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Numerical Analysis of Unsymmetrical Bending of Visco-Elastic Shells of Revolution*

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This paper is concerned with a numerical analysis of visco-elastic shells of revolution under unsymmetrical loads. The numerical method of elastic analysis of shells of revolution developed by Budiansky is extended to deal with a wide range of visco-elastic problems of the quasi-static type. The visco-elastic behaviour of a material is represented by a 4-parameter model. As a numerical examples two problems of a cylindrical shell and a conical drum composed of multishells subjected to stepwise displacements with respect to time are analyzed.

1. Introduction

It is important in engineering design of machinery and structures made of plastics, concrete, etc. that the visco-elastic behaviour in shell structures is studied. In this paper as one problem of the visco-elastic behaviour in shell structures the axi-symmetrical shells under unsymmetrical loads of importance in practical use are discussed. As to the visco-elastic analysis of an axi-symmetric problem, of shells there are many investigations.⁽¹⁾⁻⁽⁵⁾ Few studies of the unsymmetrical problems, however, are reported, in spite of importance in engineering.⁽⁶⁾ On the other hand the elastic analysis of unsymmetrical bending of shells of revolution has often been made as seen in the numerical method by Budiansky,⁽⁷⁾ by Kalnins,⁽⁸⁾ and so on. Although the visco-elastic behaviour is non-linear with respect to time, the relations between the increments on each calculating stage may be regarded as linear. Therefore the fundamental relations between the increments can be solved by applying the method of elastic analysis. In the present paper the numerical method of elastic analysis by Budiansky is extended to deal with the visco-elastic problems. The visco-elastic behaviour of materials is described using a 4-parameter model. But the numerical method is easily applicable for the materials represented by a finite series of Kelvin elements coupled with the 4-parameter model.

As numerical examples two problems of a cylindrical shell and a conical drum composed of multishells subjected to stepwise displacements with respect to time are treated and the variations of stresses and deformations with time are shown.

2. Nomenclature (refer to Figs.1-3)

a : reference length, b : $1/(1-\nu^2)$,
 d : $1/12(1-\nu^2)$, E, E_i : Young's modulus,
 F : force, h : thickness of shell,
 M : moment, M_i, M_0, M_{i0} : bending moments
per unit length (m_i, m_0 : Fourier coefficients for M_i, M_0 , and so forth), $\bar{M}_{i0}$:
modified twisting moment per unit length ($\bar{m}_{i0}$),
 N_i, N_0, N_{i0} : membrane force per unit length (n_i, n_0), $\bar{N}_{i0}$: modified membrane
shear force per unit length ($\bar{n}_{i0}$), $\hat{N}_{i0}$:
effective membrane shear force per unit length in boundary ($\hat{n}_{i0}$), P_i, P_0, P_c : shell
loads per unit area (p_i, p_0, p_c), Q_i, Q_0 :
transverse shear force per unit length (q_i, q_0),
 $\hat{Q}_i$: effective transverse shear force per unit length in boundary ($\hat{q}_i$),
 R_i, R_0 : radii of curvature, r : distance from shell to axis, s : meridional shell
coordinate, $t, \Delta t$: time and increment of time, respectively, U_i, U_0, U_c : displacements (u_i, u_0, u_c),
 γ : ρ'/ρ , Δ : interval of mesh points, $\varepsilon_i, \varepsilon_0, \varepsilon_{i0}$: strains,
 $\varepsilon_{im}, \varepsilon_{om}, \varepsilon_{i0m}$: membrane strains ($\varepsilon_{im}, \varepsilon_{om}, \varepsilon_{i0m}$),
 η_i, η_0 : viscosity coefficients, K_i, K_0, K_{i0} : bending distortions (k_i, k_0, k_{i0}),
 ν : Poisson's ratio, ξ : nondimensional meridional coordinates ($=s/a$), ξ, θ, ζ : coordinates,
 ρ : r/a , σ_0 : reference stress, $\sigma_i, \sigma_0, \sigma_{i0}$: stresses (s_i, s_0, s_{i0}),
 Φ_i, Φ_0 : rotations (φ_i, φ_0), ω_i, ω_0 : $a/R_i, a/R_0$, subscript, a : refers to visco-elasticity, $\cdot$: $\partial/\partial t$, $\prime$: $d/d\xi$

3. Analytical Formulations

3.1 Fundamental equations

If the middle surface of shells of revolution is given by $r=r(s)$, where r is the distance from the axis and s is the meridional distance measured from a boundary along the middle surface, the relations between the nondimensional curvatures $\omega_i (=a/R_i)$, $\omega_0 (=a/R_0)$ and the nondimensional radius $\rho (=r/a)$ become as follows

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equations of the 4-parameter model in uni-axial stress are written as follows in the incremental form.

$$\left. \begin{aligned} \dot{\sigma} &= E_s \dot{\epsilon} - \dot{\sigma}_s, \quad \dot{\sigma}_s = \left(\frac{E_s}{\eta_s} + \frac{E_s}{\eta_i} \right) \sigma - \frac{E_s E_i}{\eta_i} \epsilon_i \\ \epsilon_i(t) &= \frac{\sigma}{E_i} (1 - e^{-(E_i/\eta_i) \Delta t}) + \epsilon_i(t - \Delta t) e^{-(E_i/\eta_i) \Delta t} \end{aligned} \right\} \quad (8)$$

where E_s , E_i and η_s , η_i are elastic moduli of the spring and viscosity coefficients of the dashpot, respectively. If Eqs.(8) are extended to the plane stress state assumed in the ordinary shell theory under the assumptions that the solid is isotropic and that Poisson's ratios for elastic deformations and viscous deformations are identical and constant, then Eqs.(8) become as follows

$$\left. \begin{aligned} \begin{Bmatrix} \dot{\sigma}_\xi \\ \dot{\sigma}_\theta \\ \dot{\sigma}_{\theta\theta} \end{Bmatrix} &= E_s \bar{\nu} \begin{Bmatrix} \dot{\epsilon}_\xi \\ \dot{\epsilon}_\theta \\ \dot{\epsilon}_{\theta\theta} \end{Bmatrix} - \begin{Bmatrix} \dot{\sigma}_{\xi s} \\ \dot{\sigma}_{\theta s} \\ \dot{\sigma}_{\theta\theta s} \end{Bmatrix} \\ \begin{Bmatrix} \dot{\sigma}_{\xi s} \\ \dot{\sigma}_{\theta s} \\ \dot{\sigma}_{\theta\theta s} \end{Bmatrix} &= E_s \left(\frac{1}{\eta_s} + \frac{1}{\eta_i} \right) \begin{Bmatrix} \sigma_\xi \\ \sigma_\theta \\ \sigma_{\theta\theta} \end{Bmatrix} - \frac{E_s E_i}{\eta_i} \bar{\nu} \begin{Bmatrix} \epsilon_{\xi i} \\ \epsilon_{\theta i} \\ \epsilon_{\theta\theta i} \end{Bmatrix} \\ \begin{Bmatrix} \epsilon_{\xi i} \\ \epsilon_{\theta i} \\ \epsilon_{\theta\theta i} \end{Bmatrix} &= \frac{1}{E_i} (\bar{\nu})^{-1} \begin{Bmatrix} \sigma_\xi \\ \sigma_\theta \\ \sigma_{\theta\theta} \end{Bmatrix} (1 - e^{-(E_i/\eta_i) \Delta t}) \\ &+ \begin{Bmatrix} \epsilon_{\xi i}(t - \Delta t) \\ \epsilon_{\theta i}(t - \Delta t) \\ \epsilon_{\theta\theta i}(t - \Delta t) \end{Bmatrix} e^{-(E_i/\eta_i) \Delta t} \end{aligned} \right\} \quad (9)$$

where

$$\bar{\nu} = \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \quad (\bar{\nu})^{-1} = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix}$$

Substituting Eqs.(7) into the 1st relations of Eqs.(9) and solving them about stresses, the stresses are

$$\left. \begin{aligned} \dot{\sigma}_\xi &= \frac{E_s}{1-\nu^2} (\dot{\epsilon}_{\xi m} + \nu \dot{\epsilon}_{\theta m} + \zeta (\dot{K}_\xi + \nu \dot{K}_\theta)) - \dot{\sigma}_{\xi s} \\ \dot{\sigma}_\theta &= \frac{E_s}{1-\nu^2} (\dot{\epsilon}_{\theta m} + \nu \dot{\epsilon}_{\xi m} + \zeta (\dot{K}_\theta + \nu \dot{K}_\xi)) - \dot{\sigma}_{\theta s} \\ \dot{\sigma}_{\theta\theta} &= \frac{E_s}{1+\nu} (\dot{\epsilon}_{\theta\theta m} + \zeta \dot{K}_{\theta\theta}) - \dot{\sigma}_{\theta\theta s} \end{aligned} \right\} \quad (10)$$

From Eqs.(10) the membrane forces and the bending moments are as follows.

$$\left. \begin{aligned} \dot{N}_\xi &= \frac{E_s h}{1-\nu^2} (\dot{\epsilon}_{\xi m} + \nu \dot{\epsilon}_{\theta m}) - \dot{N}_{\xi s} \\ \dot{N}_\theta &= \frac{E_s h}{1-\nu^2} (\dot{\epsilon}_{\theta m} + \nu \dot{\epsilon}_{\xi m}) - \dot{N}_{\theta s} \\ \dot{N}_{\xi\theta} &= \frac{E_s h}{1+\nu} \dot{\epsilon}_{\xi\theta m} - \dot{N}_{\xi\theta s} \\ \dot{M}_\xi &= \frac{E_s h^3}{12(1-\nu^2)} (\dot{K}_\xi + \nu \dot{K}_\theta) - \dot{M}_{\xi s} \\ \dot{M}_\theta &= \frac{E_s h^3}{12(1-\nu^2)} (\dot{K}_\theta + \nu \dot{K}_\xi) - \dot{M}_{\theta s} \\ \dot{M}_{\xi\theta} &= \frac{E_s h^3}{12(1+\nu)} \dot{K}_{\xi\theta} - \dot{M}_{\xi\theta s} \end{aligned} \right\} \quad (11)$$

where

$$\left. \begin{aligned} \dot{N}_{\xi s} &= \int_{-h/2}^{h/2} \dot{\sigma}_{\xi s} d\zeta, \quad \dot{N}_{\theta s} = \int_{-h/2}^{h/2} \dot{\sigma}_{\theta s} d\zeta \\ \dot{N}_{\xi\theta s} &= \int_{-h/2}^{h/2} \dot{\sigma}_{\xi\theta s} d\zeta \\ \dot{M}_{\xi s} &= \int_{-h/2}^{h/2} \zeta \dot{\sigma}_{\xi s} d\zeta, \quad \dot{M}_{\theta s} = \int_{-h/2}^{h/2} \zeta \dot{\sigma}_{\theta s} d\zeta \\ \dot{M}_{\xi\theta s} &= \int_{-h/2}^{h/2} \zeta \dot{\sigma}_{\xi\theta s} d\zeta \end{aligned} \right\} \quad (12)$$

A complete set of field equations for the 29 independent variables: $\dot{N}_\xi, \dot{N}_\theta, \dot{N}_{\xi\theta}, \dot{M}_\xi, \dot{M}_\theta, \dot{M}_{\xi\theta}, \dot{U}_\xi, \dot{U}_\theta, \dot{U}_\zeta, \dot{\epsilon}_{\xi m}, \dot{\epsilon}_{\theta m}, \dot{\epsilon}_{\xi\theta m}, \dot{K}_\xi, \dot{K}_\theta, \dot{K}_{\xi\theta}, \dot{\Phi}_\xi, \dot{\Phi}_\theta, \dot{\sigma}_{\xi s}, \dot{\sigma}_{\theta s}, \dot{\sigma}_{\xi\theta s}, \dot{\sigma}_{\xi s}, \dot{\sigma}_{\theta s}, \dot{N}_{\xi s}, \dot{N}_{\theta s}, \dot{N}_{\xi\theta s}, \dot{M}_{\xi s}, \dot{M}_{\theta s}, \dot{M}_{\xi\theta s}$ now is given by 29 equations, (2), (4)-(6) and (9)-(12).

3.2 Nondimensional equations

The 29 independent variables now will be expanded into Fourier series. Let σ_0 be a reference stress level, and solutions of the field equations will be obtained in the following forms.

$$\left. \begin{aligned} \dot{N}_\xi &= \sigma_0 h \sum_{n=0}^{\infty} \dot{n}_\xi^{(n)} \cos n\theta, \quad \dot{N}_{\xi s} = \sigma_0 h \sum_{n=0}^{\infty} \dot{n}_{\xi s}^{(n)} \cos n\theta, \quad \dot{N}_\theta = \sigma_0 h \sum_{n=0}^{\infty} \dot{n}_\theta^{(n)} \cos n\theta \\ \dot{N}_{\theta s} &= \sigma_0 h \sum_{n=0}^{\infty} \dot{n}_{\theta s}^{(n)} \cos n\theta, \quad \dot{N}_{\xi\theta} = \sigma_0 h \sum_{n=1}^{\infty} \dot{n}_{\xi\theta}^{(n)} \sin n\theta, \quad \dot{N}_{\xi\theta s} = \sigma_0 h \sum_{n=1}^{\infty} \dot{n}_{\xi\theta s}^{(n)} \sin n\theta \\ \dot{M}_\xi &= \frac{\sigma_0 h^3}{a} \sum_{n=0}^{\infty} \dot{m}_\xi^{(n)} \cos n\theta, \quad \dot{M}_{\xi s} = \frac{\sigma_0 h^3}{a} \sum_{n=0}^{\infty} \dot{m}_{\xi s}^{(n)} \cos n\theta, \quad \dot{M}_\theta = \frac{\sigma_0 h^3}{a} \sum_{n=0}^{\infty} \dot{m}_\theta^{(n)} \cos n\theta \\ \dot{M}_{\theta s} &= \frac{\sigma_0 h^3}{a} \sum_{n=0}^{\infty} \dot{m}_{\theta s}^{(n)} \cos n\theta, \quad \dot{M}_{\xi\theta} = \frac{\sigma_0 h^3}{a} \sum_{n=1}^{\infty} \dot{m}_{\xi\theta}^{(n)} \sin n\theta, \quad \dot{M}_{\xi\theta s} = \frac{\sigma_0 h^3}{a} \sum_{n=1}^{\infty} \dot{m}_{\xi\theta s}^{(n)} \sin n\theta \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} \dot{U}_\xi &= \frac{a\sigma_0}{E_s} \sum_{n=0}^{\infty} \dot{u}_\xi^{(n)} \cos n\theta, \quad \dot{U}_\theta = \frac{a\sigma_0}{E_s} \sum_{n=1}^{\infty} \dot{u}_\theta^{(n)} \sin n\theta, \quad \dot{U}_\zeta = \frac{a\sigma_0}{E_s} \sum_{n=0}^{\infty} \dot{u}_\zeta^{(n)} \cos n\theta \\ \dot{\epsilon}_{\xi m} &= \frac{\sigma_0}{E_s} \sum_{n=0}^{\infty} \dot{\epsilon}_{\xi m}^{(n)} \cos n\theta, \quad \dot{\epsilon}_{\theta m} = \frac{\sigma_0}{E_s} \sum_{n=0}^{\infty} \dot{\epsilon}_{\theta m}^{(n)} \cos n\theta, \quad \dot{\epsilon}_{\xi\theta m} = \frac{\sigma_0}{E_s} \sum_{n=1}^{\infty} \dot{\epsilon}_{\xi\theta m}^{(n)} \sin n\theta \\ \dot{K}_\xi &= \frac{\sigma_0}{aE_s} \sum_{n=0}^{\infty} \dot{k}_\xi^{(n)} \cos n\theta, \quad \dot{K}_\theta = \frac{\sigma_0}{aE_s} \sum_{n=0}^{\infty} \dot{k}_\theta^{(n)} \cos n\theta, \quad \dot{K}_{\xi\theta} = \frac{\sigma_0}{aE_s} \sum_{n=1}^{\infty} \dot{k}_{\xi\theta}^{(n)} \sin n\theta \\ \dot{\Phi}_\xi &= \frac{\sigma_0}{E_s} \sum_{n=0}^{\infty} \dot{\phi}_\xi^{(n)} \cos n\theta, \quad \dot{\Phi}_\theta = \frac{\sigma_0}{E_s} \sum_{n=1}^{\infty} \dot{\phi}_\theta^{(n)} \sin n\theta \end{aligned} \right\} \quad (14)$$

$$\left. \begin{aligned} \dot{u}_\xi &= \sigma_0 \sum_{n=0}^{\infty} \dot{u}_{\xi}^{(n)} \cos n\theta, & \dot{u}_{\xi a} &= \sigma_0 \sum_{n=0}^{\infty} \dot{u}_{\xi a}^{(n)} \cos n\theta, & \dot{u}_\theta &= \sigma_0 \sum_{n=0}^{\infty} \dot{u}_\theta^{(n)} \cos n\theta \\ \dot{u}_{\theta a} &= \sigma_0 \sum_{n=0}^{\infty} \dot{u}_{\theta a}^{(n)} \cos n\theta, & \dot{u}_{\xi \theta} &= \sigma_0 \sum_{n=1}^{\infty} \dot{u}_{\xi \theta}^{(n)} \sin n\theta, & \dot{u}_{\xi \theta a} &= \sigma_0 \sum_{n=1}^{\infty} \dot{u}_{\xi \theta a}^{(n)} \sin n\theta \end{aligned} \right\} \dots\dots(15)$$

Also expanding the loads into Fourier series,

$$\dot{P}_\xi = \frac{\sigma_0 h}{a} \sum_{n=0}^{\infty} \dot{p}_\xi^{(n)} \cos n\theta, \quad \dot{P}_\theta = \frac{\sigma_0 h}{a} \sum_{n=1}^{\infty} \dot{p}_\theta^{(n)} \sin n\theta, \quad \dot{P}_c = \frac{\sigma_0 h}{a} \sum_{n=0}^{\infty} \dot{p}_c^{(n)} \cos n\theta \dots\dots\dots(16)$$

Substituting Eqs.(13)-(16) into above 29 equations and eliminating appropriately the variables, the resultant set for $u_\xi^{(n)}$, $u_\theta^{(n)}$, $u_c^{(n)}$ and $m_\xi^{(n)}$ then can be written as

$$\left. \begin{aligned} a_1 \dot{u}_\xi'' + a_2 \dot{u}_\xi' + a_3 \dot{u}_\xi + a_4 \dot{u}_\theta' + a_5 \dot{u}_\theta + a_6 \dot{u}_c' + a_7 \dot{u}_c + a_8 \dot{m}_\xi' + a_9 \dot{m}_\xi &= c_1 \\ a_{10} \dot{u}_\xi' + a_{11} \dot{u}_\xi + a_{12} \dot{u}_\theta' + a_{13} \dot{u}_\theta + a_{14} \dot{u}_c' + a_{15} \dot{u}_c + a_{16} \dot{m}_\xi' + a_{17} \dot{m}_\xi + a_{18} \dot{m}_\xi &= c_2 \\ a_{19} \dot{u}_\xi' + a_{20} \dot{u}_\xi + a_{21} \dot{u}_\theta' + a_{22} \dot{u}_\theta + a_{23} \dot{u}_c' + a_{24} \dot{u}_c + a_{25} \dot{m}_\xi' + a_{26} \dot{m}_\xi + a_{27} \dot{m}_\xi &= c_3 \\ a_{30} \dot{u}_\xi' + a_{31} \dot{u}_\xi + a_{32} \dot{u}_\theta' + a_{33} \dot{u}_\theta + a_{34} \dot{u}_c' + a_{35} \dot{u}_c + a_{36} \dot{m}_\xi &= c_4 \end{aligned} \right\} \dots\dots(17)$$

where the superscript (n) on Fourier coefficients will be omitted for convenience, and $a_1 \sim a_{36}$ and $c_1 \sim c_4$ are constants determined from shell form, materials and loads.

3.3 Boundary conditions and discontinuity conditions

In Sanders' theory, the expressions for virtual work per unit length at the boundaries $s=0$ and s are

$$\pm(N_\xi U_\xi + \hat{N}_\theta U_\theta + \hat{Q}_\xi U_\xi + M_\xi \Phi_\xi) \dots\dots\dots(18)$$

where $\hat{N}_\theta$ and $\hat{Q}_\xi$ are effective membrane and transverse shear forces, respectively, per unit length (Fig.1), which are written as

$$\left. \begin{aligned} \hat{N}_{\xi\theta} &= \hat{N}_{\xi\theta} + \frac{1}{2} \left(\frac{3}{R_\theta} - \frac{1}{R_\xi} \right) \hat{M}_{\xi\theta} \\ \hat{Q}_\xi &= \frac{1}{a\rho} \left[\frac{\partial}{\partial \xi} (\rho \hat{M}_\xi) + 2 \frac{\partial \hat{M}_{\xi\theta}}{\partial \theta} - \rho' \hat{M}_\theta \right] \end{aligned} \right\} \dots\dots(19)$$

Expanding these into Fourier series yields

$$\left. \begin{aligned} \hat{N}_{\xi\theta} &= \sigma_0 h \sum_{n=1}^{\infty} \hat{n}_{\xi\theta} \sin n\theta \\ \hat{Q}_\xi &= \sigma_0 h \sum_{n=0}^{\infty} \hat{q}_\xi \cos n\theta \end{aligned} \right\} \dots\dots\dots(20)$$

Substituting Eqs.(20) and (13) into Eqs.(19), the relations between the Fourier coefficients become

$$\left. \begin{aligned} \hat{n}_{\xi\theta} &= \hat{n}_{\xi\theta} + \frac{\lambda^2}{2} (3\omega_\theta - \omega_\xi) \hat{m}_{\xi\theta} \\ \hat{q}_\xi &= \lambda^2 \left[\hat{m}_\xi' + \gamma(\hat{m}_\xi - \hat{m}_\theta) + \frac{2n}{\rho} \hat{m}_{\xi\theta} \right] \end{aligned} \right\} \dots\dots\dots(21)$$

Therefore, $\hat{n}_\xi$, $\hat{n}_{\xi\theta}$, $\hat{q}_\xi$ and $\hat{\varphi}_\xi$ may be represented by u_ξ , u_θ , u_c , m_ξ as follows

$$\left. \begin{aligned} \hat{n}_\xi &= b_1 \dot{u}_\xi' + b_2 \dot{u}_\xi + b_3 \dot{u}_\theta' + b_4 \dot{u}_\theta + d_1 \\ \hat{n}_{\xi\theta} &= b_5 \dot{u}_\xi + b_6 \dot{u}_\theta' + b_7 \dot{u}_\theta + b_8 \dot{u}_c' + b_9 \dot{u}_c + d_2 \\ \hat{q}_\xi &= b_{10} \dot{u}_\xi + b_{11} \dot{u}_\theta' + b_{12} \dot{u}_\theta + b_{13} \dot{u}_c' + b_{14} \dot{u}_c \\ &\quad + b_{15} \dot{m}_\xi' + b_{16} \dot{m}_\xi + d_3 \\ \hat{\varphi}_\xi &= -\dot{u}_c' + \omega_\xi \dot{u}_\xi \end{aligned} \right\} \dots\dots(22)$$

where $b_1 \sim b_{16}$ and $d_1 \sim d_3$ are the constants determined from shell shapes, materials and loads.

Next, the differential equations (17) are not valid at points in the shell where discontinuities in geometry occur. Accordingly, special transition equations will be required which relate the solution and its derivative on either side of a discontinuity. With plus and minus superscripts denoting values just beyond and ahead of a discontinuity, respectively,

the conditions of geometrical compatibility are

$$\dot{u}_\xi^+ = \dot{u}_\xi^-, \quad \dot{u}_\theta^+ = \dot{u}_\theta^-, \quad \dot{u}_c^+ = \dot{u}_c^-, \quad \dot{\varphi}_\xi^+ = \dot{\varphi}_\xi^- \quad (23)$$

and equilibrium requires that

$$\hat{n}_\xi^+ = \hat{n}_\xi^-, \quad \hat{n}_{\xi\theta}^+ = \hat{n}_{\xi\theta}^-, \quad \hat{q}_\xi^+ = \hat{q}_\xi^-, \quad \hat{m}_\xi^+ = \hat{m}_\xi^- \quad \dots(24)$$

The differential equations (17), the boundary conditions (22), and the discontinuity conditions (23), (24) will be united into a set of appropriate finite difference equations.

4. Numerical method

Suppose that p discontinuity locations $s_1, s_2, \dots, s_p$ occur in the range $(0, s)$ of the shell as shown in Fig.4, the regions $(0, s_1)$, (s_1, s_2) , $\dots$, (s_p, s) are subdivided into $N_1-1, N_2-1, \dots, N_p-1$ equal segments, respectively, and the endpoints of the segments two indices j are given running from 1 at $s=0$ to $\text{ton}(=\sum N)$ at $s=s$, then the increments in the nondimensional variable ξ are

$$\left. \begin{aligned} \Delta_1 &= s_1/a(N_1-1) \\ \Delta_2 &= (s_2-s_1)/a(N_2-1) \\ &\vdots \\ \Delta_{p+1} &= (s-s_p)/a(N_{p+1}-1) \end{aligned} \right\} \dots\dots\dots(25)$$

in the successive regions bounded by discontinuities.

Fig.4 Mesh points

The differential equations (17) will be written in a finite difference form at all stations except the discontinuity stations and boundaries ($j=1, n$) on the basis of the usual central difference formulas:

$$Z_j' = \frac{Z_{j+1} - Z_{j-1}}{2\Delta}, \quad Z_j'' = \frac{Z_{j+1} - 2Z_j + Z_{j-1}}{\Delta^2} \dots (26)$$

For the boundary points ($j=1, n$) and the discontinuity points (e.g. $j=m, m+1$)

$$\left. \begin{aligned} Z_1' &= \frac{-3Z_1 + 4Z_2 - Z_3}{2\Delta_1} \\ Z_n' &= \frac{3Z_n - 4Z_{n-1} + Z_{n-2}}{2\Delta_n} \\ Z_m' &= \frac{3Z_m - 4Z_{m-1} + Z_{m-2}}{2\Delta^-} \\ Z_{m+1}' &= \frac{-3Z_{m+1} + 4Z_{m+2} - Z_{m+3}}{2\Delta^+} \end{aligned} \right\} \dots (27)$$

are employed. Where Δ^- and Δ^+ are the intervals ahead and beyond station $j = m$, respectively.

Applying above difference formulas for the fundamental equations, the boundary ones and the discontinuity ones, the following simultaneous equations having n unknowns with respect to Z_j may be obtained, where $Z^* = \{u_t, u_\theta, u_c, m_t\}$

$$A_j Z_{j-1} + B_j Z_j + C_j Z_{j+1} = D_j \dots (28)$$

$$\left. \begin{aligned} (AM)_n Z_{n-2} + (BM)_n Z_{n-1} + (CM)_n Z_n \\ + (DM)_{n+1} Z_{n+1} + (EM)_{n+1} Z_{n+2} \\ + (FM)_{n+1} Z_{n+3} = 0 \\ (GM)_n Z_{n-2} + (HM)_n Z_{n-1} + (IM)_n Z_n \\ + (JM)_{n+1} Z_{n+1} + (KM)_{n+1} Z_{n+2} \\ + (LM)_{n+1} Z_{n+3} = (SM)_n - (SM)_{n+1} \end{aligned} \right\} \dots (29)$$

$$\left. \begin{aligned} (A1)Z_1 + (B1)Z_2 + (C1)Z_3 = (D1) \\ (AN)Z_{n-2} + (BN)Z_{n-1} + (CN)Z_n = (DN) \end{aligned} \right\} \dots (30)$$

Now, the above numerical method is concerned with a handling of the spacial aspect of the problem. In the visco-elastic problem, however, the temporal aspect also must be considered. The solutions at any stage of problem are obtained by integration of the incremental values of internal forces and displacements for the time increment Δt . The validity of values of Δt are inspected with the convergency of the solutions for the variations of Δt .

Fig.5 Creep compliance

5. Numerical results and discussions

As numerical examples of unsymmetrical bending of shells of revolution the problems of a cylindrical shell and a conical drum under a step displacement with respect to time are treated.

The material constants employed in calculations are

$$\left. \begin{aligned} E_g &= 2500 \text{ kg/mm}^2, \quad E_i = 2000 \text{ kg/mm}^2 \\ \eta_g &= 100000 \text{ kg h/mm}^2, \quad \eta_i = 7000 \text{ kg h/mm}^2 \\ \nu &= 0.3 \end{aligned} \right\} (31)$$

When we apply at $t=0$ hr. a constant stress σ for the 4-parameter model with the above material constants and put $\sigma=0$ at $t=100$ hr., the variation of ϵ with time becomes as shown in Fig.5. It is found from Fig.5 that the curve becomes linear by about 10 hours and, when we put $\sigma=0$ at 100 hr., the elastic strain recovers at once and then the visco-elastic strain due to the Kelvin model recovers gradually. But the strain due to a dashpot with the viscosity coefficient η_g is maintained forever as plastic strain.

5.1 Cylindrical shell

Since the cylindrical shell shown in Fig. 6 is symmetric with respect to the plane C, the deformations are antisymmetric with respect to this plane and therefore we calculate for only AC segment. For AC

$$\left. \begin{aligned} \Delta = \frac{l}{r_m(n-1)}, \quad \rho=1, \quad \rho'=0, \quad \gamma=0 \\ \omega_\theta=1, \quad \omega_t=\omega_t'=0 \end{aligned} \right\} \dots (32)$$

The boundary conditions for plane A at $t=0$ are

$$U_t=0, \quad U_\theta=-U_0 \sin \theta, \quad U_c=U_0 \cos \theta, \quad U_c'=0 \quad (33)$$

Putting $E_i U_0 / a \sigma_0 = 1$, Eqs.(33) become

$$u_t=0, \quad u_\theta=-1, \quad u_c=1, \quad u_c'=0 \quad (34)$$

At $t>0$

$$\dot{u}_t=\dot{u}_\theta=\dot{u}_c=\dot{u}_c'=0 \quad (35)$$

Fig.6 Cylindrical shell

For plane C, at $t=0$:

$$u_t' = 0, \quad u_\theta = -\frac{1}{2}, \quad u_c = \frac{1}{2}, \quad m_t = 0 \quad (36)$$

At $t > 0$

$$\dot{u}_t' = \dot{u}_\theta = \dot{u}_c = \dot{m}_t = 0 \quad (37)$$

Moment M and shearing force F occur with displacement U_0 and these are given as follows,

$$\begin{aligned} \dot{M} &= \int_0^{2\pi} (-\dot{M}_t) \cos \theta r_m d\theta + \int_0^{2\pi} (-\dot{N}_t) r_m^2 \cos \theta d\theta \\ \dot{F} &= \int_0^{2\pi} (-\dot{Q}_t) r_m \cos \theta d\theta + \int_0^{2\pi} \dot{N}_t r_m \sin \theta d\theta \end{aligned} \quad (38)$$

Substituting Eqs. (13), (20), (21) into Eqs. (38) and considering Eqs. (32), (34), (35), Eqs. (38) become for $t \geq 0$

$$\begin{aligned} \frac{\dot{M}}{E_\theta h^2 U_0} &= -\pi \frac{r_m}{a} \left\{ \frac{h}{c} \dot{m}_t + \frac{r_m}{h} b (\dot{u}_t' - \dot{n}_{t0}) \right\} \\ \frac{\dot{F}}{E_\theta h U_0} &= \pi \frac{r_m}{a} \left[\frac{1-\nu}{2} \left\{ b + 3\lambda^2 d \omega_\theta \left(\frac{3\omega_\theta}{4} - \frac{1}{\rho} \right) \right\} \dot{u}_\theta' \right. \\ &\quad \left. - \lambda^2 \dot{m}_t' - \dot{n}_{t0} + \lambda^2 \left(\frac{2n}{\rho} - \frac{3}{2} \omega_\theta \right) \dot{m}_{t0} \right] \end{aligned} \quad (39)$$

Where $(\cdot)$ and $(\cdot)_0$ are omitted for $t=0$.

Table 1 shows the convergency of the elastic solutions due to a number of mesh-points and Table 2 shows the effect of β on the convergency of the visco-elastic solutions. In Tables 1 and 2 suffixes C and A refer to points C and A in shell, respectively. It is found from the results in the tables that selection of $n=201$, and $\beta=30$ in calculations is valid. Thus in the further numerical calculations the authors employ 201 as n and 30 as β considering the capacity of the computer and the computing time.

Figure 7 shows the deformation at $t=0$ due to the shearing displacement U_0 . The deformation scarcely proceeds with time. Figure 8 shows the meridional distributions of stresses in shell at $t=0$ due to the displacement U_0 , where σ_t and σ_θ are the values at $\theta=0^\circ$ and $\sigma_{t\theta}$ at $\theta=90^\circ$. It is seen from Fig. 8 that $\sigma_{t\theta}^{\text{out}}$ is large at boundaries and decrease rapidly with s , while $\sigma_{t\theta}$ is constant through all the range. Figure 9 illustrates the variations of me-

ridional distributions of $\sigma_{t\theta}^{\text{out}}$ with time.

$\sigma_{t\theta}^{\text{out}}$ relaxes rapidly in 10 hours and becomes very small at 100 hours. The variations with time of σ_t through thickness are shown in Fig. 10. The distributions become linear because the material is a linear visco-elastic body. Figure 11 is the distribution of M_t , which is positive at $s/r_m=0 \sim 0.15$ and negative at $0.15 \sim 0.6$. Figures 12 show the variations of moment M and shearing force F with time. Both M and F relax rapidly in initial 10 hours and after that decrease gradually.

Fig.7 Deformation ($t=0$)

Fig.8 Meridional distributions of stresses ($t=0$)

Fig.9 Meridional distributions of σ_t^{out}

Table 1 Effect of n on convergency of solutions ($t=0$)

n	M	F	u_{tC}	m_{tA}	σ_{tA}^{out}	$\sigma_{\theta A}^{\text{out}}$
51	-18.01	0.5414	0.08296	1.079	0.3652	0.1096
101	-18.03	0.5413	0.08298	1.061	0.3622	0.1086
151	-18.04	0.5412	0.08299	1.057	0.3616	0.1085
201	-18.04	0.5412	0.08299	1.056	0.3613	0.1084
301	-18.04	0.5412	0.08299	1.055	0.3612	0.1084
401	-18.04	0.5412	0.08299	1.055	0.3612	0.1084
501	-18.04	0.5412	0.08299	1.055	0.3612	0.1084

Table 2 Effect of β on convergency of solutions ($n=201$, $t=0.1$ hr.)

β	M	F	u_{tC}	σ_{tA}^{out}	$\sigma_{\theta A}^{\text{out}}$	$\sigma_{t\theta A}^{\text{out}}$
30	-17.30	0.5211	0.08299	0.3480	0.1044	0.1593
100	-17.30	0.5212	0.08299	0.3480	0.1044	0.1593
150	-17.30	0.5212	0.08299	0.3480	0.1044	0.1593
200	-17.30	0.5212	0.08299	0.3480	0.1044	0.1593

Fig.10 Distributions of σ_ξ through thickness (point A)

Fig.11 Meridional distributions of M_ξ

(a) M

(b) F

Fig.12 Variations of M and F with time

5.2 Conical drum

The conical drum employed in this calculation is composed of cylindrical, toroidal and conical shells as shown in Fig.13. We omit description of the boundary conditions and discontinuity conditions because of space limitations, and discuss only a few calculating results.

Figure 14 shows the deformation at $t=0$ due to the shearing displacement U_0 . The deformation scarcely proceeds with time. Figure 15 shows the meridional distributions of σ_ξ^{out} , which become large at fixed boundaries and decrease with time. Figure 16 shows the meridional distributions of $\sigma_\theta^{\text{out}}$, which have maximum values in lower toroidal part. Figure 17 illustrates the distributions of M_ξ , which are large at the fixed boundaries and in the lower toroidal part.

Fig.13 Conical drum

Fig.14 Deformation ($t=0$)

Fig.15 Meridional distributions of σ_t^{out} . Fig.17 Meridional distributions of M_t . Fig.16 Meridional distributions of σ_r^{out} .

6. Conclusions

In the present paper the authors described a numerical analysis of visco-elastic shells of revolution under unsymmetrical loads. The visco-elastic behaviour of a material was represented by 4-parameter models composed of springs and dashpots. The basic differential equations derived for incremental values were numerically solved by a finite difference method, and the solutions at any stage were obtained by integration of the incremental values. The time increment Δt is controlled as the

strain increments for each step become constant. As numerical examples two problems of a cylindrical shell and a conical drum composed of multi-shells subjected to stepwise displacements with respect to time were analyzed.

In numerical calculations FACOM 230-60 in Data Processing Center, Nagoya University was used.

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