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Transient Heat Conduction in Anisotropic Solids*

(1st Report, Fundamental Theory)

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The transient heat conduction problem in an anisotropic solid is difficult to solve analytically, and it seems most convenient to obtain the solution by numerical methods such as the finite difference method.

In this paper, the following have been made clear.

- (1) Considering the accuracy of the solution and the stability of the finite difference equations, it is shown to be desirable to convert the anisotropic field into an isotropic one by enlarging the field along the principal axis, and then to apply the finite difference method.
- (2) The equation for modifying heat flux at the boundary, accompanied with the above conversion, is proposed, and the method to obtain the solution, which was reported in the former papers, is shown to be applicable to the anisotropic field.
- (3) The actual transient temperature change in two-dimensional multi-crystalline carbon, which is regarded as orthotropic solid, agrees well with the finite difference solution by the method prescribed in this paper.

1. Introduction

Recently, the study of the transient heat conduction problem in anisotropic solids is very important in the field of industry.

They include various crystals, such as orthorhombic, tetragonal, hexagonal, monoclinic and triclinic crystals, compound materials (the majority of which are orthotropic systems), laminated materials and semi-conductors.[6] The two-dimensional equation of transient heat conduction in an anisotropic solid [1] is

$$c\rho \frac{\partial \theta}{\partial t} = \lambda_1 \frac{\partial^2 \theta}{\partial x^2} + \lambda_2 \frac{\partial^2 \theta}{\partial y^2} + \lambda_3 \frac{\partial^2 \theta}{\partial x \partial y} \dots \dots (1)$$

where λ_1 , λ_2 and λ_3 are coefficients determined from the thermal conductivities for the principal axes of anisotropic materials and the direction of principal axes. In heat conduction problems of anisotropic solids, the relations $\lambda_1 \neq \lambda_2$ and $\lambda_3 \neq 0$ are usually satisfied in Eq.(1), and the analytical approaches of Eq.(1) are very difficult, so the numerical methods such as the finite difference using the experimental data of λ_ξ and λ_η are considered promising.

In this paper, an application of the finite difference method to the transient

two-dimensional heat conduction problems in orthotropic solids is investigated theoretically and confirmed by experiments.

Nomenclature

T	: numerical solution of temperature [°C]
θ	: analytical solution or experimental value of temperature [°C]
q	: heat flux [Kcal/m ² h]
a	: thermal diffusivity [m ² /h]
λ	: thermal conductivity [Kcal/mh°C]
c	: specific heat [Kcal/Kg°C]
ρ	: density [Kg/m ³]
x, y	: coordinate of the location [m]
ξ, η	: coordinate of the principal axis [m]
θ, ψ	: angle [°]
t	: time [h]
$\Delta x, \Delta y$	: finite difference for x and y [m]
Δt	: finite difference for time t [h]
$\Delta \xi, \Delta \eta$	: finite difference for ξ and η [m]

Suffixes

x	: value in x direction
y	: value in y direction
η	: value in the direction of the principal axis η
ξ	: value in the direction of the principal axis ξ
upper pAt	: value at t = pAt
lower mAx, nAy	: value at x = mAx, y = nAy

2. The finite difference method in anisotropic temperature field

To discuss the finite difference method in an anisotropic temperature field, authors consider a segment AB which is placed in the anisotropic field of principal axes ξ and η as shown in Fig.1. The heat flux q_n in the direction normal to this segment is defined

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Fig.1 Relation between the principal axes η , ξ and p - n coordinate system.

as

$$q_n = -(\lambda_\eta \sin^2 \theta + \lambda_\xi \cos^2 \theta) \frac{\partial g}{\partial n} + (\lambda_\eta - \lambda_\xi) \sin \theta \cos \theta \frac{\partial g}{\partial p} \quad (2)$$

where the coordinate axes n and p are defined as the coordinates which identify the normal and the tangential direction of the segment AB. The finite difference equations in anisotropic solids are obtained by converting the above equation of heat flux into a difference equation, summing the flux along the boundary of the small mesh element and equating with heat which corresponds to the temperature change of the element. In the above equation of q_n , the term $\partial g / \partial p$ in addition to the term $\partial g / \partial n$ is involved with the case of anisotropic heat conduction.

In applying the finite difference method directly to the anisotropic field, two methods of dividing the anisotropic temperature field into meshes can be considered:

- (1) Dividing along the principal axes.
- (2) Dividing along the boundaries of two-dimensional field.

In the method (1) above, the third term in the right hand side of Eq.(1) disappears*, but it is difficult to divide the temperature field near the boundary surfaces into square meshes of same dimensions with required accuracy[3]-[5].

In the method (2) above, there is a merit that the temperature field can be divided along the boundary, but the sides of each mesh usually do not fit with the direction of the principal axes, therefore the finite difference equations are apt to be complex. The problem is described on an example shown in Fig.2. An anisotropic temperature field is divided into square meshes inclining from the principal axes and authors discuss to convert the heat flux q_y across side CD of the mesh into a finite difference equation according to Eq.(2). Since the coordinates n and p which are perpendicular and parallel to the segment CD are chosen on y and x axes, the heat flux q_y is obtained from Eq.(2).

$$* \quad q_\eta = -\lambda_\eta \frac{\partial g}{\partial \eta}, \quad q_\xi = -\lambda_\xi \frac{\partial g}{\partial \xi}$$

$$q_y = q_n = -(\lambda_\eta \sin^2 \alpha + \lambda_\xi \cos^2 \alpha) \frac{\partial g}{\partial n} + (\lambda_\eta - \lambda_\xi) \sin \alpha \cos \alpha \frac{\partial g}{\partial p} \quad (3)$$

In the above equation, the term $\partial g / \partial n$ is approximated by Eq.(4)

$$\frac{\partial g}{\partial n} = \frac{\partial g}{\partial y} = \frac{T_{m+1, (n+1)} dy - T_{m-1, n} dy}{\Delta y} \quad (4)$$

The term $\partial g / \partial p$ is approximated, for example, using the four surrounding nodal points as follows, because there is no proper nodal point on the extended line of the segment CD.

$$\frac{\partial g}{\partial p} = \frac{\partial g}{\partial x} = \frac{[T_{(m+1), (n+1)} dx + T_{(m+1), n} dy] / 2 - [T_{(m-1), (n+1)} dx + T_{(m-1), n} dy] / 2}{2 \Delta x} \quad (5)$$

In Eq.(5), the signs in the terms

$T_{(m+1), (n+1)} dx + T_{(m+1), n} dy$ and $T_{(m-1), (n+1)} dx + T_{(m-1), n} dy$ are opposite**, and the finite difference solution of the simple explicit form doesn't converge stably. Furthermore, in the case of the triangular meshes, the transformation of the term $\partial g / \partial p$ into a finite difference equation is usually considered difficult.

It is concluded that direct application of the finite difference method to the anisotropic temperature field is not adequate considering the accuracy of division in the neighbourhood of the boundary and the stability of the finite difference equation.

In this paper, a numerical method of

Fig.2 An example of dividing the anisotropic temperature field by the meshes of rectangle.

heat conduction in an anisotropic solid is proposed, that is, finite difference method is applied for the temperature field after converting the anisotropic temperature field into an isotropic one.

- (1) In Sec.3, a method to convert the

** This corresponds to the fact that converting the term $\partial^2 g / \partial x \partial y$ in Eq.(1) into a finite difference equation according to the symbol in Fig.2, the term becomes

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{1}{4 \Delta x \Delta y} [T_{(m+1), (n+1)} dy - T_{(m+1), n} dy + T_{(m-1), (n+1)} dy - T_{(m-1), n} dy]$$

and the temperature terms with opposite signs appear.

anisotropic temperature field into an isotropic one is presented and accuracies of the numerical results by finite difference equations in both converted and unconverted temperature fields are compared and discussed.

(2) In Sec.4, a method of modification of heat flux across the boundary surface considering the distortion of the boundary surface by the above transformation of the temperature field is proposed.

(3) In Sec.5, experiments for measuring the thermal properties of multi-crystalline carbon for both principal axes are described, the temperature responses in multi-crystalline carbon are calculated by the above method and the numerical results are compared with experimental results. The method to obtain a numerical solution of two-dimensional temperature field within the required accuracy which authors discussed in the former report [2]-[5] is extended to the anisotropic field.

3. Transformation of anisotropic temperature field into isotropic field

3-1. Transformation to the coordinates system of principal axes

Between two cartesian coordinate systems shown in Fig.3, the following relations are satisfied.

$$\begin{cases} \frac{\partial \vartheta}{\partial X} = \frac{\partial \vartheta}{\partial x} \cos \beta + \frac{\partial \vartheta}{\partial y} \sin \beta \end{cases} \quad (6)$$

$$\begin{cases} \frac{\partial \vartheta}{\partial Y} = -\frac{\partial \vartheta}{\partial x} \sin \beta + \frac{\partial \vartheta}{\partial y} \cos \beta \end{cases} \quad (7)$$

For λ_1 , λ_2 and λ_3 defined in Eq.(1), choosing the value of β as

$$\tan 2\beta = \lambda_3 / (\lambda_1 - \lambda_2) \quad (0 \leq \beta \leq \frac{\pi}{2}) \quad (8)$$

Eq.(1) can be transformed as

$$c\rho \frac{\partial \vartheta}{\partial t} = \lambda_1 \frac{\partial^2 \vartheta}{\partial X^2} + \lambda_2 \frac{\partial^2 \vartheta}{\partial Y^2} \quad (9)$$

where, $\lambda_1, \lambda_2 > 0$

$$\begin{cases} \lambda_1 = (\lambda_1 - \lambda_2) / (2 \cos 2\beta) + (\lambda_1 + \lambda_2) / 2 \end{cases} \quad (10)$$

$$\begin{cases} \lambda_2 = (\lambda_1 + \lambda_2) / 2 - (\lambda_1 - \lambda_2) / (2 \cos 2\beta) \end{cases} \quad (11)$$

In Eq.(9), the new cartesian coordinates X, Y coincide to the principal axes ξ, η of the anisotropic temperature field. Therefore, Eq.(9) can be rewritten as***

Fig.3 Relation between the x-y coordinates system and the principal coordinates X-Y.

$$c\rho \frac{\partial \vartheta}{\partial t} = \lambda_\eta \frac{\partial^2 \vartheta}{\partial \eta^2} + \lambda_\xi \frac{\partial^2 \vartheta}{\partial \xi^2} \quad (9)'$$

Further, using the following well-known transformation

$$\eta' \equiv \sqrt{\lambda_\xi / \lambda_\eta} \eta \quad (12)$$

Eq.(9)' becomes

$$c\rho \frac{\partial \vartheta}{\partial t} = \lambda_\xi \frac{\partial^2 \vartheta}{\partial (\eta')^2} + \lambda_\xi \frac{\partial^2 \vartheta}{\partial \xi^2} \quad (13)$$

From Eq.(13), it is expected that by enlarging the coordinate $\sqrt{\lambda_\xi / \lambda_\eta}$ times along the principal axis η , the anisotropic field can be converted into an isotropic field of thermal conductivity λ_ξ .

3-2. Examination of the above transformation by an example

Authors examine the above conclusion by an example shown in Table 1. By comparing the numerical solution in the anisotropic temperature field with that in the isotropic temperature field after the transformation of Eq.(9), it is found that heat conduction problems in anisotropic solids can be treated as heat conduction problems in isotropic solids by the method proposed in Sec.3-1.

Since the temperature field in Table 1 is symmetrical about the axes η and ξ , authors divide the domain $\eta \geq 0, \xi \geq 0$ by the meshes parallel to the principal axes as shown in Fig.4(a)****. The finite difference equation are shown in Appendix 1.

The anisotropic field of circular boundary is transformed to the field of the isotropic elliptical boundary according to the method in Sec.3-1 as shown in Fig.4(a). As shown in Fig.4(b), the shape of the boundary of an anisotropic temperature field is deformed by the above transformation, but two-dimensional isotropic temperature field of arbitrary shape can be divided into meshes of almost the same size along the boundary, by combining three kinds of meshes: rectangular meshes, triangular meshes and meshes divided by circles and radial lines[4][5][7]. For example, the field of elliptic boundary is divided into meshes as shown in Fig.4(b).

Table 1 An example.

Fundamental equation	$c\rho \frac{\partial \vartheta}{\partial t} = \lambda_\eta \frac{\partial^2 \vartheta}{\partial \eta^2} + \lambda_\xi \frac{\partial^2 \vartheta}{\partial \xi^2}, \lambda_\eta / \lambda_\xi = 4$ ($t \geq 0, \eta^2 + \xi^2 \leq R^2$)
Initial condition	$\vartheta = 0$ ($t = 0, \eta^2 + \xi^2 \leq R^2$)
Boundary condition	$\vartheta = 1$ ($t \geq 0, \eta^2 + \xi^2 = R^2$)

*** In all anisotropic solids, Eq.(1) is transformed into Eq.(9) using Eq.(8) but for the case of orthotropic solids, the coordinates X, Y in Eq.(9) agree with the principal axes of anisotropic solids.

**** In the case of Table 1, although there is a drawback that the size of mesh is irregular, the temperature field can be divided along the boundary only with meshes parallel to the principal axes.

Table 2 Calculated results of Table 1.

Location	4	5	6	7	8
$\frac{q_1 l}{R^2} = 0.01$	0.222 0.231	0.080 0.074	0.256 0.249	0.017 0.016	0.005 0.004
0.02	0.415 0.415	0.217 0.208	0.503 0.491	0.118 0.114	0.047 0.044
0.03	0.537 0.534	0.327 0.317	0.641 0.629	0.251 0.244	0.122 0.117
0.04	0.625 0.621	0.417 0.406	0.728 0.717	0.372 0.364	0.208 0.202
0.05	0.691 0.687	0.495 0.483	0.786 0.777	0.473 0.465	0.294 0.287
0.06	0.743 0.739	0.562 0.550	0.828 0.820	0.557 0.549	0.376 0.369
0.07	0.784 0.780	0.621 0.609	0.860 0.853	0.625 0.617	0.451 0.444
0.08	0.817 0.814	0.671 0.660	0.884 0.878	0.681 0.674	0.519 0.511

Lower figures show the results of Fig. 4(a)
upper figures show the results of Fig. 4(b)

(a) Anisotropic field (b) Isotropic field of
 $\lambda_\eta = 4\lambda_\xi$ Thermal conductivity λ_ξ

Fig. 4 An example of temperature field divided into meshes.

(a) Anisotropic temperature field (b) Isotropic temperature field
Fig. 5 Variation of the boundary by the transformation

The finite difference equations for meshes of Fig. 4(b) are shown in Appendix 2.

Transient temperature variations are calculated at the points 4 - 8 of Fig. 4(a) about anisotropic field by the finite difference method. The results are shown in Table 2, where lower figures show the calculated results of Fig. 4(a) and upper figures show those of the transformed isotropic field of Fig. 4(b). The finite differences of location and time are selected such that the numerical solutions are obtained having the required accuracy [4] [5] of less than $\pm 3\%$ of maximum changes of temperature.

Table 2 shows that the temperature variations calculated directly about an anisotropic temperature field agree well with those of an isotropic temperature

field after the transformation.

It is clear that heat conduction problems in an anisotropic temperature field can be treated as heat conduction problems in an isotropic temperature field by the transformation proposed in Sec. 3-1.

4. Modification of heat flux at the boundary through transformation of temperature field

After the transformation proposed in Sec. 3-1, the volume of the temperature field increases to $\sqrt{\lambda_\xi/\lambda_\eta}$ times that of anisotropic field, but usually the area of the boundary surface does not. When the boundary condition is not that of the prescribed boundary temperature, the heat flux at the boundary must be modified by the above transformation.

4-1. Equation for the modification of heat flux at the boundary

Consider a small right-angled triangle shown in Fig. 5(a), the hypotenuse of which has a length l , the other two sides being assumed to be parallel to the directions of principal axes η and ξ . The triangle is converted into a triangle shown in Fig. 5(b) by the transformation of Sec. 3-1. Defining the terms of heat flux q and q' , length l , l' , l_η and l_ξ , and angle θ and ψ as shown in Fig. 5, we calculate the relation between q and q' for which the temperature variations in two triangles are equivalent.

Defining the volumes of the above triangles per unit thickness before and after the transformation as V and V' respectively, condition for the temperature variations of both triangles to be equal is

$$q/V = q'/V' \quad (14)$$

By using the following relations

$$V' = \sqrt{\lambda_\xi/\lambda_\eta} V \quad (15)$$

$$l'^2 = (l_\eta)^2 + (l_\xi)^2 = (l')^2 [(\lambda_\eta/\lambda_\xi) \cos^2 \varphi + \sin^2 \varphi] \quad (16)$$

equation (14) becomes

$$\frac{q'}{q} = \sqrt{\frac{\lambda_\xi}{\lambda_\eta}} \frac{\sin^2 \varphi + \cos^2 \varphi}{\sin^2 \varphi + \cos^2 \varphi} \quad (17)$$

The values of q/q' are calculated for $\lambda_\xi/\lambda_\eta = 0.01 \sim 100$ and $\psi = 0 \sim 90^\circ$, and shown in Fig. 6. By the transformation to the direction of η according to Eq. (12), the heat flux across the boundary surface which makes an angle ψ with the principal axis η is modified by Eq. (17).

4-2. Examination of the correction factor q/q' by an example

The problem in Table 1, replacing the boundary condition to the constant heat flux, is considered as an example. In the same way of Sec. 3-2, the circular anisotropic temperature field and the elliptic isotropic temperature field are divided as shown in Fig. 4(a), (b) respectively. In the case of Fig. 4(b), the boundary heat flux is modified according to Eq. (17) and the temperature variations are calculated by the finite difference method. The finite diff-

for $\lambda_{\xi}/\lambda_{\eta} < 1.0$: the values of q'/q are shown in the left side.
for $\lambda_{\xi}/\lambda_{\eta} > 1.0$: the values are shown in the right side.

Fig.6 The value of q'/q .

Fig.7 The extruded cylindrical multi-crystalline carbon and the specimen for measuring the thermal diffusivity.

Table 3 Calculated results

Location	1	2	3	8	9
$\frac{a_{\xi}}{R^2} = 0.05$	1.41 1.43	1.71 1.72	2.41 2.46	0.22 0.22	0.76 0.76
0.06	1.57 1.60	1.94 1.95	2.73 2.78	0.33 0.33	0.90 0.90
0.07	1.74 1.76	2.17 2.18	3.03 3.07	0.45 0.45	1.04 1.04
0.08	1.91 1.93	2.39 2.40	3.31 3.36	0.58 0.58	1.19 1.19

where, $qR/\lambda_{\xi} = 10$. Upper figures show the results by the transformation and the modified boundary heat flux, lower figures show the results of Fig.4(a).

reference equations are shown in Appendix 1 and Appendix 2.

Transient temperature variations at the points 1 - 3, 8 and 9 in Fig.4(b) are calculated by the finite difference method using the modified boundary heat flux. The results are shown in Table 3, where, $qR/\lambda_{\xi} = 10$. Lower figures in Table 3 are the values calculated for anisotropic temperature field in Fig.4(a), and upper figures are the values of Fig.4(b). Both solutions agree well. In this calculation, truncation error of temperature is negligible [5].

Table 3 shows that an anisotropic heat conduction problem of which boundary condition is not the boundary condition of the prescribed temperature is converted into an isotropic heat conduction problem by the transformation in Sec.3-1 and the modification of the boundary heat flux in Sec.4-1.

5.Confirmation by experiments

Some typical solutions of the finite difference by the method presented above are compared with the experiments of transient heat conduction in anisotropic multi-crystalline carbon to verify the method.

The extruded multi-crystalline carbon, which is an orthotropic solid, has different thermal conductivities in the direction of extrusion and in the direction perpendicular to it. In this section, the thermal diffusivities of the anisotropic specimen are determined by measurements concerning the directions of the two-principal axes, then the experimental results of the two-dimensional transient temperature variations

in the multi-crystalline carbon are compared with the solutions of temperature variation by the finite difference method presented in this paper using the above mentioned thermal properties.

5-1.Thermal properties in the directions of two principal axes

Two kinds of cylindrical specimens which have 7 - 10mm diameter and 70 - 80mm length are cut off from the multi-crystalline carbon along the directions of two principal axes, as shown in Fig.7. Using each specimen, their thermal diffusivities in the directions of principal axes are measured by the step-wise heating method.

The experimental results of thermal diffusivities are shown in Fig.8, where a_{η} and a_{ξ} respectively are the thermal diffusivities along two principal axes (parallel and perpendicular to the direction of the extrusion). From Fig.8, the anisotropic ratio is

$$\frac{\lambda_{\eta}}{\lambda_{\xi}} = \frac{a_{\eta}}{a_{\xi}} = \frac{186 - 0.186(273 + \theta)}{173 - 0.252(273 + \theta)}$$

and near the room-temperature, it is

$$\lambda_{\eta}/\lambda_{\xi} \approx 1.34 \quad (18)$$

5-2.Experimental apparatus of temperature measurements in anisotropic specimens

A tetragonal prism of 60×60×12mm is cut off from the multi-crystalline carbon as shown in Fig.9(b) as a specimen. In this specimen, the angle between the principal axis ξ and the heating surface (one of the sides of the tetragon) is chosen to be 45°. In Fig.9(b), the numerals 1 - 5 show the locations of thermocouple (0.1mm ϕ Cu-Co) in the specimen. Poly-methyl methacrylate (PMMA) plates, the thermal conductivity of which is about one-hundredth of that of the multi-crystalline carbon, are used as insulators. Because the multi-crystalline carbon is a very good conductor of heat, utmost care is taken in order that the thermal contact resistance between the thermocouples and the specimen may be minimized.

The experimental equipment for transient heat conduction in anisotropic solids is shown schematically in Fig.10. After holding the temperature of the specimen

Fig.8 Experimental results of thermal diffusivities in the direction of principal axes.

constant in a constant temperature bath, the specimen was heated step-wise from the surface by a radiant heater, and the temperature responses in the specimen after the heating were measured.

5-3. Comparison of experimental results with solutions of the finite difference method

The transient temperature responses in the anisotropic specimen (used in Sec.5-2) are calculated by the finite difference method presented in Sec.3 and 4 to compare with the experimental results.

From Eq.(18),

$$\sqrt{\lambda_{\eta}}/\lambda_{\xi} \approx 1.18$$

And the anisotropic specimen $oabc$ in Fig.11 is transformed to an isotropic field $oa'b'c'$ of thermal conductivity λ_{η} by enlarging the space 1.18 times along the direction of principal axis ξ (perpendicular to the direction of extrusion). Two-dimensional isotropic field such as $oa'b'c'$ in Fig.11 is divided into meshes of rectangle and triangle****, and the temperature responses in the anisotropic specimen can be calculated in the transformed isotropic field $oa'b'c'$ by the finite difference method. The thermal properties and the sizes of mesh used in the calculation are shown in Table 4, and the finite difference equations are shown in Appendix 2.

The boundary conditions of field $oa'b'c'$ in Fig.11 are :
at surface oc' : prescribed temperature (measured temperature by the thermocouples at the location 1 - 3 in Fig.9(b))
at sides $c'b'$ and oa' : adiabatic (insulated surface by PMMA plate)
at bottom $a'b'$: adiabatic (insulated surface by PMMA plate).

Isothermal lines shown in Fig.12 are obtained by the above boundary conditions, where the thermal diffusivity used in this calculation is the mean value between 15°C and 30°C. The tendency that the isothermal lines incline by the anisotropy of the thermal conductivity is shown****. In Fig.13, the experimental results of the transient heat conduction at the locations **** In the case of the rhombic temperature field in Fig.11, the temperature field can be divided along the boundary only with the mesh parallel to the principal axes.

Fig.9 The specimen.

Fig.10 Experimental equipments for the measurement of temperature responses.

Fig.11 The transformed configuration of the specimen.

4 and 5 in Fig.9(b) and the finite difference solutions of the anisotropic specimen when the heating surface makes an angle $\psi = 45^\circ$ with the principal axis ξ are shown. From Fig.13, satisfactory coincidence is found between the analytical results and the experimental results. In Fig.14, the temperature differences $\Delta\vartheta_{mn}$ at the symmetric locations in the anisotropic specimen in Fig.13 are shown, and the infl-

**** In the anisotropic solids, the isothermal lines are not always perpendicular to the adiabatic surface.

Fig.12 Transient temperature distributions in the anisotropic specimen.

Fig.13 Experimental results and numerical results in anisotropic specimen

Fig.14 Temperature differences at the symmetric locations.

uence of anisotropy is indicated clearly. If the specimen is an isotropic solid, those temperature differences must be zero.

6. Conclusions

A finite difference method for heat conduction problem in the anisotropic materials is described, and comparing the numerical solutions with the experimental results, the following conclusions are drawn :

(1) For calculating the transient temperature variation in anisotropic solids by the finite difference method, transformation of the anisotropic field by Eqs.(8) and (12) to the isotropic field is shown to be recommendable, and the two-dimensional isotropic field by the above transformation should be divided into meshes along the boundary by combining the three kinds of meshes : rectangular, triangular and meshes divided by circles and radial lines; and by applying the method reported in the authors' former papers, temperature solutions of necessary accuracies can be obtained.

(2) In the heat conduction problems of boundary conditions except the prescribed surface temperature, the boundary heat flux should be modified by Eq.(17) following the above transformation.

(3) The transient temperature responses by experiments using multi-crystalline carbon, which is regarded as an orthotropic solid, agreed well with the finite difference solutions by the method presented in this paper.

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Table 4 Thermal properties and sizes of the finite difference.

λ_n / λ_ξ	1.34
$\sqrt{\lambda_n / \lambda_\xi}$	1.18
The number of division in oa'	6
The number of division in oc'	6
$\Delta \xi$	8.34mm
$\Delta \eta$	7.07mm
Δt	0.05s
Thermal diffusivity a	0.45m ² /h

Fig.15 Elements of rectangular meshes in anisotropic temperature field.

search. The computation in this research was performed by the Computation Center of Tokyo University.

Appendix 1. The explicit finite difference equation in dividing the anisotropic temperature field into meshes parallel to the principal axes

As shown in Fig.15(a), a rectangular mesh having the sides parallel to the principal axes is considered, and the explicit finite difference equation is

$$T^{(p+1)\Delta t} = T^{p\Delta t} + \sum_{i=1}^4 S_i q_i \Delta t / (c\rho \Delta \eta \Delta \xi) \dots (19)$$

where $S_1 = S_3 = \Delta \eta$, $S_2 = S_4 = \Delta \xi$.

When the element is inside the temperature field, and q_i is the heat flux from the neighbouring element, heat flux q_i is $q_i = \lambda_i (T_i^{p\Delta t} - T^{p\Delta t}) / \Delta L_i$

where $\lambda_1 = \lambda_3 = \lambda_\xi$, $\lambda_2 = \lambda_4 = \lambda_\eta$.

When the element is located at the boundary surface of constant heat flux, heat flux q_i is $q_i = \text{const.}$

For the cases of other boundary conditions, heat flux q_i becomes as shown in the former papers [2][5].

When the neighbourhood of the boundary surface is divided into elements of right-angled triangle as shown in Fig.16, the finite difference equation is

$$T^{(p+1)\Delta t} = T^{p\Delta t} + \frac{2q_1 \Delta t}{c\rho \Delta \eta \Delta \xi} + \frac{2\lambda_\xi \Delta t}{c\rho \Delta \xi \Delta L_1} (T_1^{p\Delta t} - T^{p\Delta t}) + \frac{2\lambda_\eta \Delta t}{c\rho \Delta \eta \Delta L_2} (T_2^{p\Delta t} - T^{p\Delta t}) \dots (20)$$

where $l = \sqrt{(\Delta \eta)^2 + (\Delta \xi)^2}$

The value of q in Eq.(20) is equal to that of rectangular element. For each element above, the nodal point is selected on the intersection of lines which pass midpoint of each side and are perpendicular to the side.

At the nodal point on the boundary of constant temperature, the finite difference equation is

$$T^{(p+1)\Delta t} = T^{p\Delta t} = \text{const.}$$

Arranging the right hand side of Eqs.-(19) and (20) with respect to the terms $T^{p\Delta t}$ and $T_i^{p\Delta t}$, and selecting the time difference Δt such that the coefficient of $T^{p\Delta t}$ is not negative, a stable solution is obtained.

Appendix 2. The explicit finite difference equation for the element of rectangular and triangular meshes

The explicit finite difference equation

Fig.16 Mesh elements dividing an anisotropic temperature field.

(a) Rectangular mesh

(b) Triangular mesh

ion for the element of rectangular and triangular (an acute angled or a right-angled) meshes shown in Fig.17 is

$$T^{(p+1)\Delta t} = T^{p\Delta t} + \sum_{i=1}^M S_i q_i \Delta t / (c\rho V) \dots (21)$$

where M is number of sides of each element.

The value of V is

for rectangular element: $V = S_1 \times S_2$

for triangular element: $V = \frac{1}{2} (S_1 + S_2 + S_3) l$

Each nodal point is selected on the intersection of lines which pass midpoint of each side and are perpendicular to the side.

The heat flux q_i has the following values.

(1) When the side is in contact with the neighbouring element (the nodal point temperature T_i), setting the interval of nodal point ΔL_i , the heat flux is

$$q_i = \lambda (T_i^{p\Delta t} - T^{p\Delta t}) / \Delta L_i$$

(2) When the side is a boundary of constant heat flux,

$q_i = q_0$ where q_0 is the constant value decided by the angle between the boundary surface and the direction of principal axis η .

(3) The heat flux q_i under other boundary conditions becomes as shown in the former papers [2][5].

Under the boundary condition of constant temperature, the finite difference equation is

$$T^{(p+1)\Delta t} = T^{p\Delta t} = T_0 \text{ (constant value)}$$

Arranging the right hand side of Eq.-(21) with respect to the terms $T^{p\Delta t}$, $T_i^{p\Delta t}$ ($i = 1, 2, 3, \dots, M$), and selecting the time difference Δt such that the coefficient of $T^{p\Delta t}$ is not negative, a stable solution is obtained.

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