axially symmetric load by taking $\omega/\omega_0=0.1$. The values at $\omega/\omega_0=0$ in Figs.3,4 and 7 denote the results of the plate subjected to statical loads. We can see that the bending moment M_θ in the plate under the statical load becomes infinity from these figures.

5. Conclusions

In this paper, we investigated the dynamical behaviors of a plate with a hole subjected to dynamical loads at the hole by using the improved plate theory in which the effects of thickness-shear and rotatory inertia are considered. The expressions of the bending moments and shearing forces in the plate subjected to dynamical loads at the hole were obtained. As numerical examples, three cases of the plate excited by the sinusoidal distributed loads, concentrated loads and axially symmetric load were discussed, and the effects of the frequencies of the loads and the ratio of radius of the hole to the thickness of the plate on the

bending moment were examined. The result by the classical plate theory was also obtained and the differences in the results between improved plate theory and classical one were clarified.

The numerical results were obtained by using NEAC-2200 Model 500 Computer of the Tohoku University Computing Center.

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