Viscoelastic fluids such as polymer solutions show anomalous phenomena that are not observed in purely viscous fluids. In such situations, relating with heat transfer and possibility of using a hot-wire anemometer method in such fluids, the problem of the flow around a circular cylinder is a basic one to be solved.

In this report, the problem of the flow field and heat transfer of a two-dimensional steady laminar flow past a cylinder located perpendicularly to the uniform flow of a dilute polymer solution is treated analytically. From the results of the analysis the following are obtained: (1) the drag coefficient is larger than that of the solvent, (2) the separation point moves backward compared with that of the solvent because of the visco-elastic behaviour, (3) the heat transfer coefficient is small compared with that of the solvent when Reynolds number is relatively small such as for a flow past a hot-wire anemometer.

1. Introduction

Viscoelastic fluids such as high polymer solutions exhibit various anomalous phenomena that are not observed in purely viscous fluids. From a viewpoint of application and of engineering of fluid mechanics great attention is paid to these phenomena and it has become necessary to make clear such fluid flow behaviour.

Of these fluid flows, two-dimensional flow around a circular cylinder is a basic one to be analyzed with relation to various heat transfer problem and to the possibility of using a hot-wire anemometer for velocity measurement. In spite of these circumstances, such researches are few both experimentally and analytically.

On the other hand, such experimental and analytical researches on Newtonian fluid are accumulated from early times (12) some of which are experimental ones on drag coefficients of a circular cylinder located in a uniform flow by Wieselsberger (13), by Tritton in detail (12), and experimental one on stream lines etc. by F. Howan (5). Besides analytical researches on flow fields around a circular cylinder were reported by Lamb for slow flow (33) by Imai for slow and moderate speed flow (11) and by the theory with boundary-layer approximation for high speed flow (10). In recent years numerical analyses with the aid of computer are reported on intermediate speed flow (15).

Experimental researches on heat transfer from a circular cylinder located in a uniform flow are reported by many authors using air flow. Mean heat-transfer coefficients were reported by R. Hilpert (4) and others. Researches on local heat-transfer coefficients along a cylinder surface were reported by Eckert et al. for relatively slow flow (9) and by Schmidt et al. for fast flow (6). Nevertheless results in water or oil flow are few and were reported experimentally by Davis (2) and by Piert (7).

This report treats analytically two-dimensional steady laminar flow and temperature-fields around, and forced heat transfer from, a circular cylinder located in a uniform flow of a dilute polymer solution.

Nomenclature (those defined in the text may not follow the one defined below.)

\[ a \] : radius of a circular cylinder
\[ \beta_{0} \] : first kind of Rivlin-Ericksen acceleration tensor
\[ \beta_{0}^{\text{II}} \] : second kind of Rivlin-Ericksen acceleration tensor
\[ c_{d} \] : specific heat at constant pressure
\[ c_{w} \] : drag coefficient
\[ \tilde{v} \] : metric tensor
\[ h \] : mesh difference
\[ n \] : index which indicates non-Newtonian viscosity
\[ N_{c} \] : local Nusselt number
\[ N_{\tau} \] : mean Nusselt number
\[ P \] : pressure
\[ p \] : dimensionless pressure = $P/\rho U^2$
\[ P_{r} \] : Prandtl number
\[ q \] : heat flux vector
\[ r \] : dimensionless radius
\[ R \] : Reynolds number = $2aU/\nu$
\[ S \] : index which indicates elastic behaviour
\[ \tau \] : etc.: physical component of a stress
2. Basic Equations

In this report, a two-dimensional steady laminar flow around a circular cylinder located in a uniform flow of a dilute polymer solution is considered. The fluid is assumed to be incompressible, and moreover, characteristic values of the fluid are assumed not to depend on temperature variation.

Let the axis of the cylinder be at the origin, the X-axis being parallel to a uniform flow, the Y-axis being perpendicular to the X-axis, Fig.1. Equations of motion in X- and Y-directions are respectively

\[ \rho \left( \frac{\partial U}{\partial X} + \nu \frac{\partial U}{\partial Y} \right) = -\frac{\partial P}{\partial X} + \frac{\partial T_{XX}}{\partial X} + \frac{\partial T_{XY}}{\partial Y} \]  
\[ \rho \left( \frac{\partial V}{\partial Y} + \nu \frac{\partial V}{\partial X} \right) = -\frac{\partial P}{\partial Y} + \frac{\partial T_{XY}}{\partial X} + \frac{\partial T_{YY}}{\partial Y} \]

According to the Denn model which represents the rheological behaviour of viscoelastic fluids rather well, deviatoric stress tensor \( T' \) is expressed as

\[ T' = \mu(\nabla U + \nabla U^T) - \frac{2}{3} \mu \nabla \cdot U \nabla U \]

where \( B_{00}^N \) and \( B_{00}^N \) are the first and second kind of Rivlin-Ericksen acceleration tensors respectively. For incompressible fluids, \( B_{00}^N \) and \( B_{00}^N \) become

\[ \dot{B}_{00} = \frac{\partial \varepsilon_{e}}{\partial t} + \varepsilon_{e} \dot{\varepsilon}_{e} \]

\[ \dot{B}_{00} = \frac{\partial \varepsilon_{e}}{\partial t} + \varepsilon_{e} \dot{\varepsilon}_{e} \]

\[ \ddot{B}_{00} = \frac{\partial \varepsilon_{e}}{\partial t} + \varepsilon_{e} \dot{\varepsilon}_{e} \]

\[ s \] is a second invariant of \( B_{00} \) and is defined as

\[ S = \frac{1}{2} \dot{B}_{00} \]

Equations (7) and (8), hence the Denn model (3) contains four material constants \( \mu, \lambda, n \) and \( s \). Of these constants, \( n \) and \( s \) take values in general in the following ranges:

\[ 0 < s < 1, \ 0 < s < 2 \]

In the case of a dilute polymer solution it is appropriate to assume \( n=1 \) (moreover when \( \lambda=0 \), the Denn model represents Newtonian fluids) and \( s=2 \) (over an appropriate shear rate). In this report calculation is limited to the case \( s=2 \), considering the capacity of computer and computation time. In the case \( n=1 \) and \( s=2 \), Eq. (3) becomes

\[ T' = \mu \dot{B}_{00} - \lambda B_{00}^N \]

Equation of continuity is

\[ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \]

And an energy equation for a steady flow becomes

\[ \rho c ( U \dot{\theta} + V \dot{\theta}) = \frac{1}{2} \nabla T = \frac{1}{2} \frac{d \nabla}{d t} \]

When temperature difference between the fluid and the cylinder surface is appreciable as in the case of using a hot-wire velocimeter and the free stream velocity is not so large (e.g. below 5 m/s in water), it is considered that Eckert number \( E = \frac{\dot{U} \dot{\theta} \Omega}{\dot{U} \dot{\theta} \Omega} \leq 1 \)

For this reason a dilute polymer solution with relatively weak viscoelastic behaviour the following relation holds:

\[ B_{00} T < \rho \dot{U} (s - \dot{\theta}) \]

and the second term of right hand side of Eq. (11) can be neglected. Fourier's law

\[ q = \kappa \nabla \theta \]

is adopted for heat flux and if dependence on temperature variation for \( \kappa \) is neg-
\[ u = \frac{\phi'}{\alpha}, \quad v = \frac{\theta'}{\alpha} \]  

where \( \alpha = \lambda^2/\rho a^2 \), \( \beta = 2\mu a/\nu \), \( \psi = \nu/\rho \). Similarly, dimensionless form of Eq. (13) is (if surface temperature is not constant, suitable value may be substituted for \( \Theta \)).

\[ \frac{\partial^2 \psi}{\partial \xi^2} + \frac{b \psi}{\partial \eta} = \frac{2}{R_0^2} \frac{\partial \psi}{\partial \eta} \]  

The following analyses are divided into three parts, i.e. in the case of low Reynolds number, moderate Reynolds number and high Reynolds number. For such purpose, the flow behaviour far from the cylinder must be known. When transformation

\[ x = z = \exp(z(t)) \quad r = \exp(z(t)) \]  

is introduced, asymptotic expansions of \( \psi \) and \( \xi \) for large \( r \) become 

\[ \psi = y + \frac{e^z}{\sqrt{2\pi}} \text{erf}(z) - \frac{e^z R \sqrt{2\pi}}{16 \sqrt{\pi}} e^{\frac{1}{2} \frac{e^z}{\sqrt{2\pi}}} (\text{erf}(\sqrt{2\pi}z) - 1) - e^z \text{erf}(Q) + \frac{e^z}{\sqrt{2\pi}} Q \]  

where

\[ R = \sqrt{0.5 \frac{h}{a \sin(0.5 \xi)}, \quad e = \theta = \frac{h}{a \sin(0.5 \xi)}} \]  

Although these expressions correspond to Newtonian fluid flow, in the case \( e \approx 0 \) these formulae are similar at least up to the second term of the right hand side of Eq. (22) and the first of Eq. (23). For this reason these expressions may be introduced in the case of flow of a dilute polymer solution. (The effect of \( \epsilon \) is included in \( c_0 \) implicitly.)

Since the surface temperature is frequently constant as for the hot-wire velocimeter, the assumption that the surface temperature is constant (i.e. \( \Theta \) const.) is adopted. For temperature behaviour far from the cylinder, first approximate solution is obtained with the perturbation method (infinite series are approximated by erf- and tanh-functions). Then asymptotic solution is

\[ \psi = y + \frac{e^z}{\sqrt{2\pi \pi}} \text{erf}(z) - \frac{e^z R \sqrt{2\pi}}{16 \sqrt{\pi}} e^{\frac{1}{2} \frac{e^z}{\sqrt{2\pi}}} (\text{erf}(\sqrt{2\pi}z) - 1) - e^z \text{erf}(Q) + \frac{e^z}{\sqrt{2\pi}} Q \]  

where

\[ R = \sqrt{0.5 \frac{h}{a \sin(0.5 \xi)}, \quad e = \theta = \frac{h}{a \sin(0.5 \xi)}} \]  

\[ \beta = 1 + \frac{\mu a}{\gamma}, \quad A_n = (2n+1)R \frac{h}{a \sin(0.5 \xi)} + A_n \]  

\( N \) indicates mean Nusselt number and Nusselt number based on a radius \( a \) is defined as

\[ N_a = \frac{\beta}{\beta_a} \]

Boundary conditions are for the velocity field

\[ u = v = 0 \quad (r = 0) \quad \psi = 0 \quad (r = \infty) \]

and for the temperature field

\[ \psi = 0 \quad (r = 0) \quad \psi = 0 \quad (r = \infty) \]

At the surface the following relations hold:

\[ \begin{align*}
\frac{\partial \psi}{\partial r}, & = 0, \quad \frac{\partial \psi}{\partial r}, & = 0, \quad \{T_{\delta}\}, = 0, \quad \{T_{\gamma}\}, = 0 \\
\frac{\partial \psi}{\partial r}, & = 0, \quad \frac{\partial \psi}{\partial r}, = \frac{2}{R_0} \sin(\gamma), \quad \{
\end{align*} \]

Hence the coefficients of pressure drag \( c_{dp} \) and of friction drag \( c_{dd} \) become

\[ c_{dp} = -\frac{1}{R_0^2} \int_0^\infty \sin(\gamma) d\eta d\gamma - \frac{1}{R_0^2} \int_0^\infty \cos(\gamma) d\eta d\gamma \]

\[ c_{dd} = -\frac{1}{R_0^2} \int_0^\infty \sin(\gamma) d\gamma d\eta = -\frac{1}{R_0^2} \int_0^\infty \cos(\gamma) d\gamma d\eta \]

respectively. And the coefficient of drag \( c_d \) becomes

\[ c_d = c_{dp} + c_{dd} = \frac{4}{R_0^2} \int_0^\infty \sin(\gamma) d\gamma d\gamma - \int_0^\infty \cos(\gamma) d\gamma d\eta \]
3. Analysis

3.1 Case of low Reynolds number

In this case the solution may be obtained by direct numerical computation of Eq. (17), but here the solution is obtained with Oseen's approximation. Let the difference between u and free stream velocity $u^*$ be $u$ and neglect the product $u$ and $v$ etc. as higher order infinitesimals in the equations of motion. Then the equations of motion become

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The solutions of Eqs. (28) - (30) are expressed as

$$u = -ax \frac{\partial}{\partial x} \phi + 2 \frac{\partial}{\partial y} \phi$$

$$v = -bx \frac{\partial}{\partial x} \phi + \frac{\partial}{\partial y} \phi$$

$$\rho = \frac{\partial}{\partial y} \phi$$

where

$$\phi = 0$$

$$\frac{\partial}{\partial y} \phi = 0$$

$$\frac{\partial}{\partial x} \phi = 0$$

With the transformation $x = r \gamma$, Eq. (35) becomes

$$a(x - 2a) \frac{\partial}{\partial x} \phi + (2ax - 3a) \frac{\partial}{\partial y} \phi = 0$$

where suffix x indicates partial differentiation with respect to x. Since it is rather difficult to get an analytical solution of Eq. (36), the approximation $f_{\infty} = 0.5 f_0$ may be introduced. This means that the operation is averaged with respect to the angle when f is a function of r only. Moreover the same assumption is appropriate when $\gamma \leq 1$. Hence Eq. (36) becomes

$$a(x - 2a) \frac{\partial}{\partial x} \phi + (2ax - 3a) \frac{\partial}{\partial y} \phi = 0$$

Choosing a such that

$$a(x - 2a)^{1+3} = (2a - 3a)^{1-1}$$

we obtain

$$f = 0$$

where

$$\beta = (x - 2a) \frac{\partial}{\partial x} \phi + (2ax - 3a) \frac{\partial}{\partial y} \phi$$

$$a(x - 2a) \frac{\partial}{\partial x} \phi + (2ax - 3a) \frac{\partial}{\partial y} \phi = 0$$

Since $f = 0$, the solution of Eq. (38) is expressed as

$$f \propto \text{Bessel}(\beta)$$

where $\text{Bessel}(\beta)$ is a modified Bessel function of zeroth order. From the fact that $\alpha \propto (1/n)$, and $\beta \propto (1/n)$, Laurent expansions of u/C and v/C with respect to Re are (in ascending power of Re and up to the zeroth)

$$u = \frac{\alpha}{C} \frac{1}{(1/n) R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]$$

$$v = \frac{\alpha}{C} \frac{1}{(1/n) R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right]$$

Under the boundary condition ($r = 1$: $u = -1$ and $v = 0$), the following are sufficient in the range cited before:

$$\phi = \frac{1}{R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]$$

$$\frac{\partial}{\partial y} \phi = 0$$

Near the cylinder

$$\frac{\partial}{\partial x} \phi = \frac{1}{R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right]$$

Vorticity $\zeta$ becomes

$$\zeta = \frac{3}{2} \frac{1}{R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]$$

And the drag coefficient $d_s$ is

$$d_s = \frac{1}{2} \frac{1}{R_i} \int_0^{2\pi} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} d\theta$$

$$\int_0^{2\pi} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} d\theta$$

Equation (48) is expressed in the following form (i.e. asymptotic expansion with respect to $Re = 0$ and power series expansion with respect to $\epsilon$):

$$r = \frac{1}{R_i} \left[ \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]$$

$$\frac{\partial}{\partial y} \phi = 0$$

For temperature field, transformation of Eq. (20) into $(\xi, \eta)$-coordinates gives

$$\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} = \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial \xi} \phi \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \eta} \phi \right)$$

Since solving this linear partial differential equation analytically is rather difficult, numerical method is applied to it. Derivatives are approximated by difference method (mainly based on intermediate difference) and $h$ denotes mesh interval. If we select the closing error as $O(h^2)$, then the 11th mesh point (i, j)

$$\int_{i-1}^{i+1} \frac{\partial^2 \theta}{\partial \xi^2} = \frac{1}{2} \left( \frac{\partial \theta}{\partial \xi} \right)_{i+1} + \frac{1}{2} \left( \frac{\partial \theta}{\partial \xi} \right)_{i-1}$$

$$\int_{j-1}^{j+1} \frac{\partial^2 \theta}{\partial \eta^2} = \frac{1}{2} \left( \frac{\partial \theta}{\partial \eta} \right)_{j+1} + \frac{1}{2} \left( \frac{\partial \theta}{\partial \eta} \right)_{j-1}$$

and so on. Boundary conditions are

$$\xi = 0: \frac{\partial \theta}{\partial \eta} = 0 \quad (\text{symmetry})$$

$$\eta = 0: \frac{\partial \theta}{\partial \xi} = 0 \quad (\text{symmetry})$$
In this section $\omega=40$ and $\xi=29h$. However since it is almost impossible due to the capacity and computation time of the computer to solve the simultaneous equations derived from Eq. (50) by difference method, we consider the following equations instead:

$$\tilde{\theta}(i,j+1)-\tilde{\theta}(i,j)+\tilde{\theta}(i,j-1)$$

$$=\frac{\partial}{\partial t} \left[ \tilde{\theta}(i+1,j)-\tilde{\theta}(i-1,j) \right]$$

$$-\frac{\partial^2}{\partial x^2} \left[ \tilde{\theta}(i,j+1)-\tilde{\theta}(i,j-1) \right]$$

$$-\frac{\partial^2}{\partial x^2} \left[ \tilde{\theta}(i+1,j)-\tilde{\theta}(i-1,j) \right]$$

(51)

where $2i+1, 2j+1$. Similar equations are used at $j=1$ and $j=41$ under boundary conditions. Equation (51) is solved by the accelerated line Liebmann method. The process is as follows. First assume suitable value for $\tilde{\theta}$, and calculate $\tilde{\theta}$ according to Eq. (24). Secondly suitable values for $\theta$ are assumed at all mesh points. Next for fixed $i$, we calculate the right hand side of Eq. (51), solve then for $\tilde{\theta}(i,j)$ ($j=1,2,41$) and replace the value $\theta(i,j)$ with the value

$$\tilde{\theta}(i,j)+\omega(\tilde{\theta}(i,j)-\theta(i,j))$$

where $\omega$ is an acceleration factor and in this section $\omega=1$. After one cycle of iteration $\omega=2,3,4,5,29,28,27,26,25$, $N_0$ is computed and $\tilde{\theta}$ is modified according to Eq. (24). This cycle is then repeated until values of $\tilde{\theta}$ converge. Criteria for convergence are as follows:

- necessary condition:
  $$|\tilde{\theta}(i,j)-\tilde{\theta}(i,j)|<0.01$$

- convergence condition:
  $$\text{relative variation of } N_0 \leq 0.02$$

Local Nusselt number $N$ becomes according to a forward difference formula

$$N=\frac{1}{2h} \left[ \tilde{\theta}(2,j)-\tilde{\theta}(3,j)-\tilde{\theta}(1,j) \right]$$

(52)

3.2 Case of moderate Reynolds number

When $1 \leq Re_{c1}<1000$, application of Oseen's approximation is not valid and application of boundary-layer approximation near the surface of the cylinder is not appropriate either. Hence the velocity field is obtained through numerical solution of Eqs. (17) - (19) by difference approximation. Equations used are from Eqs. (17) and (18)

$$\frac{\partial}{\partial x} \left( \frac{\partial \tilde{\theta}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \tilde{\theta}}{\partial y} \right)$$

$$= \frac{\partial}{\partial t} \left[ \tilde{\theta}(i+1,j)-\tilde{\theta}(i-1,j) \right]$$

$$-\frac{\partial^2}{\partial x^2} \left[ \tilde{\theta}(i,j+1)-\tilde{\theta}(i,j-1) \right]$$

$$-\frac{\partial^2}{\partial x^2} \left[ \tilde{\theta}(i+1,j)-\tilde{\theta}(i-1,j) \right]$$

(53)

$$\tilde{\theta}(i,j)$$

Boundary conditions are

$$\tilde{\theta}(x=0) = \tilde{\theta}(x)$$

$$\tilde{\theta}(y=0) = \tilde{\theta}(y)$$

$$\tilde{\theta}(x=\text{symmetry of flow})$$

$$\tilde{\theta}(y=\text{symmetry of flow})$$

Solving procedure by the accelerated line Liebmann method is the same as in 3.1, that is, we regard the left-hand sides of Eqs. (53) and (54) as unknown (partly known) and the right-hand sides as known. Here $h=40$ and $\xi=30h$. The order of computation is reverse to that of temperature field, that is, the calculation is iterated with the order $i=36, \ldots, 4, 3$ and after $i=3, \varphi_0(2, j)$ is computed under the boundary condition, then the calculation is continued with the order $i=4, \ldots, 36$. After this one cycle we calculate vorticity over the surface with Eq. (53), next calculate $\varphi_c$ and modify $\varphi$ and $\xi$. This process is a little different from that of Takami et al. (53). The acceleration factors $\omega_\varphi, \omega_\xi$ for $\varphi, \xi$ lie in the range

$$0<\omega_\varphi<0.5, \quad 1.0<\omega_\xi<1.5$$

Criteria for convergence are as follows:

- necessary condition:
  $$|\tilde{\theta}(i,j)-\tilde{\theta}(i,j)|<0.001$$

- convergence condition:
  $$\text{relative variation of } \varphi_c \text{ in 20 cycles}<0.002$$

3.3 Case of high Reynolds number

Even if we adopt the difference method for equations of motion, difficulties arise from the consideration for wake flow, and so the velocity field is obtained through boundary-layer approximation. Only in this section let the forward stagnation point be at the origin, $x$-axis being along the surface of the cylinder, $y$-axis being perpendicular to the $x$-axis. When $\varphi$ and $\psi$ denote physical components of velocity in $x$- and $y$-directions respectively and an effect of curvature is neglected, the boundary-layer equation becomes

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = 0$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y}$$

(55)
\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = - \nabla p + \mu \Delta u + \mu \frac{\partial (T_r - T_{av})}{\partial y} + \frac{\partial}{\partial y} T_{av} \frac{d}{dx} P_r \]  

With setting \( \lambda = \mu' \) in Eq. (55), we obtain

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = - \nabla p + \mu \Delta u + \mu \frac{\partial (T_r - T_{av})}{\partial y} + \frac{\partial}{\partial y} T_{av} \frac{d}{dx} \mu' \]  

Where \( U_0(x) \) denotes the velocity outside the boundary layer. In a two-dimensional flow, from the equation of continuity, the following holds:

\[ \frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} \right) = 0 \]  

When boundary-layer approximation is applied to \( \theta = 0 \), we obtain

\[ \frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} \right) = 0 \]  

Hence laminar boundary-layer approximation gives

\[ \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \frac{\partial}{\partial y} \left( \mu' \frac{d}{dx} \right) \]  

Integration of Eq. (59) from 0 to \( h \) (dh/dx) 0 with respect to \( y \) gives

\[ \int_0^h \left[ u - u_0 \right] dy = \int_0^h \left[ u - u_0 \right] dy = \int_0^h \frac{\partial}{\partial y} \left( \mu' \frac{d}{dx} \right) dy \]  

Where \( \tau_0 \) means the shear stress on the surface:

\[ \tau_0 = \frac{\partial u}{\partial y} \]  

When elastic behaviour of the solution is not so great as in a dilute polymer solution, the second term of the right-hand side of Eq. (60) may be estimated from the analogy of Newtonian fluid flow. Hence the relation

\[ \int_0^h \frac{\partial u}{\partial y} dy = \frac{1}{2} \int_0^h \left( U_0^2 - u^2 \right) dy \]  

May hold. With this approximation (e.g. when \( \epsilon \approx 0.01 \)) we obtain

\[ \frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \frac{\partial}{\partial y} \left( \mu' \frac{d}{dx} \right) \]  

Velocity \( u \) is assumed to be of the form

\[ u = u_0 f(y_1(x), z) = u_0 f(y_1(x), z) = \theta(x, z) \]  

Boundary conditions used here are

\[ f(0, z) = 0, \quad f(1, z) = 1, \quad \frac{\partial f}{\partial y}(1, z) = \frac{\partial f}{\partial y}(1, z) = 0 \]  

Since elastic behaviour of a dilute polymer solution is relatively weak, the following is assumed:

\[ \mu' \frac{\partial u}{\partial y} \]  

Hence we obtain

\[ \frac{\partial}{\partial x} f(x, z) = - \frac{\partial}{\partial x} \mu' \frac{\partial u}{\partial y} \]  

\[ f(x, z) = \mu' \frac{\partial u}{\partial y} \]  

From the form of the boundary conditions, the following is assumed:

\[ f(x, z) = P(x) + \theta G' \]  

\[ F(0) = G(0) = 0, \quad F'(0) = -1, \quad G'(0) = 0 \]  

\[ F(1) = 1, \quad G'(1) = 0 \]  

Hence Eq. (63) becomes

\[ \psi'(0.5 A' + 1.5 A' \psi + 2.5 A' \psi') + \psi F'(0) + \theta F(0) + \psi A' \]  

\[ + 2 \psi A' + \theta A' + \theta A' \psi A' \]  

\[ - \frac{1}{2} \psi A' \psi A' \psi A' = 0 \]  

At \( x=0 \), \( \psi = 0 \) and

\[ - \frac{1}{2} \psi \psi' \psi + \theta A' \psi + \theta A' \psi \psi' + \theta A' \psi \]  

\[ + \theta A' \psi \psi' + \theta A' \psi \]  

Where

\[ A' = \int_0^1 \left( 1 - F \right) dA, \quad A'' = \int_0^1 \left( 1 - 2F \right) dA \]  

\[ A''' = \int_0^1 \left( 1 - 3F \right) dA, \quad A^{(4)} = \int_0^1 \left( 1 - 4F \right) dA \]  

Moreover if relatively minute terms (having a coefficient \( \lambda \) in Eq. (70) omitted, Eq. (70) becomes

\[ \psi'(0.5 A' + 1.5 A' \psi + 2.5 A' \psi') + \psi F'(0) + \theta F(0) + \psi A' \]  

\[ + 2 \psi A' + \theta A' + \theta A' \psi A' \]  

\[ - \frac{1}{2} \psi A' \psi A' \psi A' = 0 \]  

Temperature field: Since it is expected that the temperature field will show an aspect of a boundary-layer type, dimensionless temperature \( \theta \) becomes in boundary-layer approximation

\[ \theta = \frac{\partial u}{\partial y} \]  

Integrating Eq. (73) from 0 to \( \infty \) with respect to \( y \), we obtain

\[ \int_0^\infty \left( \frac{\partial u}{\partial y} \right) dy = - \frac{\partial}{\partial y} \left( \mu' \frac{d}{dx} \right) \]  

The stream function \( \psi \) and dimensionless temperature \( \theta \) are nearly \( \xi = 0 \) expanded in the following forms:

\[ \psi = \psi F'(0) + \psi F(0) \]  

\[ \theta = \theta(0) - \frac{1}{2} \theta(0)^2 F'(0) + \frac{1}{2} \theta(0)^2 F'(0) \]
(The preceding two equations depend on the coordinate system Eq.(21).) Therefore on the analogy of a thermal boundary-layer along a flat plate, \( \theta \) is assumed to be
\[
\theta = 1 - 2x + 2x^2 - \pi x^3 (0 \leq x \leq 1, \pi x = \varphi / \theta)
\]  

From Eq. (75), Eq. (74) becomes
\[
\left\{ \begin{array}{l}
\psi + \frac{1}{2} \frac{U_1^2}{U_0} \frac{\partial \psi}{\partial x} = P_1(x) \frac{\partial \psi}{\partial x} \\
\frac{1}{U_0} \frac{\partial P_1(x)}{\partial x} + \frac{1}{2} \frac{U_1^2}{U_0^2} = -\frac{1}{P_1(x)} \frac{\partial P_1(x)}{\partial x}
\end{array} \right.
\]  

where \( \lambda = \pi \theta \).

\[
P_1(x) = x \int_{x_0}^{x} \psi(x') \beta(x') dx'
\]

\[
P(x) = x \int_{x_0}^{x} g(x') \beta(x') dx'
\]

Initial conditions are (when \( x = x_0 \) and \( \varphi = \varphi_0 \) at \( x = 0 \))
\[
\left[ \begin{array}{c}
\psi \\
P \\
\end{array} \right]_{x=x_0} = \left[ \begin{array}{c}
\psi_0 \\
P_0 \\
\end{array} \right]
\]  

The local Nusselt number \( \text{Nu} \) (based on a radius \( a \)) is given by
\[
\text{Nu} = \frac{2 \pi a}{X} \frac{2}{\pi a} \left( \frac{1}{\pi a} \right)^{1/2} \left( \frac{\partial \psi}{\partial a} \right)_{a}
\]

A numerical example is given by the following conditions:

\[
U_0 = 20, \quad \alpha = \sin(a/\theta), \quad \epsilon = \pi a R
\]

\[
F = 200, \quad \pi a R = 2 + 2.5 \pi a R, \quad G = 4(1 - \pi a R)^2
\]

And the function \( P_1(x) \) etc. become
\[
P_1(x) = \left\{ \begin{array}{l}
\frac{2}{15} x^3 + \frac{3}{140} x^4 + \frac{1}{180} x^5 (0 \leq x \leq 1) \\
\frac{3}{15} x^3 + \frac{2}{15} x^4 + \frac{1}{140} x^5 + \frac{1}{180} x^6 (1 \leq x)
\end{array} \right.
\]

\[
P(x) = \left\{ \begin{array}{l}
\frac{1}{80} x^3 + \frac{1}{840} x^4 + \frac{1}{1680} x^5 (0 \leq x \leq 1) \\
\frac{1}{2} x^3 + \frac{1}{180} x^4 + \frac{1}{840} x^5 + \frac{1}{3024} x^6 (1 \leq x)
\end{array} \right.
\]

\[
\text{Nu} \times 10^{-10} = 2 X^2 (\psi - \cos(a/\theta))^{1/2}
\]

4. Results of the analysis

Stream lines and equi-vorticity lines (Re=20, \( \epsilon = 0 \)) are shown in Fig. 2. And relative stream lines are shown in Fig. 3. Vorticity distributions along the surface of the cylinder, which contribute greatly to the drag coefficient, are shown in Fig. 4 (in the case Re=20) and in Fig. 5 (in the case Re=1). In these cases due to the viscoelastic behaviour of the solution the absolute value of vorticity near the front portion of the cylinder increases compared with that of a solvent. In the case Re=20, the separation point is moved backward due to the viscoelastic behaviour (at the separation point \( \zeta = 0 \)). In Fig. 6 velocity distribution at the section ( \( \psi = \text{const} \) ) are shown (Re=20). From these it is found that near the cylinder the circumferential component of velocity increases and a little decreases far away compared with that in Newtonian fluid flow. In Fig. 7 ratios of drag coefficients to those in Newtonian fluid (solvent) flow are shown in the case Re=1 and in Fig. 8 drag coefficients are shown in the case Re=20.
These show the increment of drag due to the viscoelastic behaviour and are consistent with experimental results by Acosta et al. In Fig. 9 values of \( c_0 \) are shown under the assumption \( c=0 \). The difference between Eqs. (48) and (49) is hardly remarkable. The temperature fields in Newtonian fluid (i.e., \( c=0 \)) are shown in Figs. 10-12. In Fig. 13 isotherms photographed by Eckert et al. are shown (in air, \( Re=1260 \)). In the flows with separation (i.e., in Figs. 10 and 13) analytical and experimental results may have the same trend. It is also shown that heat transfer by convection is relatively violent outside the separating streamline and that heat transfer by conduction is violent in the front portion of the cylinder. In order to show the effect of viscoelas-

![Fig. 7 Drag coefficient (Re=20)](image)

![Fig. 8 Drag coefficient (Re=20)](image)

![Fig. 9 Drag coefficient](image)

![Fig. 10 Isotherms (Re=20, c=0, Pr=7.01)](image)

![Fig. 11 Isotherms (Re=1, c=0, Pr=7.01)](image)

![Fig. 12 Isotherms (Re=0.5, c=0, Pr=7.01)](image)

![Fig. 13 Isotherms (from an interference photograph by E. Eckert and E. Soehngen, Re=1260)](image)

![Fig. 14 Local Nusselt number (Re=20, Pr=7.01)](image)

![Fig. 15 Local Nusselt number (Re=20, Pr=7.01)](image)

![Fig. 16 Local Nusselt number (Re=1, Pr=7.01)](image)
ticity on heat transmission, distributions of local Nusselt number in the cases \( R_e \leq 1 \), \( R_e = 20 \), \( R_e > 1 \) (Pr=7.01) are shown in Figs. 14, 15, 16 respectively. Fig. 14 shows the reduction of heat transmission due to viscoelastic behaviour, which has the same trend as in the experimental results by Acosta et al.\(^{10}\) When \( R_e = 20 \), the variation of heat transmission is not necessarily clear, but heat transmission will vary to some extent when \( \varepsilon \) increases. The circumstance \( R_e > 1 \) will appear when \( a \gg \sqrt{\frac{1}{\rho}} \) and in this case heat transmission does not vary a great deal compared with that in Newtonian fluid flow, which is coincident with the prediction by Metzner et al.\(^{10}\). Next in order to show the effect of characteristic values on heat transmission, the aspects of \( Nu \) (\( \varepsilon = 0 \)) with variation of \( Pr \) are shown in Figs. 17, 18, 19. The variation of characteristic values will produce comparable effects with those due to viscoelasticity. In these circumstances, when \( R_e > 1 \), characteristic values will be converted at the temperature \( \Theta = 0.5(\Theta_0 + \Theta_e) \) and when \( 0 \leq R_e \leq 1 \) for the velocity field and \( Pr \) for the temperature field and when \( R_e > 1 \), they will be converted at \( \Theta = \Theta_0 \).

From a viewpoint of these, experimental and analytical results for water are shown in Fig. 20. In the case \( R_e > 1 \), boundary-layer thickness and flow measure at the forward stagnation point are shown in Figs. 21, 22 respectively.

5. Conclusions

The problem of a flow field around and heat transfer from a circular cylinder located perpendicularly in a two-dimensional steady laminar flow of a dilute polymer solution is treated analytically. As the result it is found that due to the viscoelastic behaviour of the solution the drag coefficient increases compared with that in a solvent and that a separation point is moved a little backward if it exists. And it is predicted that in the case of relatively low Reynolds number such as in using a hot-wire velocimeter heat transmission decreases, whereas in the case of high Reynolds number it does not vary a great deal.

Since in a dilute polymer solution the dependence of relative heat transmission rate (to that in a solvent) on the kind and concentration of polymers varies according to Reynolds number, in the case of using a hot-wire velocimeter it is necessary to correct the relation between standard velocity and output at many standard values in each solution.

Numerical computation is executed with HITAC 8500/8400 computer at Tokyo Institute of Technology.

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