Approximate Synthesis of Plane Four-Bar Mechanism for Function Generation*

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By extending the five point approximation and the fifth order approximation presented by F. Freudenstein, an analytical method is established for the synthesis of a plane four-bar mechanism to generate approximately a desired function over a finite interval. This method may be regarded as a general method in the synthesis where the input and output scale factors are specified in advance.

Moreover, in order to reduce the structural error of the function generator designed by using this method, a new numerical procedure for respacing precision points is suggested on the basis of the tenth degree polynomial closely approximating the structural error.

1. Introduction

In recent times a number of analytical methods for the synthesis of plane mechanisms have been discussed in the literature. Freudenstein's method^{(1),(2)} (the five point approximation and the fifth order approximation) is a powerful procedure for designing a plane four-bar mechanism such that the functional relationship between the input and output link rotations may be sufficiently close approximations to a desired function. Suh-Radcliffe's method⁽³⁾ is a useful technique for the same purpose. However, the numerical procedure for determining the design parameters consists in solving a system of nonlinear equations or a system of six linear equations with complex coefficients and a cubic equation. Consequently their actual computer program involves lengthy and complicated calculations.

In this report, an analytical method is presented for the synthesis of a plane four-bar mechanism to generate approximately a desired function over a finite interval. By using this synthesis method, the design parameters are determined such that the differential coefficients $g_i^{(k)}$ ($k = 0 \sim m_i$) of the generated function at precision points χ_i ($i = 1 \sim n$) may coincide with the ones of the desired function. Where $1 \leq n \leq 5$, and m_i ($i = 1 \sim n$) is zero or a positive integer such that $\sum_{i=1}^n m_i = 5 - n$. Therefore it may be regarded as a general method in the synthesis where two scale factors are specified in advance. The numerical procedure for determining the five design parameters is established in such a concise form as consists in solving two systems of five linear equations and a cubic equation. The five point approximation and the fifth order approximation may be derived

from the synthesis method in this report as its special cases.

Moreover, in order to reduce the structural error of the function generator designed by using this method, a new numerical procedure for respacing precision points is suggested on the basis of the tenth degree polynomial closely approximating the structural error.

2. Basic equation

Consider a rectangular co-ordinate system $O-XY$ the directions of whose X and Y axes are denoted by the unit vectors i and j respectively. Arrange a plane four-bar mechanism having the four revolute pairs whose center points are A, B, C, D such that A may coincide with the origin of the system $O-XY$ and the fixed link DA may lie on the X axis (refer to Fig.1). Let the fixed link DA , the driving link AB , the coupler link BC , and the driven link CD be denoted by the vectors $\overrightarrow{DA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{DC}$ respectively.

Here, we have

$$\left. \begin{aligned} \overrightarrow{DA} &= Z_a i \\ \overrightarrow{AB} &= Z_b \cos \phi i + Z_b \sin \phi j \\ \overrightarrow{BC} &= Z_c \cos \theta i + Z_c \sin \theta j \\ \overrightarrow{DC} &= Z_d \cos \psi i + Z_d \sin \psi j \end{aligned} \right\} \dots \dots \dots (1)$$

The positive real numbers Z_a, Z_b, Z_c, Z_d indicate the distance between two center points of adjacent revolute

Fig.1 Vector representation of the plane four-bar mechanism.

* Received 4th May, 1972.

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pairs D, A; A, B; B, C; C, D respectively. They are referred to as the link lengths of the four-bar mechanism. The input angle ϕ , the angle θ subtended by the coupler link and the fixed link, and the output angle ψ are defined by counterclockwise angles measured from the X axis to vectors $\overline{AB}$, $\overline{BC}$, and $\overline{DC}$ respectively.

Because two adjacent link of four-bar mechanism are connected by the revolute pair and the four links construct a closed loop, the following equation holds

$$\overline{DA} + \overline{AB} + \overline{BC} = \overline{DC} \quad (2)$$

Rearranging the equation obtained by substituting Eq.(1) into Eq.(2) as follows

$$(Z_4 + Z_1 \cos \phi - Z_2 \cos \psi) i + (Z_1 \sin \phi - Z_2 \sin \psi) j = -Z_3 \cos \theta i - Z_3 \sin \theta j$$

and squaring both sides of the above equation, we have the following equation which expresses the functional relationship between the input angle ϕ and the output angle ψ :

$$2Z_1 Z_3 \cos \phi - 2Z_1 Z_2 \cos \psi - 2Z_2 Z_3 \cos (\phi - \psi) + Z_1^2 + Z_2^2 - Z_3^2 = 0 \quad (3)$$

Equation (3) may be solved for the purpose of a displacement analysis of the four-bar mechanism⁽⁴⁾, that is, ψ is found explicitly as a function of ϕ in the form $\psi = G_l(\phi)$ ($l = 1$ or 2):

$$G_1(\phi) = 2 \tan^{-1} \left\{ \frac{\sin \phi + T_1 \sqrt{T_2 - (T_3 + \cos \phi)^2}}{T_4 + T_3 \cos \phi} \right\} \quad (4.1)$$

$$G_2(\phi) = 2 \tan^{-1} \left\{ \frac{\sin \phi - T_1 \sqrt{T_2 - (T_3 + \cos \phi)^2}}{T_4 + T_3 \cos \phi} \right\} \quad (4.2)$$

Where

$$T_1 = \frac{Z_2}{Z_4}, \quad T_2 = \left(\frac{Z_1 Z_2}{Z_4 Z_3} \right)^2, \quad T_3 = \frac{Z_2^2 + Z_1^2 - Z_3^2 - Z_4^2}{2Z_1 Z_3},$$

$$T_4 = \frac{(Z_2 + Z_4)^2 + Z_1^2 - Z_3^2}{2Z_1 Z_4}, \quad T_5 = 1 + \frac{Z_2}{Z_4}$$

The output angle of such a mechanism is determined from Eq.(4.1) if in this mechanism the center point C lies on the opposite side to the center point A with respect to the straight line BD when ϕ belongs to the interval $[0^\circ, 180^\circ]$, or C lies on the same side as A with respect to BD when ϕ belongs to $[180^\circ, 360^\circ]$. The output angle of such a mechanism is determined from Eq.(4.2) if in this mechanism C lies on the same side as A with respect to BD when ϕ belongs to $[0^\circ, 180^\circ]$, or C lies on the opposite side to A with respect to BD when ϕ belongs to $[180^\circ, 360^\circ]$.

Choosing the length of the fixed

link as a unit length and dividing both sides of Eq.(3) by $2Z_1 Z_3$ yields

$$R_1 \cos \phi - R_2 \cos \psi + R_3 = \cos (\phi - \psi) \quad (5)$$

Where

$$R_1 = \frac{1}{Z_4} \quad (6.1)$$

$$R_2 = \frac{1}{Z_3} \quad (6.2)$$

$$R_3 = \frac{1 + Z_1^2 - Z_2^2 + Z_4^2}{2Z_1 Z_4} \quad (6.3)$$

Taking the m th order derivative of Eq.(5) with respect to ϕ , we have the basic equation for the analytical synthesis method of the plane four-bar mechanism as follows

$$R_1(d_1 \cos \phi + d_2 \sin \phi) + R_2(d_3 \cos \psi + d_4 \sin \psi) + R_3 d_5 = d_6 \cos (\phi - \psi) + d_7 \sin (\phi - \psi) \quad (7.m)$$

Where $d_1, d_2, \dots, d_7$ are the expressions summarized at the $m+1$ th row in Table 1.

3. Synthesis equations

Consider the problem of generating approximately the desired function

$$y = f(x), \quad x_1 \leq x \leq x_f \quad (8)$$

by means of the functional relationship between the input and output angles.

By using the two variables χ , ψ corresponding to the input and output angles ϕ , ψ according to the equation

$$\left. \begin{aligned} \phi &= \phi_p + r_\phi(x - x_p) \\ \psi &= \psi_p + r_\psi(y - y_p) \end{aligned} \right\} \quad (9)$$

the functional relationship between the input and output variables of the four-bar mechanism is expressed in the form:

$$g_l(x) = \frac{G_l(r_\phi x - r_\psi y_p + \phi_p; Z_1, Z_2, Z_3, Z_4) - \psi_p}{r_\psi} + y_p \quad (l=1 \text{ or } 2) \quad (10)$$

Where r_ϕ is the input scale factor, r_ψ is the output scale factor, and (ϕ_p, ψ_p) are the input and output angles corresponding to the values (x_p, y_p) of the desired function.

The function to satisfy the specified condition, $g_1(x)$ or $g_2(x)$, is hereinafter referred to as the generated function $g(x)$.

In this chapter, we secure synthesis equations for determining design parameters such that the differential coefficients $g_i^{(k)}$ ($k = 0 \sim m_i$) of the generated function at the precision point χ_i may coincide with the ones of the desired function, where m_i is zero or a positive integer.

Table 2 Expression of d_j ($j = 1 \sim 7$) in the basic equation (7.m) in the case where m takes 0, 1, ..., or 4.

	d_1	d_2	d_3	d_4	d_5	d_6	d_7
m	0	1	0	-1	0	1	0
1	0	1	0	$-\psi^{(1)}$	0	0	$1 - \psi^{(1)}$
2	1	0	$-(\psi^{(1)})^2$	$-\psi^{(2)}$	0	$(1 - \psi^{(1)})^2$	$-\psi^{(2)}$
3	0	1	$3\psi^{(1)}\psi^{(2)}$	$-(\psi^{(1)})^3 + \psi^{(3)}$	0	$3(1 - \psi^{(1)})\psi^{(2)}$	$(1 - \psi^{(1)})^3 + \psi^{(3)}$
4	1	0	$-(\psi^{(1)})^4 + 4(\psi^{(1)}\psi^{(2)})^2 + 3(\psi^{(2)})^2$	$-6(\psi^{(1)})^2\psi^{(2)} + \psi^{(4)}$	0	$(1 - \psi^{(1)})^4 + 4(1 - \psi^{(1)})\psi^{(2)} - 3(\psi^{(2)})^2$	$-6(1 - \psi^{(1)})^2\psi^{(2)} + \psi^{(4)}$

Let the values (χ_p, y_p) coincide with (χ_i, y_i) , so that the input and output angles (ϕ_i, ψ_i) corresponding to (χ_i, y_i) are expressed as follows

$$\phi_i = \phi_1 + r_\phi x_{i1}; \quad x_{i1} = x_i - x_1 \quad (11)$$

$$\psi_i = \psi_1 + r_\psi y_{i1}; \quad y_{i1} = y_i - y_1 \quad (12.0)$$

Moreover, from Eq.(9), we have

$$\frac{d\psi}{d\phi} = \frac{d\psi}{dy} \frac{dy}{dx} \frac{dx}{d\phi} = \frac{r_\psi}{r_\phi} y^{(1)}$$

$$\frac{d^2\psi}{d\phi^2} = \frac{d^2\psi}{dy^2} \frac{dy^{(1)}}{dx} \frac{dx}{d\phi} = \frac{r_\psi}{r_\phi^2} y^{(2)}$$

.....

$$\frac{d^m\psi}{d\phi^m} = \frac{d^m\psi}{dy^m} \frac{dy^{(m-1)}}{dx} \frac{dx}{d\phi} = \frac{r_\psi}{r_\phi^m} y^{(m)}$$

Consequently, the k th order differential coefficient of ψ with respect to ϕ at the precision angle ϕ_i corresponding to the one of $f(\chi)$ with respect to χ at the precision point is expressed as follows

$$\psi_i^{(k)} = \frac{r_\psi}{r_\phi^k} y_i^{(k)}, \quad (k=1 \sim m_i) \quad (12.k)$$

Substituting the values $(\phi_i, \psi_i^{(k)})$ into Eqs.(7.0)~(7. m_i) yields the following equations for determining design parameters such that $g_i^{(k)}$ ($k=0 \sim m_i$) may coincide with $f_i^{(k)}$ ($k=0 \sim m_i$) respectively

$$\begin{aligned} R_1(d_{m1} \cos(\phi_1 + r_\phi x_{i1}) + d_{m2} \sin(\phi_1 + r_\phi x_{i1})) \\ + R_2(d_{m3} \cos(\psi_1 + r_\psi y_{i1}) + d_{m4} \sin(\psi_1 + r_\psi y_{i1})) \\ + R_3 d_{m5} = d_{m5} \cos(\phi_1 + r_\phi x_{i1} - \psi_1 - r_\psi y_{i1}) \\ + d_{m7} \sin(\phi_1 + r_\phi x_{i1} - \psi_1 - r_\psi y_{i1}) \\ (m=0 \sim m_i) \quad (13.m) \end{aligned}$$

Where d_{mj} ($j=1 \sim 7$) are the values of d_j ($j=1 \sim 7$) at the $m+1$ th row in Table 1 when $\psi^{(k)}$ ($k=1 \sim m_i$) take values $r_\psi f_i^{(k)}/r_\phi^k$ ($k=1 \sim m_i$).

Since Eqs.(13. m) ($m=0 \sim m_i$) contain seven design parameters $R_1, R_2, R_3, \phi_1, \psi_1, r_\phi, r_\psi$, a synthesis method is theoretically possible such that the generated function satisfies seven specified conditions, that is, $g_i^{(k)}$ ($k=0 \sim m_i, i=1 \sim n$) may coincide with $f_i^{(k)}$ ($k=0 \sim m_i, i=1 \sim n$) respectively. Where $1 \leq n \leq 7$, and m_i ($i=1 \sim n$) is zero or a positive integer such that $\sum_{i=1}^n m_i = 7 - n$. However it is impractical to solve Eqs.(13) for $R_1, R_2, R_3, \phi_1, \psi_1, r_\phi, r_\psi$ which is a system of simultaneous nonlinear equation with seven unknowns.

In the succeeding chapter, we investigate the synthesis method in the case where two scale factors r_ϕ, r_ψ are specified in advance.

4. General precision derivative approximation

Specify two scale factors r_ϕ, r_ψ and space precision points χ_i ($i=1 \sim n, 1 \leq n \leq 5$) such that

$$x_1 < x_2 < \dots < x_n < x_f \quad (14)$$

Moreover give greatest numbers of order m_i ($i=1 \sim n$) such that

$$\sum_{i=1}^n m_i = 5 - n, \quad (1 \leq n \leq 5) \quad (15)$$

In this case, we present an analytical method for the synthesis of plane four-bar mechanisms such that the differential coefficients $g_i^{(k)}$ ($k=0 \sim m_i$) of the generated function at precision points χ_i ($i=1 \sim n$) may coincide with the ones of the desired function. Then numerical procedure for determining the five design parameters $Z_b, Z_c, Z_d, \phi_1, \psi_1$ is established in such a concise form as consists in solving two system of five linear equations and a cubic equation.

Since Eq.(13. m) contains the higher order differential coefficients from 0th to m th at the precision point χ_i , the synthesis equations (13. m) ($m=0 \sim m_i$) are considered a set of m_i+1 equations which specify the conditions for making the 0th order, the 1th order, ..., and the m_i th order differential coefficients of the generated function at m_i+1 points $\chi_i^*, \chi_{i+1}^*, \dots$, and $\chi_{i+m_i}^*$ which are equal to χ_i , coincide with the ones of the desired function, respectively. Taking note of this fact, we can establish the numerical procedure for determining the design parameters in a general form by using (χ_i^*, m_i^*) ($i=1 \sim 5$) decided from (χ_i, m_i) ($i=1 \sim n$) according to the flow chart in Fig.2.

The number of sets which consist of greatest numbers of order m_i ($i=1 \sim n$) satisfying Eq.(15) is sixteen as shown in Table 2, so that there are sixteen sets of m_i^* ($i=1 \sim 5$). Therefore it is seen that the synthesis method in this chapter is composed of the five point approximation and the fifth order approximation which are the case [1] and the case [16] in Table 2 respectively and the other fourteen methods.

Put

$$\phi_{i1} = r_\phi(x_i^* - x_1^*), \quad \psi_{i1} = r_\psi(y_i - y_1)$$

and let d_{ij} ($j=1 \sim 7$) be values of d_j ($j=1 \sim 7$) at the m_i^*+1 th row in Table 1 when $\psi^{(k)}$ ($k=1 \sim m_i^*$) take values $r_\psi f_i^{(k)}/r_\phi^k$ ($k=1 \sim m_i^*$) respectively. In this case, the synthesis equation are expressed as a system of five nonlinear equations in the form :

Fig.2 Flow chart of the procedure for determining (χ_i^*, m_i^*) ($i=1 \sim 5$) from (χ_i, m_i) ($i=1 \sim n$).

$$\begin{aligned}
& R_1(d_{11} \cos(\phi_1 + \phi_{i1}) + d_{12} \sin(\phi_1 + \phi_{i1})) \\
& + R_2(d_{13} \cos(\phi_1 + \phi_{i1}) + d_{14} \sin(\phi_1 + \phi_{i1})) \\
& + R_3 d_{15} = d_{16} \cos(\phi_1 + \phi_{i1} - \phi_1 - \phi_{i1}) \\
& + d_{17} \sin(\phi_1 + \phi_{i1} - \phi_1 - \phi_{i1}) \\
& (i=1 \sim 5) \dots\dots\dots (16 \cdot i)
\end{aligned}$$

In order to solve Eqs. (16.1) ~ (16.5) for the five unknowns $R_1, R_2, R_3, \phi_1, \psi_1$, we expand their both sides by means of addition theorems in the trigonometry and introduce six new parameters which are defined by the following equations

$$\left. \begin{aligned} R_1 \cos \phi_1 &= X_1, & R_1 \sin \phi_1 &= X_2 \\ R_2 \cos \phi_1 &= X_3, & R_2 \sin \phi_1 &= X_4 \\ R_3 &= X_5 \end{aligned} \right\} \dots\dots\dots (17)$$

$$\phi_1 - \psi_1 = \alpha \dots\dots\dots (18)$$

Consequently, Eqs. (16.1) ~ (16.5) may be rewritten in the matrix form :

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \end{pmatrix} \cos \alpha + \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{pmatrix} \sin \alpha \dots\dots\dots (19)$$

Where

$$\left. \begin{aligned} a_{1j} &= d_{1j}, & a_{2j} &= d_{1j}, & b_{11} &= d_{16}, & b_{12} &= d_{17} \\ a_{21} &= d_{11} \cos \phi_{11} + d_{12} \sin \phi_{11}, & a_{22} &= -d_{11} \sin \phi_{11} + d_{12} \cos \phi_{11} \\ a_{23} &= d_{13} \cos \phi_{11} + d_{14} \sin \phi_{11}, & a_{24} &= -d_{13} \sin \phi_{11} + d_{14} \cos \phi_{11} \\ b_{21} &= d_{16} \cos(\phi_{11} - \phi_1) + d_{17} \sin(\phi_{11} - \phi_1) \\ b_{22} &= -d_{16} \sin(\phi_{11} - \phi_1) + d_{17} \cos(\phi_{11} - \phi_1) \\ & (i=2 \sim 5, j=1 \sim 5) \end{aligned} \right\} \dots\dots\dots (20)$$

This system of five equations with six unknowns is indeterminate. However the examination of the above definitions of X_i ($i=1 \sim 5$) and α indicates that some of these unknowns are related to one another. From Eq. (17) and Eq. (18), we have

$$\tan \phi_1 = X_2/X_1, \quad \tan \psi_1 = X_4/X_3$$

$$\tan \alpha = \tan(\phi_1 - \psi_1) = \frac{\tan \phi_1 - \tan \psi_1}{1 + \tan \phi_1 \tan \psi_1}$$

Hence the compatibility equation to be added to the system of five equations (19) is set up as follows

$$\tan \alpha = \frac{X_2 X_3 - X_1 X_4}{X_1 X_3 + X_2 X_4} \dots\dots\dots (21)$$

Since Eq. (19) is a linear equation with respect to five unknowns X_i ($i=1 \sim 5$), we can express X_i as a function of α in the form :

$$\left. \begin{aligned} X_1 &= (E_1 \cos \alpha + F_1 \sin \alpha)/H, & X_2 &= (E_2 \cos \alpha + F_2 \sin \alpha)/H \\ X_3 &= (E_3 \cos \alpha + F_3 \sin \alpha)/H, & X_4 &= (E_4 \cos \alpha + F_4 \sin \alpha)/H \\ X_5 &= (E_5 \cos \alpha + F_5 \sin \alpha)/H \end{aligned} \right\} \dots\dots\dots (22)$$

Where E_i, F_i ($i=1 \sim 5$), H are determinants of the fifth order and representing its column by the diagonal element, they are expressed as follows

$$\left. \begin{aligned} E_1 &= |b_{11} & a_{22} & a_{33} & a_{44} & a_{55}|, & F_1 &= |b_{12} & a_{22} & a_{33} & a_{44} & a_{55}| \\ E_2 &= |a_{11} & b_{21} & a_{33} & a_{44} & a_{55}|, & F_2 &= |a_{11} & b_{22} & a_{33} & a_{44} & a_{55}| \\ E_3 &= |a_{11} & a_{22} & b_{31} & a_{44} & a_{55}|, & F_3 &= |a_{11} & a_{22} & b_{32} & a_{44} & a_{55}| \\ E_4 &= |a_{11} & a_{22} & a_{33} & b_{41} & a_{55}|, & F_4 &= |a_{11} & a_{22} & a_{33} & b_{42} & a_{55}| \\ E_5 &= |a_{11} & a_{22} & a_{33} & a_{44} & b_{51}|, & F_5 &= |a_{11} & a_{22} & a_{33} & a_{44} & b_{52}| \\ H &= |a_{11} & a_{22} & a_{33} & a_{44} & a_{55}| \end{aligned} \right\} \dots\dots\dots (23)$$

Table 2 Sixteen cases where m_i ($i=1 \sim n$) satisfy Eq. (15).

	m_1	m_2	m_3	m_4	m_5	m_1^*	m_2^*	m_3^*	m_4^*	m_5^*
(1)	0	0	0	0	0	0	0	0	0	0
(2)	1	0	0	0	0	0	1	0	0	0
(3)	0	1	0	0	0	0	0	1	0	0
(4)	0	0	1	0	0	0	0	0	1	0
(5)	0	0	0	1	0	0	0	0	0	1
(6)	1	1	0	0	0	0	1	0	1	0
(7)	1	0	1	0	0	0	1	0	0	1
(8)	0	1	1	0	0	0	0	1	0	1
(9)	2	0	0	0	0	0	1	2	0	0
(10)	0	2	0	0	0	0	0	1	2	0
(11)	0	0	2	0	0	0	0	0	1	2
(12)	1	2	0	0	0	0	1	0	1	2
(13)	2	1	0	0	0	0	1	2	0	1
(14)	3	0	0	0	0	0	1	2	3	0
(15)	0	3	0	0	0	0	0	1	2	3
(16)	4	0	0	0	0	0	1	2	3	4

After rewriting Eq. (21) as

$$\tan \alpha (X_1 X_3 + X_2 X_4) - (X_2 X_3 - X_1 X_4) = 0$$

substitute X_i ($i=1 \sim 4$) from Eq. (22) into the above equation and divide both sides of the resultant equation by $\cos^2 \alpha / H^2$, so that we have the cubic equation for determining the value of α , in which $\tan \alpha$ is denoted as t ,

$$h_0 t^3 + h_1 t^2 + h_2 t + h_3 = 0 \dots\dots\dots (24)$$

Where

$$\left. \begin{aligned} h_0 &= F_1 F_3 + F_2 F_4 \\ h_1 &= E_1 F_3 + F_1 E_3 + E_2 F_4 + F_2 E_4 - F_2 F_3 + F_1 F_4 \\ h_2 &= E_1 E_3 + E_2 E_4 - E_2 F_3 - F_2 E_3 + E_1 F_4 + F_1 E_4 \\ h_3 &= E_1 E_4 - E_2 E_3 \end{aligned} \right\} \dots\dots\dots (25)$$

Determining the real root T_α of Eq. (24) and denoting the principal value of $\tan^{-1} T_\alpha$ by α_0 , the unknown α may take the following two values :

$$\alpha_0, \quad \alpha_0 + 180^\circ \dots\dots\dots (26)$$

In the strict sense, the value of α must be determined such as to satisfy the following equations

$$\cos(\phi_1 - \psi_1) = \cos \alpha, \quad \sin(\phi_1 - \psi_1) = \sin \alpha$$

that is,

$$\frac{X_1 X_3 + X_2 X_4}{\sqrt{(X_1^2 + X_2^2)(X_3^2 + X_4^2)}} = \cos \alpha, \quad \frac{X_2 X_3 - X_1 X_4}{\sqrt{(X_1^2 + X_2^2)(X_3^2 + X_4^2)}} = \sin \alpha \dots\dots\dots (27)$$

For two values of α from Eq. (26), two values of the left side of each equation in Eq. (27) are equal, but two values of the right side are equal in magnitude and opposite in sign. Therefore we must find such a correct value of α that the following inequalities hold

$$(X_1 X_3 + X_2 X_4) \cos \alpha > 0, \quad (X_2 X_3 - X_1 X_4) \sin \alpha > 0$$

The unknowns X_i ($i=1 \sim 5$) may be evaluated by substituting the value of α determined as in the above manner into Eq. (22). Thereupon, the solutions $R_1, R_2, R_3, \phi_1, \psi_1$ of the system of five nonlinear equations are evaluated by the following equations

$$\left. \begin{aligned} R_1 &= \sqrt{X_1^2 + X_2^2}, \quad R_2 = \sqrt{X_3^2 + X_4^2}, \quad R_3 = X_5 \\ \phi_1 &= \tan^{-1} \frac{X_2}{X_1}, \quad \left(\text{where } \cos \phi_1 = \frac{X_1}{R_1}, \sin \phi_1 = \frac{X_2}{R_1} \right) \\ \phi_2 &= \tan^{-1} \frac{X_4}{X_3}, \quad \left(\text{where } \cos \phi_2 = \frac{X_3}{R_2}, \sin \phi_2 = \frac{X_4}{R_2} \right) \end{aligned} \right\} \quad (28)$$

Finally, if the values of R_1, R_2, R_3 obtained from Eq.(28) satisfy the inequality

$$1 + \frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{2R_3}{R_1 R_2} > 0$$

three link lengths Z_b, Z_c, Z_d of the four-bar mechanism are evaluated as follows

$$Z_b = \frac{1}{R_2}, \quad Z_c = \sqrt{1 + \frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{2R_3}{R_1 R_2}}, \quad Z_d = \frac{1}{R_1} \quad (29)$$

5. Five point approximation and fifth order approximation

One case in the synthesis method established in chapter 4, namely, the case (1) in Table 2 is the five point approximation presented by F. Freudenstein, where the 0th order differential coefficient is specified at each precision point χ_i ($i=1 \sim 5$).

In this case, Eq.(20) becomes

$$\left. \begin{aligned} a_{11} &= -a_{13} = a_{25} = a_{35} = a_{45} = a_{55} = b_{11} = 1 \\ a_{12} &= a_{14} = b_{12} = 0 \\ a_{i1} &= \cos \phi_{i1}, \quad a_{i2} = -\sin \phi_{i1}, \quad a_{i3} = -\cos \phi_{i1}, \quad a_{i4} = \sin \phi_{i1} \\ b_{i1} &= \cos(\phi_{i1} - \phi_{11}), \quad b_{i2} = -\sin(\phi_{i1} - \phi_{11}), \quad (i=2 \sim 5) \end{aligned} \right\} \quad (20)'$$

so that Eq.(19) may be rewritten in the form :

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} & 1 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1 \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \end{bmatrix} \cos \alpha + \begin{bmatrix} 0 \\ b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{bmatrix} \sin \alpha$$

The above equation is equivalent to the following two equations

$$\begin{aligned} X_1 - X_3 + X_5 &= \cos \alpha \\ \begin{bmatrix} a_{21}-1 & a_{22} & a_{23}+1 & a_{24} \\ a_{31}-1 & a_{32} & a_{33}+1 & a_{34} \\ a_{41}-1 & a_{42} & a_{43}+1 & a_{44} \\ a_{51}-1 & a_{52} & a_{53}+1 & a_{54} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} &= \begin{bmatrix} b_{21}-1 \\ b_{31}-1 \\ b_{41}-1 \\ b_{51}-1 \end{bmatrix} \cos \alpha + \begin{bmatrix} b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{bmatrix} \sin \alpha \end{aligned}$$

Consequently, putting

$$\left. \begin{aligned} E_1 &= |b_{21}-1 \quad a_{22} \quad a_{23}+1 \quad a_{24}|, \quad E_2 = |a_{21}-1 \quad b_{11}-1 \quad a_{43}+1 \quad a_{54}| \\ E_3 &= |a_{21}-1 \quad a_{22} \quad b_{41}-1 \quad a_{54}|, \quad E_4 = |a_{21}-1 \quad a_{32} \quad a_{43}+1 \quad b_{51}-1| \\ F_1 &= |b_{22} \quad a_{32} \quad a_{43}+1 \quad a_{54}|, \quad F_2 = |a_{21}-1 \quad b_{32} \quad a_{43}+1 \quad a_{54}| \\ F_3 &= |a_{21}-1 \quad a_{32} \quad b_{42} \quad a_{54}|, \quad F_4 = |a_{21}-1 \quad a_{32} \quad a_{43}+1 \quad b_{52}| \\ H &= |a_{21}-1 \quad a_{32} \quad a_{43}+1 \quad a_{54}| \\ E_5 &= -E_1 + E_3 + H, \quad F_5 = -F_1 + F_3 \end{aligned} \right\} \quad (23)'$$

we may use Eq.(20)' and Eq.(23)' in place of Eq.(20) and Eq.(23) respectively in the five point approximation.

One case in the synthesis method in chapter 4, namely, the case (16) in Table 2 is the fifth order approximation presented by F. Freudenstein, where the higher order differential coefficients from 0th to 4th are specified at the precision point χ_1 .

In this case, Eq.(20) becomes

$$\left. \begin{aligned} a_{11} &= a_{31} = a_{51} = a_{22} = a_{42} = -a_{13} = a_{15} = b_{11} = 1 \\ a_{31} &= a_{41} = a_{12} = a_{32} = a_{52} = a_{23} = a_{14} = a_{25} = a_{35} = a_{45} = a_{55} = b_{21} = b_{12} = 0 \\ a_{33} &= -p_1^2, \quad a_{43} = 3p_1 p_2, \quad a_{53} = -p_1^4 + 4p_1 p_2 + 3p_2^2 \\ a_{24} &= -p_1, \quad a_{34} = -p_2, \quad a_{44} = -p_1^3 + p_3, \quad a_{54} = -6p_1^2 p_2 + p_4 \\ b_{31} &= (1-p_1)^2, \quad b_{41} = 3(1-p_1)p_2, \quad b_{51} = (1-p_1)^4 + 4(1-p_1)p_2 - 3p_2^2 \\ b_{22} &= 1-p_1, \quad b_{32} = -p_2, \quad b_{42} = (1-p_1)^3 + p_3, \quad b_{52} = -6(1-p_1)^2 p_2 + p_4 \\ p_k &= \frac{r_k}{r_k^k} f_1^{(k)} \quad (k=1 \sim 4) \end{aligned} \right\} \quad (20)''$$

so that Eq.(19) may be rewritten in the form :

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & a_{24} & 0 \\ 1 & 0 & a_{33} & a_{34} & 0 \\ 0 & 1 & a_{43} & a_{44} & 0 \\ 1 & 0 & a_{53} & a_{54} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ b_{31} \\ b_{41} \\ b_{51} \end{bmatrix} \cos \alpha + \begin{bmatrix} 0 \\ b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{bmatrix} \sin \alpha$$

The above equation is equivalent the following four equations

$$\begin{aligned} X_1 - X_3 + X_5 &= \cos \alpha \\ X_2 + a_{24} X_4 &= b_{22} \sin \alpha \\ X_1 + a_{33} X_3 + a_{34} X_4 &= b_{31} \cos \alpha + b_{32} \sin \alpha \\ \begin{bmatrix} a_{43} & a_{44} - a_{24} \\ a_{53} - a_{33} & a_{54} - a_{34} \end{bmatrix} \begin{bmatrix} X_3 \\ X_4 \end{bmatrix} &= \begin{bmatrix} b_{41} \\ b_{51} - b_{31} \end{bmatrix} \cos \alpha + \begin{bmatrix} b_{42} - b_{22} \\ b_{52} - b_{32} \end{bmatrix} \sin \alpha \end{aligned}$$

Consequently, putting

$$\left. \begin{aligned} E_3 &= \begin{vmatrix} b_{41} & a_{44} - a_{24} \\ b_{51} - b_{31} & a_{54} - a_{34} \end{vmatrix} \\ E_4 &= \begin{vmatrix} a_{43} & b_{41} \\ a_{53} - a_{33} & b_{51} - b_{31} \end{vmatrix} \\ F_3 &= \begin{vmatrix} b_{42} - b_{22} & a_{44} - a_{24} \\ b_{52} - b_{32} & a_{54} - a_{34} \end{vmatrix} \\ F_4 &= \begin{vmatrix} a_{43} & b_{42} - b_{22} \\ a_{53} - a_{33} & b_{52} - b_{32} \end{vmatrix} \\ H &= \begin{vmatrix} a_{43} & a_{44} - a_{24} \\ a_{53} - a_{33} & a_{54} - a_{34} \end{vmatrix} \\ E_1 &= -(a_{33} E_3 + a_{34} E_4) + b_{31} H \\ E_2 &= -a_{24} E_4 \\ F_1 &= -(a_{33} F_3 + a_{34} F_4) + b_{32} H \\ F_2 &= -a_{24} F_4 + b_{22} H \\ E_5 &= -E_1 + E_3 + H \\ F_5 &= -F_1 + F_3 \end{aligned} \right\} \quad (23)''$$

we may use Eq.(20)'' and Eq.(23)'' in place of Eq.(20) and Eq.(23) respectively in the fifth order approximation.

6. Respacing of precision points

In this report, the structural error of the function generator designed through the synthesis is defined as the percentage of the deviation of the generated function from the desired function to the range of variations of the desired function.

$$\varepsilon(x) = \frac{g(x) - f(x)}{\max\{f(x)\} - \min\{f(x)\}} \times 100 \quad (30)$$

It is unpracticable to describe exactly the structural error $\varepsilon(x)$ in such a function which contains explicitly precision points χ_i ($i=1 \sim n$) as parameters.

$$\varepsilon(x) = \varepsilon(x; x_1, x_2, \dots, x_n), \quad x_k \leq x \leq x_f \quad (31)$$

However, $\varepsilon(x)$ may be approximated with acceptable accuracy by a tenth degree polynomial containing χ_i ($i=1 \sim n$) as

parameters, In this chapter, in order to reduce the structural error of the function generator, a new numerical procedure for respacing precision points is suggested on the basis of the tenth degree polynomial closely approximating the structural error.

Define five degree polynomials $U(\chi)$ and $V(\chi)$ as

$$u(x) = (x - x_1^*)(x - x_2^*)(x - x_3^*) \times (x - x_4^*)(x - x_5^*) \dots (32)$$

$$v(x) = \mu_0 x^5 + \mu_1 x^4 + \mu_2 x^3 + \mu_3 x^2 + \mu_4 x + \mu_5 \dots (33)$$

Here we adopt the tenth degree polynomial $E(\chi)$ defined as the product of $U(\chi)$ and $V(\chi)$ for the above mentioned purpose

$$e(x) = u(x)v(x) \dots (34)$$

Where the coefficients μ_j ($j = 0 \sim 5$) in Eq.(33) are determined such that $E(\chi)$ passes through the six points $(\bar{\chi}_i, \bar{\varepsilon}_i)$ ($i = 0 \sim 5$) on $E(\chi)$ which are yielded by the process described at the end of this chapter.

Then we assume that greatest values $\bar{\varepsilon}_j$ ($j = 0 \sim n$) of $|E(\chi)|$ in intervals $[\chi_j, \chi_{j+1}]$ ($j = 0 \sim n$) obtained from dividing the specified interval $[\chi_0, \chi_n]$ by precision points χ_i ($i = 1 \sim n$) are expressed as a function of χ_i ($i = 1 \sim n$) in the form :

$$\bar{\varepsilon}_j = \bar{\varepsilon}_j(x_1, x_2, \dots, x_n) \quad (j = 0 \sim n)$$

Where $\chi_0 = \chi_5$ and $\chi_{n+1} = \chi_f$.

In this case, on the basis of rate of variation of $\bar{\varepsilon}_j$ ($j = 0 \sim n$) with respect to χ_i ($i = 1 \sim n$), we attempt to estimate the spacing of precision points such that all of the greatest values of $|E(\chi)|$ in each interval may become equal.

The structural error becomes zero at each precision point and generally takes a maximum or a minimum in intervals $[\chi_j, \chi_{j+1}]$ ($j = 1 \sim n-1$) and varies monotonically in intervals $[\chi_0, \chi_1]$ and $[\chi_n, \chi_{n+1}]$. Then, let real roots of the nine degree equation $E'(\chi) = 0$ in intervals $[\chi_j, \chi_{j+1}]$ ($j = 1 \sim n-1$) be denoted by $\bar{\chi}_j$ ($j = 1 \sim n-1$) and put $\bar{\chi}_0 = \chi_0$, $\bar{\chi}_n = \chi_{n+1}$.

Moreover, rewrite Eq.(32) as follows

$$u(x) = x^5 + \lambda_1 x^4 + \lambda_2 x^3 + \lambda_3 x^2 + \lambda_4 x + \lambda_5 \dots (32')$$

Consequently, the greatest value of $|E(\chi)|$ in each interval is evaluated by means of the following equation

$$\bar{\varepsilon}_j = |(\mu_0 \bar{\chi}_j^5 + \mu_1 \bar{\chi}_j^4 + \mu_2 \bar{\chi}_j^3 + \mu_3 \bar{\chi}_j^2 + \mu_4 \bar{\chi}_j + \mu_5) \times (\lambda_1 \bar{\chi}_j^4 + \lambda_2 \bar{\chi}_j^3 + \lambda_3 \bar{\chi}_j^2 + \lambda_4 \bar{\chi}_j + \lambda_5)| \quad (j = 0 \sim n) \dots (35)$$

Where

$$\left. \begin{aligned} \lambda_1 &= -(x_1^* + x_2^* + x_3^* + x_4^* + x_5^*) \\ \lambda_2 &= x_1^* x_2^* + x_1^* x_3^* + x_1^* x_4^* + x_1^* x_5^* + x_2^* x_3^* \\ &\quad + x_2^* x_4^* + x_2^* x_5^* + x_3^* x_4^* + x_3^* x_5^* + x_4^* x_5^* \\ \lambda_3 &= -(x_1^* x_2^* x_3^* + x_1^* x_2^* x_4^* + x_1^* x_2^* x_5^* \\ &\quad + x_1^* x_3^* x_4^* + x_1^* x_3^* x_5^* + x_1^* x_4^* x_5^* \\ &\quad + x_2^* x_3^* x_4^* + x_2^* x_3^* x_5^* + x_2^* x_4^* x_5^* \\ &\quad + x_3^* x_4^* x_5^*) \\ \lambda_4 &= x_1^* x_2^* x_3^* x_4^* + x_1^* x_2^* x_3^* x_5^* + x_1^* x_2^* x_4^* x_5^* \\ &\quad + x_1^* x_3^* x_4^* x_5^* + x_2^* x_3^* x_4^* x_5^* \\ \lambda_5 &= -x_1^* x_2^* x_3^* x_4^* x_5^* \end{aligned} \right\} \dots (36)$$

The coefficients λ_i ($i = 1 \sim 5$) in Eq.(32)' are a function of χ_i^* ($i = 1 \sim 5$) as shown in Eq.(36). Besides real roots $\bar{\chi}_j$ ($j = 1 \sim n-1$) are considered to be a

function of χ_i^* ($i = 1 \sim 5$) because they satisfy the equation $E'(\bar{\chi}_j) = 0$ which contains λ_i ($i = 1 \sim 5$) as parameters, that is,

$$\begin{aligned} &(5\bar{\chi}_j^4 + 4\lambda_1 \bar{\chi}_j^3 + 3\lambda_2 \bar{\chi}_j^2 + 2\lambda_3 \bar{\chi}_j + \lambda_4) \\ &\times (\mu_0 \bar{\chi}_j^5 + \mu_1 \bar{\chi}_j^4 + \mu_2 \bar{\chi}_j^3 + \mu_3 \bar{\chi}_j^2 + \mu_4 \bar{\chi}_j + \mu_5) \\ &+ (\bar{\chi}_j^5 + \lambda_1 \bar{\chi}_j^4 + \lambda_2 \bar{\chi}_j^3 + \lambda_3 \bar{\chi}_j^2 + \lambda_4 \bar{\chi}_j + \lambda_5) \\ &\times (5\mu_0 \bar{\chi}_j^4 + 4\mu_1 \bar{\chi}_j^3 + 3\mu_2 \bar{\chi}_j^2 + 2\mu_3 \bar{\chi}_j + \mu_4) = 0 \end{aligned} \dots (37)$$

The five values χ_i^* ($i = 1 \sim 5$) coincide with one of precision points χ_i ($i = 1 \sim n$). Therefore, differentiating partially both sides of Eq.(35) with respect to χ_i , we have the following equation

$$\frac{\partial \bar{\varepsilon}_j}{\partial \chi_i} = \text{sign}[e(\bar{\chi}_j)] \{ w_{ij} v_j + (u_i' v_j + u_j v_i') \frac{\partial \bar{\chi}_j}{\partial \chi_i} \} \quad (j = 0 \sim n) \dots (38)$$

Where

$$\left. \begin{aligned} u_j &= u(\bar{\chi}_j), \quad u_j' = u'(\bar{\chi}_j) \\ v_j &= v(\bar{\chi}_j), \quad v_j' = v'(\bar{\chi}_j) \\ w_{ij} &= (m_i + 1) (\lambda_{i1} \bar{\chi}_j^4 + \lambda_{21} \bar{\chi}_j^3 + \lambda_{31} \bar{\chi}_j^2 + \lambda_{41} \bar{\chi}_j + \lambda_{51}) \\ \lambda_{i1} &= -1, \quad \lambda_{21} = -\lambda_1 - \lambda_i \\ \lambda_{31} &= -(\lambda_{21}^2 - (\lambda_4 - \lambda_i^2)) / 2 \\ \lambda_{41} &= -\lambda_4 (\lambda_1 - \frac{1}{\lambda_i}), \quad \lambda_{51} = \lambda_5 / \lambda_i \\ \lambda_6 &= \sum_{i=1}^5 x_i^* x_i^2, \quad \lambda_7 = \sum_{i=1}^5 \frac{1}{x_i^*} \end{aligned} \right\} \dots (39)$$

Since $\bar{\chi}_j$ ($j = 1 \sim n-1$) satisfies Eq.(37), we have

$$u_j' v_j + u_j v_j' = 0 \quad (j = 1 \sim n-1)$$

Hence the second term in $\{ \}$ of Eq.(38) becomes zero in the case where the subscript j takes 1, 2, ..., or $n-1$. Moreover, from the assumption that $\bar{\chi}_0$ and $\bar{\chi}_n$ have nothing to do with χ_i ($i = 1 \sim n$), we have

$$\frac{\partial \bar{\varepsilon}_j}{\partial \chi_i} = 0 \quad (j = 0 \text{ or } n) \dots (40)$$

Hence the second term in $\{ \}$ of Eq.(38) also becomes zero in the case where j takes 0 or n . Consequently, after determining the value $\bar{\chi}_j$ ($j = 0 \sim n$) to yield the greatest value of $|E(\chi)|$ in each interval, we may evaluate the value of $\partial \bar{\varepsilon}_j / \partial \chi_i$ by means of the following equation

$$\frac{\partial \bar{\varepsilon}_j}{\partial \chi_i} = \text{sign}[e(\bar{\chi}_j)] w_{ij} v_j \quad (i = 1 \sim n, \quad j = 0 \sim n) \dots (41)$$

When small changes of precision points χ_i ($i = 1 \sim n$) are assumed to be $\Delta \chi_i$ ($i = 1 \sim n$), the small change of the greatest value $\bar{\varepsilon}_j$ of $|E(\chi)|$ in each interval may be expressed as follows

$$\Delta \bar{\varepsilon}_j = \frac{\partial \bar{\varepsilon}_j}{\partial \chi_1} \Delta \chi_1 + \frac{\partial \bar{\varepsilon}_j}{\partial \chi_2} \Delta \chi_2 + \dots + \frac{\partial \bar{\varepsilon}_j}{\partial \chi_n} \Delta \chi_n \dots (42)$$

If the tenth degree polynomial $E(\chi)$ approximates the structural error $\varepsilon(\chi)$ with acceptable accuracy, we may use $\partial \bar{\varepsilon}_j / \partial \chi_i$ instead of $\partial \varepsilon_j / \partial \chi_i$ in the right side of Eq.(42).

In this report, assuming that the structural error is minimized in a case where $\bar{\varepsilon}_j$ ($j = 0 \sim n$) in the succeeding synthesis becomes equal to the mean value of ones at the present time, we determine small changes $\Delta \chi_i$ ($i = 1 \sim n$) of precision points for reducing the structural error as the approximate solution of the system of $n+1$ equations with n unknowns

$$\frac{\partial \bar{\varepsilon}_j}{\partial x_1} \Delta x_1 + \frac{\partial \bar{\varepsilon}_j}{\partial x_2} \Delta x_2 + \dots + \frac{\partial \bar{\varepsilon}_j}{\partial x_n} \Delta x_n = \Delta \bar{\varepsilon}_j$$

$$(j=0 \sim n)$$

Where

$$\Delta \bar{\varepsilon}_j = \bar{\varepsilon}_n - \bar{\varepsilon}_j, \quad \bar{\varepsilon}_n = \sum_{j=0}^n \bar{\varepsilon}_j / (n+1)$$

Then the approximate solution $\Delta \chi_i$ ($i=1 \sim n$) is evaluated such as to minimize the value Q defined by the following equation

$$Q = \sum_{j=0}^n \left(\Delta \bar{\varepsilon}_j - \sum_{i=1}^n \frac{\partial \bar{\varepsilon}_j}{\partial x_i} \Delta x_i \right)^2 \quad (43)$$

Differentiating partially both sides of Eq. (43) with respect to $\Delta \chi_i$ yields

$$\frac{\partial Q}{\partial \Delta x_i} = -2 \left\{ \sum_{j=0}^n \Delta \bar{\varepsilon}_j \frac{\partial \bar{\varepsilon}_j}{\partial x_i} - \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial \bar{\varepsilon}_j}{\partial x_i} \frac{\partial \bar{\varepsilon}_j}{\partial x_j} \right) \Delta x_j \right\} \quad (44)$$

Since values of $\Delta \chi_i$ ($i=1 \sim n$) satisfying n equations $\partial Q / \partial \Delta \chi_i = 0$ ($i=1 \sim n$) minimize Q , small changes of precision points may be determined by solving the following simultaneous equation

$$\sum_{i=1}^n r_{ij} \Delta x_j = s_i \quad (i=1 \sim n) \quad (45)$$

Where

$$r_{ij} = \sum_{l=0}^n \frac{\partial \bar{\varepsilon}_l}{\partial x_i} \frac{\partial \bar{\varepsilon}_l}{\partial x_j}, \quad s_i = \sum_{l=0}^n \Delta \bar{\varepsilon}_l \frac{\partial \bar{\varepsilon}_l}{\partial x_i}$$

We choose the values $(\tilde{x}_i, \tilde{\varepsilon}_i)$ ($i=0 \sim 5$) in the following process which decide the coefficients μ_j ($j=0 \sim 5$) in Eq. (33).

Firstly, with respect to χ_i^* ($i=1 \sim 5$) and χ_j ($j=1 \sim n$) (i) in the case where $x_i^* = x_j$ holds, we put

$$\bar{x}_{i-1} = \bar{x}_{j-1}$$

(ii) in the case where $x_i^* = x_{i+1}^* = x_j$,

$$\bar{x}_{i-1} = \bar{x}_{j-1}, \quad \bar{x}_i = (\bar{x}_{j-1} + x_j) / 2$$

(iii) in the case where

$$x_i^* = x_{i+1}^* = x_{i+2}^* = x_j,$$

$$\bar{x}_{i-1} = \bar{x}_{j-1}, \quad \bar{x}_i = (\bar{x}_{j-1} + x_j) / 2$$

$$\bar{x}_{i+1} = (x_j + \bar{x}_j) / 2$$

(iv) in the case where

$$x_i^* = x_{i+1}^* = x_{i+2}^* = x_{i+3}^* = x_j,$$

$$\bar{x}_{i-1} = \bar{x}_{j-1}, \quad \bar{x}_i = (2\bar{x}_{j-1} + x_j) / 3$$

$$\bar{x}_{i+1} = (\bar{x}_{j-1} + 2x_j) / 3, \quad \bar{x}_{i+2} = (x_j + \bar{x}_j) / 2$$

(v) in the case where

$$x_1^* = x_2^* = x_3^* = x_4^* = x_5^* = x_1,$$

$$\bar{x}_0 = \bar{x}_5, \quad \bar{x}_1 = (2\bar{x}_0 + x_1) / 3, \quad \bar{x}_2 = (\bar{x}_0 + 2x_1) / 3$$

$$\bar{x}_3 = (2x_1 + \bar{x}_1) / 3, \quad \bar{x}_4 = (x_1 + 2\bar{x}_1) / 3$$

Finally, we put

$$\bar{x}_5 = \bar{x}_n$$

The coefficients μ_j ($j=0 \sim 5$) are determined by solving the following simultaneous equation

$$\bar{x}_i^5 \mu_0 + \bar{x}_i^4 \mu_1 + \bar{x}_i^3 \mu_2 + \bar{x}_i^2 \mu_3 + \bar{x}_i \mu_4 + \mu_5 = \bar{v}_i$$

$$(i=0 \sim 5) \quad (46)$$

Where

7. Numerical examples

In order to synthesize a plane four-bar mechanism to generate approximately the sine function in the interval $[0^\circ, 90^\circ]$, we choose the desired function

$$y = \sin x, \quad 0^\circ \leq x \leq 90^\circ$$

and specify two scale factors as

$$r_\phi = 1.0, \quad r_\psi = 90.0$$

Here we adopt seven cases of the specifications of greatest numbers of order, namely, {1}, {2}, {7}, {9}, {13}, {14}, {16} in Table 2.

For each case, three cycles of numerical computations which consist of the determination of the five design parameters and the respacing of precision points are performed. The numerical results are summarized in Table 3. For the cases except {14.C} in Table 3, one set of the five design parameters $Z_b, Z_c, Z_d, \phi_1, \psi_1$ is determined as shown in each row in Table 3 and a function generator is obtained which generates approximately the desired function by means of $g_2(\chi)$. For the case {14.C}, three sets of the five design parameters are determined. One is shown in Table 3. The others are

{14-c-ii}

$$Z_b = 2.9208815, \quad \phi_1 = 197.17239^\circ$$

$$Z_c = 1.0310399, \quad \phi_1 = 236.79662^\circ$$

$$Z_d = 1.8033975$$

{14-c-iii}

$$Z_b = 2.9122773, \quad \phi_1 = 206.37513^\circ$$

$$Z_c = 0.80079157, \quad \phi_1 = 240.62528^\circ$$

$$Z_d = 2.1448297$$

Since the driving link of the four-bar mechanism having the design parameters in the case {14.C-ii} or {14.C-iii} can not rotate continuously in its specified range, it is impossible to make use of this mechanism as a function generator.

In the numerical computations, the greatest value in the interval $[\chi_j, \chi_{j+1}]$ among the values of $|\mathcal{E}(\chi)|$ for the values of χ which vary discretely from 0° to 90° with each incremental size 1° and the value of χ to yield this greatest value are used in place of $\bar{\varepsilon}_j$ and $\bar{\chi}_j$ respectively.

The structural error $\mathcal{E}(\chi)$, the fifth degree polynomial $V(\chi)$, and the difference between $\mathcal{C}(\chi)$ and $\mathcal{E}(\chi)$ in the case {1} of Table 3 are shown in Fig.3, Fig.4, and Fig.5 respectively.

8. Conclusions

(1) The analytical synthesis method is established in which the five design parameters are determined such that the differential coefficients $g_i^{(k)}$ ($k=0 \sim m_i$) of the generated function at precision points χ_i ($i=1 \sim n$) may coincide with the ones of the desired function.

(2) A new numerical procedure for respacing precision points is suggested on the basis of the tenth degree polynomial closely approximating the structural error.

Table 3 Numerical results of synthesis examples. Its 1-th, 2-th, 3-th, 4-th, 5-th, 6-th, and 7-th row are results obtained in the case {1}, {2}, {7}, {9}, {13}, {14}, and {16} of the specifications of m_i ($i = 1 \sim n$), respectively.

	x_1	x_2	x_3	x_4	x_5	Z_b	Z_c	Z_d	ϕ_1	ϕ_2	$\bar{\epsilon}_0$	$\bar{\epsilon}_1$	$\bar{\epsilon}_2$	$\bar{\epsilon}_3$	$\bar{\epsilon}_4$	$\bar{\epsilon}_5$
a	9.0	27.0	45.0	63.0	81.0	2.369 344 4	2.733 752 1	0.766 417 33	306.137 30°	263.846 67°	0.62	0.097	0.051	0.062	0.15	0.94
b	3.545 172	23.625 709	45.374 496	68.594 117	86.414 956	2.059 217 9	2.428 594 4	0.735 697 91	299.656 45	257.395 93	0.26	0.19	0.13	0.18	0.24	0.36
c	2.884 675	21.391 582	48.174 129	71.064 509	87.331 487	1.876 008 0	2.270 472 4	0.703 645 88	298.066 73	256.097 73	0.21	0.20	0.22	0.21	0.21	0.24
a	18.0	45.0	63.0	81.0		2.411 729 9	2.778 477 3	0.766 317 79	315.298 03	277.369 40	0.83	0.072	0.066	0.15	0.95	
b	12.633 451	38.663 212	68.376 691	87.302 947		2.455 818 8	2.746 344 5	0.810 864 96	310.591 98	274.448 78	0.42	0.11	0.34	0.36	0.33	
c	9.219 032	46.286 232	69.686 245	87.246 158		2.242 011 4	2.544 841 3	0.790 771 70	306.125 18	269.304 19	0.28	0.31	0.27	0.30	0.30	
a	18.0	45.0	72.0			2.422 285 2	2.790 545 9	0.768 445 99	315.092 54	276.772 39	0.84	0.075	0.095	1.3		
b	12.627 769	44.215 961	78.849 492			2.083 913 2	2.447 707 8	0.739 545 47	308.626 93	271.566 37	0.50	0.19	0.33	0.62		
c	10.069 577	48.477 873	80.874 631			1.860 943 4	2.245 851 6	0.706 995 01	304.939 36	267.939 56	0.37	0.37	0.35	0.42		
a	27.0	63.0	81.0			2.531 960 1	2.907 893 0	0.766 434 43	324.823 78	289.550 59	1.1	0.11	0.17	0.98		
b	21.457 232	62.363 105	87.762 922			2.943 095 0	3.195 763 2	0.862 929 98	321.252 29	288.041 88	0.61	0.27	0.61	0.35		
c	19.078 802	67.106 523	86.924 212			3.043 291 8	3.248 935 2	0.897 937 11	319.131 04	286.528 07	0.46	0.50	0.46	0.45		
a	27.0	72.0				2.543 054 0	2.920 549 4	0.768 497 72	324.615 04	288.950 38	1.1	0.14	1.3			
b	22.554 293	77.568 368				2.448 986 5	2.783 282 5	0.781 779 14	320.208 91	286.439 39	0.78	0.43	0.83			
c	20.731 653	79.457 944				2.450 550 9	2.759 026 1	0.795 223 80	318.547 71	285.606 79	0.64	0.63	0.67			
a	36.0	81.0				2.752 527 1	3.149 774 2	0.770 207 64	334.506 09	299.288 88	1.5	0.30	1.2			
b	31.575 116	86.163 101				3.493 400 4	3.760 406 6	0.885 744 85	332.676 26	299.534 13	0.93	0.81	0.70			
c	30.787 483	85.692 711				3.806 453 7	4.035 257 3	0.924 511 94	332.554 30	299.573 16	0.83	0.84	0.83			
a	45.0					3.064 485 2	3.494 480 5	0.782 115 83	343.421 31	305.172 13	1.9	2.6				
b	46.228 426					2.679 089 2	3.156 950 2	0.729 571 14	343.599 50	304.867 28	2.2	2.2				
c	46.191 562					2.689 636 4	3.166 107 1	0.731 098 95	343.594 15	304.877 58	2.2	2.2

Fig.3 Structural error $E(x)$ of the function generator in the case {1} in Table 3.

Fig.4 Fifth degree polynomial $V(x)$ in the case {1} in Table 3.

Fig.5 Difference between the structural error $E(x)$ and the tenth degree polynomial $e(x)$ in the case {1} in Table 3.

Acknowledgement

The author wishes to express his sincere gratitude to Professor S. Ogino of Yamagata University for his guidance and encouragement about the present report.

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