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Dynamic Stresses and Deformations in a Thick Cylinder under
Nonaxially Symmetrical Impulsive Loads

(1st Report, Analysis Based on the Two Dimensional Dynamic
Theory of Elasticity)*

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In this paper, the response of a hollow infinite circular cylinder (as a plane strain field) or a circular disk (as a plane stress field) is studied, which is subjected to a pair of impulsive line loads along its outer generators facing each other.

First, the line loads are expanded in a Fourier series and represented as a sum of a uniformly distributed pressure and pressures in the form of trigonometric functions. By the use of this method of superposition, the dynamic stresses in the cylinder or the disk are analyzed based on the two dimensional dynamic theory of elasticity by using the stress functions and the Laplace transformation.

As the result, it is shown that the maximum tensile stress is produced along the inner generators corresponding to the loaded ones and the value is about twice as large as the corresponding static result.

1. Introduction

The study of the response of hollow cylinders to impact loads is an important branch of applied mechanics concerning shell structures or ring stiffeners. The dynamic problems of hollow cylinders have therefore attracted the attention of many authors.

Hoppe has given the original solution for natural frequencies of a thin-walled cylinder of negligible axial thickness as a ring.⁽¹⁾ Thereafter, many approaches by the similar method based on the beam theory have been tried and a lot of available results in the field of mechanical vibration have been offered.⁽²⁾⁻⁽⁴⁾

Federhofer has analyzed the exact solution for frequencies of a thick cylinder as a plane strain field based on the two dimensional theory of elasticity at first.⁽⁵⁾

Meanwhile, dynamic stresses of a thick cylinder of infinite length deformed by an axially symmetric impulsive pressure has been exactly analyzed by Tranter on the basis of the dynamic theory of elasticity making use of the Laplace transformation⁽⁶⁾ and numerically calculated by Suzuki.⁽⁷⁾ Also, a similar problem has been analyzed by one of the authors using the displacement potentials and the phenomenological

explanations for the numerical results have been given with respect to elastic wave propagations and natural vibrations.⁽⁸⁾

On the other hand, the non-axisymmetric stress field of a hollow cylinder due to impulsive loads has not been made clear enough.

The present paper is concerned with the analysis of the non-axisymmetric response of a thick-walled circular cylinder of infinite length or a thin circular plate to a pair of impact line loads along its outer generators diametrically opposite to each other. First, the line loads are expanded in a Fourier series each term of which represents a uniformly distributed pressure and pressures of trigonometric distributions on circumferential boundary. And by the use of the method of superposition, the impulsive stresses in a cylinder or a disk are analyzed on the basis of the two dimensional dynamic theory of elasticity making use of the displacement potentials and the Laplace transformation.

2. Fundamental equations

Consider a hollow cylinder subjected to diametrically opposite impulsive line loads $q(t)$ along its outer generators as shown in Fig.1(a).

Then, the state of stress may be plane stress in the case of the length l of a cylinder being short enough as compared with the thickness $(b-a)$ and may be plane strain in the case of an infinite cylinder.

Making use of the displacement components (u,v) in a polar coordinate system (r,θ) , the equations of equilibrium in axisymmetrical plane stress or strain field are written in the following way.

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$$\left. \begin{aligned} \nabla^2 u + \frac{2-\kappa}{\kappa} \frac{\partial e}{\partial r} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} &= \frac{\gamma}{gG} \frac{\partial^2 u}{\partial t^2} \\ \nabla^2 v + \frac{2-\kappa}{\kappa} \frac{1}{r} \frac{\partial e}{\partial \theta} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} &= \frac{\gamma}{gG} \frac{\partial^2 v}{\partial t^2} \end{aligned} \right\} \dots\dots\dots(1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and t is time, g is the acceleration of gravity, G is the modulus of rigidity and γ is weight per unit volume. Here, $\kappa = 1-\nu$ for a plane stress field, $\kappa = (1-2\nu)/(1-\nu)$ for a plane strain field, and ν is Poisson's ratio.

Now, by introducing displacement potentials ϕ_0 and λ_3 , we express the displacement components by the equations

$$u = \frac{\partial \phi_0}{\partial r} + \frac{1}{r} \frac{\partial \lambda_3}{\partial \theta}, \quad v = \frac{1}{r} \frac{\partial \phi_0}{\partial \theta} - \frac{\partial \lambda_3}{\partial r} \dots\dots(2)$$

Then, it is easily shown from Eqs.(1) that ϕ_0 and λ_3 must be the functions which satisfy the following wave equations.

$$\nabla^2 \phi_0 = \frac{1}{c^2} \frac{\partial^2 \phi_0}{\partial t^2}, \quad \nabla^2 \lambda_3 = \frac{1}{c_2^2} \frac{\partial^2 \lambda_3}{\partial t^2} \dots\dots\dots(3)$$

where

$$c^2 = \frac{2}{\kappa} \frac{gG}{\gamma}, \quad c_2^2 = \frac{gG}{\gamma}$$

and, c and c_2 are the dilatational and distortional wave velocities, respectively. By using Eqs.(2), the stress components are expressed as follows.

$$\left. \begin{aligned} \frac{\sigma_r}{2G} &= -\frac{1}{r} \frac{\partial \phi_0}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi_0}{\partial \theta^2} + \frac{1}{\kappa} \nabla^2 \phi_0 + \frac{1}{r} \frac{\partial^2 \lambda_3}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \lambda_3}{\partial \theta} \\ \frac{\sigma_\theta}{2G} &= \frac{1}{r} \frac{\partial \phi_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_0}{\partial \theta^2} + \frac{1-\kappa}{\kappa} \nabla^2 \phi_0 - \frac{1}{r} \frac{\partial^2 \lambda_3}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \lambda_3}{\partial \theta} \\ \frac{\tau_{r\theta}}{2G} &= \frac{1}{r} \frac{\partial^2 \phi_0}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \phi_0}{\partial \theta} + \frac{1}{2} \nabla^2 \lambda_3 - \frac{\partial^2 \lambda_3}{\partial r^2} \end{aligned} \right\} \dots\dots(4)$$

In the case of the initial conditions

$$(\phi_0)_{t=0} = \left(\frac{\partial \phi_0}{\partial t} \right)_{t=0} = 0, \quad (\lambda_3)_{t=0} = \left(\frac{\partial \lambda_3}{\partial t} \right)_{t=0} = 0 \dots\dots\dots(5)$$

the Laplace transformations of Eqs.(3) are

$$\nabla^2 \bar{\phi}_0 = \alpha^2 \bar{\phi}_0, \quad \nabla^2 \bar{\lambda}_3 = \alpha_2^2 \bar{\lambda}_3 \dots\dots\dots(6)$$

where

$$\bar{f}(p) = \int_0^\infty f(t) e^{-pt} dt$$

$$\text{and } \alpha^2 = p^2/c^2, \quad \alpha_2^2 = p^2/c_2^2.$$

Further, the Laplace transformations of Eqs.(2) and (4) are

$$\left. \begin{aligned} \bar{u} &= \frac{\partial \bar{\phi}_0}{\partial r} + \frac{1}{r} \frac{\partial \bar{\lambda}_3}{\partial \theta}, \quad \bar{v} = \frac{1}{r} \frac{\partial \bar{\phi}_0}{\partial \theta} - \frac{\partial \bar{\lambda}_3}{\partial r} \\ \frac{\bar{\sigma}_r}{2G} &= -\frac{1}{r} \frac{\partial \bar{\phi}_0}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \bar{\phi}_0}{\partial \theta^2} + \frac{1}{\kappa} \nabla^2 \bar{\phi}_0 + \frac{1}{r} \frac{\partial^2 \bar{\lambda}_3}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \bar{\lambda}_3}{\partial \theta} \\ \frac{\bar{\sigma}_\theta}{2G} &= \frac{1}{r} \frac{\partial \bar{\phi}_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{\phi}_0}{\partial \theta^2} + \frac{1-\kappa}{\kappa} \nabla^2 \bar{\phi}_0 - \frac{1}{r} \frac{\partial^2 \bar{\lambda}_3}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \bar{\lambda}_3}{\partial \theta} \\ \frac{\bar{\tau}_{r\theta}}{2G} &= \frac{1}{r} \frac{\partial^2 \bar{\phi}_0}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \bar{\phi}_0}{\partial \theta} + \frac{1}{2} \nabla^2 \bar{\lambda}_3 - \frac{\partial^2 \bar{\lambda}_3}{\partial r^2} \end{aligned} \right\} \dots\dots(7)$$

The solutions of Eqs.(6) in the form of separation of variables are given as follows.

$$\left. \begin{aligned} \bar{\phi}_0 &= \begin{pmatrix} 1 \\ \theta \end{pmatrix} \begin{pmatrix} I_0(\alpha r) \\ K_0(\alpha r) \end{pmatrix}, \quad \begin{pmatrix} \sin n\theta \\ \cos n\theta \end{pmatrix} \begin{pmatrix} I_n(\alpha r) \\ K_n(\alpha r) \end{pmatrix} \\ \bar{\lambda}_3 &= \begin{pmatrix} 1 \\ \theta \end{pmatrix} \begin{pmatrix} I_0(\alpha_2 r) \\ K_0(\alpha_2 r) \end{pmatrix}, \quad \begin{pmatrix} \sin n\theta \\ \cos n\theta \end{pmatrix} \begin{pmatrix} I_n(\alpha_2 r) \\ K_n(\alpha_2 r) \end{pmatrix} \end{aligned} \right\} \dots\dots\dots(8)$$

where the terms on the right hand side of Eqs.(8) mean an arbitrary linear combination of the product of each element in the first and second parentheses, and $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of 1st and 2nd kinds of order n .

3. Method of solution

Suppose the partially distributed uniform impulsive pressures as shown in Fig.1 (b). Then the function $P(\theta, t)$ denoting the pressure is expressed by using a Fourier series expansion as follows.

$$P(\theta, t) = 2\beta p(t) \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=2,4,\dots}^{\infty} \frac{\sin n\beta}{n\beta} \cos n\theta \right) \dots\dots\dots(9)$$

where $p(t)$ is a function of time which depends on the type of load history.

For line loads shown in Fig.1(a), Eq.(9) becomes

$$P(\theta, t) = \frac{q(t)}{b} \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=2,4,\dots}^{\infty} \cos n\theta \right) \dots\dots(10)$$

Thus, it is found that the line loads are represented as a superposition of a uniformly distributed pressure $q(t)/\pi b$ and pressures in the form of trigonometric functions $2q(t)\cos(n\theta)/\pi b$.

Fig.1 Nonaxially symmetrical loads acting on the outer surface of a cylinder or a disk

(a) The case of $n=2$ (b) The case of $n=4$

Fig.2 Pressures in the form of trigonometric functions

Fig.3 Impulsive pressure varying in the form of a step function

As the stresses of a cylinder subjected to a uniformly distributed impulsive pressure have been analysed, based on the dynamic theory of elasticity, here, we shall deal with the pressures in the form of trigonometric functions $p(t)\cos(n\theta)$.

The boundary conditions of this problem are

$$\text{at } r=a: \sigma_r^{(n)}=0, \quad \tau_{r\theta}^{(n)}=0 \quad \dots(11)$$

$$\text{at } r=b: \sigma_r^{(n)}=-p(t)\cos(n\theta), \quad \tau_{r\theta}^{(n)}=0$$

where superscript (n) corresponds to the circumferential wave number n of the pressure $p(t)\cos(n\theta)$.

Assuming that the cylinder is at rest under the unstressed condition at time $t=0$, the initial conditions are given by the equations

$$u^{(n)} = \frac{\partial u^{(n)}}{\partial t} = 0, \quad v^{(n)} = \frac{\partial v^{(n)}}{\partial t} = 0 \quad \dots(12)$$

And, for ϕ_0 and λ_3 , Eqs.(5) must be satisfied.

Therefore the Laplace transformations of Eqs.(11) in the case of the pressure with a step distribution in time, shown in Fig.3, are written in the form

$$\begin{aligned} \text{at } r=a: \sigma_r^{(n)} &= 0, \quad \tau_{r\theta}^{(n)} = 0 \\ \text{at } r=b: \sigma_r^{(n)} &= -p_0 \cos(n\theta)/p, \quad \tau_{r\theta}^{(n)} = 0 \end{aligned} \quad \dots(13)$$

Considering Eqs.(13), the general solutions of ϕ_0 and λ_3 are given as

$$\begin{aligned} \bar{\phi}_0^{(n)} &= [A_1 I_n(\alpha r) + B_1 K_n(\alpha r)] \cos n\theta \\ \bar{\lambda}_3^{(n)} &= [A_2 I_n(\alpha_2 r) + B_2 K_n(\alpha_2 r)] \sin n\theta \end{aligned} \quad \dots(14)$$

Substituting Eqs.(14) into Eqs.(7), the Laplace transformation of the stress components is

$$\left. \begin{aligned} \frac{\bar{\sigma}_r^{(n)}}{2G \cos n\theta} &= A_1 \alpha^2 F_1(\alpha r) + B_1 \alpha^2 G_1(\alpha r) \\ &\quad - A_2 \alpha_2^2 F_2(\alpha_2 r) - B_2 \alpha_2^2 G_2(\alpha_2 r) \\ \frac{\bar{\tau}_{r\theta}^{(n)}}{2G \sin n\theta} &= A_1 \alpha^2 F_2(\alpha r) + B_1 \alpha^2 G_2(\alpha r) \\ &\quad - A_2 \alpha_2^2 F_3(\alpha_2 r) - B_2 \alpha_2^2 G_3(\alpha_2 r) \end{aligned} \right\} \quad \dots(15)$$

where

$$\begin{aligned} F_1(x) &= \left(\frac{n^2+n}{x^2} + \frac{1}{\kappa} \right) I_n(x) - \frac{1}{x} I_{n-1}(x) \\ G_1(x) &= \left(\frac{n^2+n}{x^2} + \frac{1}{\kappa} \right) K_n(x) + \frac{1}{x} K_{n-1}(x) \\ F_2(x) &= \frac{n^2+n}{x^2} I_n(x) - \frac{n}{x} I_{n-1}(x) \\ G_2(x) &= \frac{n^2+n}{x^2} K_n(x) + \frac{n}{x} K_{n-1}(x) \\ F_3(x) &= \left(\frac{n^2+n}{x^2} + \frac{1}{2} \right) I_n(x) - \frac{1}{x} I_{n-1}(x) \\ G_3(x) &= \left(\frac{n^2+n}{x^2} + \frac{1}{2} \right) K_n(x) + \frac{1}{x} K_{n-1}(x) \end{aligned}$$

From Eqs.(13) and (15), the following linear algebraic equations for unknown coefficients are easily derived.

$$\begin{pmatrix} F_1(\alpha b) & G_1(\alpha b) & F_2(\alpha_2 b) & G_2(\alpha_2 b) \\ F_1(\alpha a) & G_1(\alpha a) & F_2(\alpha_2 a) & G_2(\alpha_2 a) \\ F_2(\alpha b) & G_2(\alpha b) & F_3(\alpha_2 b) & G_3(\alpha_2 b) \\ F_2(\alpha a) & G_2(\alpha a) & F_3(\alpha_2 a) & G_3(\alpha_2 a) \end{pmatrix} \begin{pmatrix} \alpha^2 A_1 \\ \alpha^2 B_1 \\ -\alpha_2^2 A_2 \\ -\alpha_2^2 B_2 \end{pmatrix} = -\frac{p_0}{2Gp} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \dots(16)$$

And by solving Eqs.(16), the four unknowns may be determined.

Here, we introduce the following dimensionless notations $\rho_1 = a/b$, $\rho = r/b$, $\lambda = \alpha b$, $\tau = ct/b$, $\mu = c/c_2 = \sqrt{2/\kappa}$. Then, the Laplace inversion formulas of Eqs.(15) are written in the form

$$\left. \begin{aligned} \frac{\sigma_r^{(n)}}{p_0 \cos n\theta} &= \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{L_3(\lambda)}{\lambda D(\lambda)} e^{\lambda \tau} d\lambda \\ \frac{\tau_{r\theta}^{(n)}}{p_0 \sin n\theta} &= \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{L_3(\lambda)}{\lambda D(\lambda)} e^{\lambda \tau} d\lambda \end{aligned} \right\} \quad \dots(17)$$

where

$$D(\lambda) = \begin{vmatrix} F_1(\lambda) & G_1(\lambda) & F_2(\mu\lambda) & G_2(\mu\lambda) \\ F_1(\rho_1\lambda) & G_1(\rho_1\lambda) & F_2(\mu\rho_1\lambda) & G_2(\mu\rho_1\lambda) \\ F_2(\lambda) & G_2(\lambda) & F_3(\mu\lambda) & G_3(\mu\lambda) \\ F_2(\rho_1\lambda) & G_2(\rho_1\lambda) & F_3(\mu\rho_1\lambda) & G_3(\mu\rho_1\lambda) \end{vmatrix}$$

$$L_s(\lambda) = \begin{vmatrix} F_1(\rho\lambda) & G_1(\rho\lambda) & F_2(\mu\rho\lambda) & G_2(\mu\rho\lambda) \\ F_1(\rho_1\lambda) & G_1(\rho_1\lambda) & F_2(\mu\rho_1\lambda) & G_2(\mu\rho_1\lambda) \\ F_3(\lambda) & G_3(\lambda) & F_3(\mu\lambda) & G_3(\mu\lambda) \\ F_2(\rho_1\lambda) & G_2(\rho_1\lambda) & F_3(\mu\rho_1\lambda) & G_3(\mu\rho_1\lambda) \end{vmatrix}$$

$$L_s(\lambda) = \begin{vmatrix} F_2(\rho\lambda) & G_2(\rho\lambda) & F_3(\mu\rho\lambda) & G_3(\mu\rho\lambda) \\ F_1(\rho_1\lambda) & G_1(\rho_1\lambda) & F_2(\mu\rho_1\lambda) & G_2(\mu\rho_1\lambda) \\ F_2(\lambda) & G_2(\lambda) & F_3(\mu\lambda) & G_3(\mu\lambda) \\ F_2(\rho_1\lambda) & G_2(\rho_1\lambda) & F_3(\mu\rho_1\lambda) & G_3(\mu\rho_1\lambda) \end{vmatrix}$$

Fig.4 Contour used in the evaluation of the integral

Now, the integrands on the right hand side of Eqs.(17) are single valued functions of λ , and they have a simple pole at the origin $\lambda=0$. It will be also shown that the poles are at $\lambda=\pm\lambda_m^{(n)}$ ($m=1,2,3,\dots; 0<\lambda_1^{(n)}<\lambda_2^{(n)}<\lambda_3^{(n)}<\dots$) which are the roots of the transcendental equation $D(\lambda)=0$, and these are all simple and purely imaginary. Then, the solutions for the stress and displacement components are given by

$$\left. \begin{aligned} \frac{2G}{p_0} \frac{u^{(n)}}{b \cos n\theta} &= u_{it}^{(n)} + 2\rho \sum_{m=1}^{\infty} \frac{N_1(\lambda_m^{(n)})}{D_0(\lambda_m^{(n)})} \cos \lambda_m^{(n)} \tau \\ \frac{2G}{p_0} \frac{v^{(n)}}{b \sin n\theta} &= v_{it}^{(n)} + 2\rho \sum_{m=1}^{\infty} \frac{N_2(\lambda_m^{(n)})}{D_0(\lambda_m^{(n)})} \cos \lambda_m^{(n)} \tau \\ \frac{\sigma_r^{(n)}}{p_0} \frac{1}{\cos n\theta} &= \sigma_{r, it}^{(n)} + 2 \sum_{m=1}^{\infty} \frac{N_3(\lambda_m^{(n)})}{D_0(\lambda_m^{(n)})} \cos \lambda_m^{(n)} \tau \\ \frac{\sigma_\theta^{(n)}}{p_0} \frac{1}{\cos n\theta} &= \sigma_{\theta, it}^{(n)} + 2 \sum_{m=1}^{\infty} \frac{N_4(\lambda_m^{(n)})}{D_0(\lambda_m^{(n)})} \cos \lambda_m^{(n)} \tau \\ \frac{\tau_{r\theta}^{(n)}}{p_0} \frac{1}{\sin n\theta} &= \tau_{r\theta, it}^{(n)} + 2 \sum_{m=1}^{\infty} \frac{N_5(\lambda_m^{(n)})}{D_0(\lambda_m^{(n)})} \cos \lambda_m^{(n)} \tau \end{aligned} \right\} \dots (18)$$

The first terms of Eqs.(18) are the residues at the origin and coincide with the solutions for the cylinder subjected to the static pressure $p_0 \cos(n\theta)$ on its outer surface, and are expressed as follows.

$$2K_0 u_{it}^{(n)} = \left(n - \frac{2}{\mu^2 - 1} \right) \left\{ \left(\frac{1}{\rho_1^{2n}} - 1 \right) + n \left(\frac{1}{\rho_1^2} - 1 \right) \right\} \frac{\rho^{n+1}}{n+1} + \left(n + \frac{2}{\mu^2 - 1} \right) \left\{ (\rho_1^{2n} - 1) + n \left(1 - \frac{1}{\rho_1^2} \right) \right\} \frac{\rho^{-n+1}}{n-1} + \left\{ n^2 (\rho_1^2 - 1) + n \left(1 - \frac{1}{\rho_1^{2n}} \right) \right\} \frac{\rho^{n-1}}{n-1} + \{ n^2 (1 - \rho_1^2) + n (1 - \rho_1^{2n}) \} \frac{\rho^{-n-1}}{n+1}$$

$$2K_0 v_{it}^{(n)} = \left(n + \frac{2\mu^2}{\mu^2 - 1} \right) \left\{ \left(1 - \frac{1}{\rho_1^{2n}} \right) + n \left(1 - \frac{1}{\rho_1^2} \right) \right\} \frac{\rho^{n+1}}{n+1} + \left(n - \frac{2\mu^2}{\mu^2 - 1} \right) \left\{ (\rho_1^{2n} - 1) + n \left(1 - \frac{1}{\rho_1^2} \right) \right\} \frac{\rho^{-n+1}}{n-1} + \left\{ n^2 (1 - \rho_1^2) + n \left(\frac{1}{\rho_1^{2n}} - 1 \right) \right\} \frac{\rho^{n-1}}{n-1} + \{ n^2 (1 - \rho_1^2) + n (1 - \rho_1^{2n}) \} \frac{\rho^{-n-1}}{n+1}$$

$$2K_0 \sigma_{r, it}^{(n)} = \left\{ n^2 \left(\frac{1}{\rho_1^2} - 1 \right) + n \left(\frac{1}{\rho_1^{2n}} + 1 - \frac{2}{\rho_1^2} \right) + 2 \left(1 - \frac{1}{\rho_1^{2n}} \right) \right\} \rho^n + \left\{ n^2 \left(\frac{1}{\rho_1^2} - 1 \right) + n \left(\frac{2}{\rho_1^2} - 1 - \rho_1^{2n} \right) + 2(1 - \rho_1^{2n}) \right\} \rho^{-n} + \left\{ n^2 (\rho_1^2 - 1) + n \left(1 - \frac{1}{\rho_1^{2n}} \right) \right\} \rho^{n-2} + \{ n^2 (\rho_1^2 - 1) + n (\rho_1^{2n} - 1) \} \rho^{-n-2}$$

$$2K_0 \sigma_{\theta, it}^{(n)} = \left\{ n^2 \left(1 - \frac{1}{\rho_1^2} \right) + n \left(3 - \frac{2}{\rho_1^2} - \frac{1}{\rho_1^{2n}} \right) + 2 \left(1 - \frac{1}{\rho_1^{2n}} \right) \right\} \rho^n + \left\{ n^2 \left(1 - \frac{1}{\rho_1^2} \right) + n \left(\rho_1^{2n} + \frac{2}{\rho_1^2} - 3 \right) + 2(1 - \rho_1^{2n}) \right\} \rho^{-n} + \left\{ n^2 (1 - \rho_1^2) + n \left(\frac{1}{\rho_1^{2n}} - 1 \right) \right\} \rho^{n-2} + \{ n^2 (1 - \rho_1^2) + n (1 - \rho_1^{2n}) \} \rho^{-n-2}$$

$$2K_0 \tau_{r\theta, it}^{(n)} = \left\{ n^2 \left(1 - \frac{1}{\rho_1^2} \right) + n \left(1 - \frac{1}{\rho_1^{2n}} \right) \right\} \rho^n + \left\{ n^2 \left(\frac{1}{\rho_1^2} - 1 \right) + n (1 - \rho_1^{2n}) \right\} \rho^{-n} + \left\{ n^2 (1 - \rho_1^2) + n \left(\frac{1}{\rho_1^{2n}} - 1 \right) \right\} \rho^{n-2} + \{ n^2 (\rho_1^2 - 1) + n (\rho_1^{2n} - 1) \} \rho^{-n-2}$$

where

$$K_0 = \left(\rho_1^n - \frac{1}{\rho_1^n} \right)^2 - n^2 \left(\rho_1 - \frac{1}{\rho_1} \right)^2$$

And, the second terms represent the superposition of the solutions of the natural vibrations for the mode of circumferential wave number n . And the functions used in Eqs.(18) are defined by the equations

$$N_1(x) = \begin{vmatrix} f_3(\rho x) & g_5(\rho x) & f_6(\mu\rho x) & g_8(\mu\rho x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu\rho_1 x) & g_2(\mu\rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu\rho_1 x) & g_3(\mu\rho_1 x) \end{vmatrix}$$

$$N_2(x) = \begin{vmatrix} f_3(\rho x) & g_6(\rho x) & f_3(\mu\rho x) & g_5(\mu\rho x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu\rho_1 x) & g_2(\mu\rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu\rho_1 x) & g_3(\mu\rho_1 x) \end{vmatrix}$$

$$N_3(x) = \begin{vmatrix} f_1(\rho x) & g_1(\rho x) & f_2(\mu\rho x) & g_2(\mu\rho x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu\rho_1 x) & g_2(\mu\rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu\rho_1 x) & g_3(\mu\rho_1 x) \end{vmatrix}$$

$$N_4(x) = \begin{vmatrix} f_4(\rho x) & g_4(\rho x) & f_2(\mu\rho x) & g_2(\mu\rho x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu\rho_1 x) & g_2(\mu\rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu\rho_1 x) & g_3(\mu\rho_1 x) \end{vmatrix}$$

$$N_5(x) = \begin{vmatrix} f_2(\rho x) & g_2(\rho x) & f_3(\mu\rho x) & g_3(\mu\rho x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu\rho_1 x) & g_2(\mu\rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu\rho_1 x) & g_3(\mu\rho_1 x) \end{vmatrix}$$

$$\begin{aligned}
D_0(x) = & \begin{vmatrix} f_{11}(x) & g_1(x) & f_2(\mu x) & g_2(\mu x) \\ f_{11}(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu \rho_1 x) & g_2(\mu \rho_1 x) \\ f_{21}(x) & g_2(x) & f_3(\mu x) & g_3(\mu x) \\ f_{21}(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu \rho_1 x) & g_3(\mu \rho_1 x) \end{vmatrix} \\
& + \begin{vmatrix} f_{11}(x) & g_{11}(x) & f_2(\mu x) & g_2(\mu x) \\ f_{11}(\rho_1 x) & g_{11}(\rho_1 x) & f_2(\mu \rho_1 x) & g_2(\mu \rho_1 x) \\ f_{21}(x) & g_{21}(x) & f_3(\mu x) & g_3(\mu x) \\ f_{21}(\rho_1 x) & g_{21}(\rho_1 x) & f_3(\mu \rho_1 x) & g_3(\mu \rho_1 x) \end{vmatrix} \\
& + \begin{vmatrix} f_1(x) & g_1(x) & f_{21}(\mu x) & g_2(\mu x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_{21}(\mu \rho_1 x) & g_2(\mu \rho_1 x) \\ f_2(x) & g_2(x) & f_{31}(\mu x) & g_3(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_{31}(\mu \rho_1 x) & g_3(\mu \rho_1 x) \end{vmatrix} \\
& + \begin{vmatrix} f_1(x) & g_1(x) & f_2(\mu x) & g_{21}(\mu x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu \rho_1 x) & g_{21}(\mu \rho_1 x) \\ f_2(x) & g_2(x) & f_3(\mu x) & g_{31}(\mu x) \\ f_2(\rho_1 x) & g_2(\rho_1 x) & f_3(\mu \rho_1 x) & g_{31}(\mu \rho_1 x) \end{vmatrix}
\end{aligned}$$

where

$$\begin{aligned}
f_1(x) &= \left(\frac{n^2+n}{x^2} - \frac{1}{\kappa} \right) J_n(x) - \frac{1}{x} J_{n-1}(x) \\
f_2(x) &= \frac{n^2+n}{x^2} J_n(x) - \frac{n}{x} J_{n-1}(x) \\
f_3(x) &= \left(\frac{n^2+n}{x^2} - \frac{1}{2} \right) J_n(x) - \frac{1}{x} J_{n-1}(x) \\
f_4(x) &= \left(\frac{n^2+n}{x^2} + \frac{1-\kappa}{\kappa} \right) J_n(x) - \frac{1}{x} J_{n-1}(x) \\
f_5(x) &= \frac{n}{x^2} J_n(x) - \frac{1}{x} J_{n-1}(x) \\
f_6(x) &= \frac{n}{x^2} J_n(x) \\
f_{11}(x) &= \left\{ \frac{n(n+1)(n+2)}{x^2} - \frac{n+\kappa}{\kappa} \right\} J_n(x) - \left(\frac{n^2+2}{x} - \frac{x}{\kappa} \right) J_{n-1}(x) \\
f_{21}(x) &= \left\{ \frac{n(n+1)(n+2)}{x^2} - n \right\} J_n(x) - \frac{3n}{x} J_{n-1}(x) \\
f_{31}(x) &= \left\{ \frac{n(n+1)(n+2)}{x^2} - \frac{n+2}{2} \right\} J_n(x) - \left(\frac{n^2+2}{x} - \frac{x}{2} \right) J_{n-1}(x)
\end{aligned}$$

Here, $g(x)$ is a function obtained by replacing $J_n(x)$ by $Y_n(x)$ in $f(x)$ and, $J_n(x)$ and $Y_n(x)$ are the Bessel functions of order n of the first and second kind, respectively. The natural circular frequencies in dimensionless form $\lambda_m^{(n)}$ are the roots of the following transcendental equation about x .

$$\begin{vmatrix} f_1(x) & g_1(x) & f_2(\mu x) & g_2(\mu x) \\ f_1(\rho_1 x) & g_1(\rho_1 x) & f_2(\mu \rho_1 x) & g_2(\mu \rho_1 x) \\ f_3(x) & g_3(x) & f_3(\mu x) & g_3(\mu x) \\ f_3(\rho_1 x) & g_3(\rho_1 x) & f_3(\mu \rho_1 x) & g_3(\mu \rho_1 x) \end{vmatrix} = 0 \quad \dots\dots\dots(19)$$

As a result, the circumferential stress σ_θ , for example, produced in the cylinder due to step line loads $q(t)=q_0 H(t)$, where $H(t)$ is unit step function, is given by the equation

$$\sigma_\theta = \sigma_\theta^{(0)} + \sum_{n=2,4,\dots}^{\infty} \sigma_\theta^{(n)} \quad \dots\dots\dots(20)$$

And if the total magnitude F_0 of line loads is provided, q_0 may be equal to $F_0/1$.

By applying Duhamel's integral to the result in the case of the step load, the results in the cases of impact loads having arbitrary time histories may be easily obtained. Furthermore, changing a and b in Eqs.(18), the results for the line loads acting on the inner surface of the cylinder may be readily shown.

4. Numerical results

Numerical calculations were carried out with use of a digital computer (HITAC 5020E, Tokyo University) for the case of $\kappa=0.7$, that is, an infinite cylinder of $\nu=0.23$ or a thin circular plate of $\nu=0.3$.

4.1 Roots of transcendental equation

The roots $\lambda_m^{(n)}$ of Eq.(19) against ρ_1 , which are found out by trial and error procedure, are shown in Fig.5 in the case of $m=1$.

The lowest frequencies $\lambda_1^{(n)}$ tend to increase as n increases or ρ_1 decreases, and $\lambda_1^{(n)}$ except $n=0$ tends to zero as ρ_1 becomes close to unity, that is, as wall-thickness of a cylinder becomes negligible. On the other hand, $\lambda_1^{(0)}$, which expresses the result of axisymmetric vibrations, tends to $\sqrt{2\kappa-\kappa^2}=0.954$ as ρ_1 becomes close to unity.

Fig.5 Natural circular frequencies of nonaxially symmetrical vibration of a cylinder or a disk

4.2 The case of pressures in the form of trigonometric functions

Generally speaking, a series of displacement components seems to converge more quickly than that of stress components, and the effects of higher modes of natural oscillations on stress components are more severe when time τ after the impact is very short. Further, it will be shown that the stress σ_θ at the inner surface of the cylinder remains zero until the time $\tau_0=1-\rho_1$ when the elastic wave emanating from the outer surface due to the impulsive pressure arrives first.

$\rho_1 = 0.8$, $\rho = \rho_1$, $\theta = 0^\circ$, $\nu = 0.3$, $m = 1 \sim 20$, Chain lines denote static solutions

Fig.6 Radial displacement and circumferential stress due to impulsive pressures, whose space distributions are in the form of trigonometric functions, with a step distribution in time

So, in Eqs.(18), the convergence of the series for $\sigma_\theta^{(n)}$ on the inner surface of the cylinder during time $\tau < \tau_0$ will be worse condition comparing with another situation.

From numerical examination, it is known that the series in Eqs.(18) converges more slowly as n becomes large or ρ_1 becomes small. For example, by the summation of the stress components with $\lambda^{(n)}$ less than 100, without reference to ρ_1 (for smaller ρ_1 , larger m is needed), the first three significant figures of the results are obtained in the case $n \leq 10$, whereas the two figures in the case $10 < n < 30$.

Under these considerations, the numerical calculations for the time variations of the radial displacement $u^{(n)}$ and the circumferential stress $\sigma_\theta^{(n)}$ on the inner surface of the cylinder are carried out and graphically shown in Figs.6 and 7. It is seen from these results that the stress and displacement are much affected mainly by the lowest mode except when n is large or ρ_1 is small, and those oscillate about the corresponding static solutions, which decrease as n becomes large or ρ_1 becomes small.

Understanding that the circumferential stress on the inner surface is the maximum

stress in the cylinder, we introduce the dynamic factor k for $\sigma_\theta^{(n)}$ defined by the ratio of the dynamic maximum value to the static one, and show the values of k against ρ_1 in Fig.8, which are evaluated using the maximum value from the first four or five peaks of stress variations in time.

The value of k increases as n becomes large and in each case equally tends to 2 as $\rho_1 \rightarrow 1$.

$n = 10$, $\rho = \rho_1$, $\theta = 0^\circ$, $\nu = 0.3$, Chain lines denote static solutions

Fig.7 Circumferential stress due to impulsive pressures, whose space distributions are in the form of trigonometric functions, with a step distribution in time

$\nu = 0.3$, Solid line denotes the case of line loads

Fig.8 Effect of n on the relation between the dynamic factor k and the ratio of the inner radius to the outer radius

4.3. The case of line loads

In Fig.9, the load distribution is shown which is approximated by the Fourier series expansion Eq.(10) truncated n to 40. Though it is poor approximation to regard Fig.9 as line loads, the stresses in the cylinder, expressed by Eq.(20), converge

(Summation of Eq.(10) with n ranging from 0 to 40.)

Fig.9 Line loads expressed by a Fourier series expansion

Maximum value in each term of Eq.(20), $\nu=0.3$, $\rho_1=0.8$
Fig.10 Convergent condition of the circumferential stress on the inner surface at the loaded points

$\rho_1=0.8$, $\rho=\rho_1$, $\theta=0^\circ$, $\nu=0.3$

Fig.11 Amplitude of the circumferential stress corresponding to each mode of the cylinder or the disk [Eqs.(18)]

with n less than 40 satisfactorily except in the vicinity of the loaded points. For example, the convergent condition of the circumferential stress at the point on the inner surface behind the loaded point is shown in Fig.10. In Fig.10, vertical axis shows the maximum value from some initial peaks of time variations.

Though the series with m as shown in Fig.11 converges more slowly as n becomes large, the results for line loads are given accurately using the summation with $\lambda^{(n)}$ less than 100 for each n and with n^m from 0 to 40, because the stress variations are governed mainly by the lowest mode in each of n , especially $n=2$.

In Figs.(12), the circumferential stress variations in time on the inner surface of the cylinder of $\rho_1=0.8$ are shown. The stresses oscillate about the corresponding static values and are mainly influenced by the lowest mode in the case of $n=2$ except at the point $\theta=45^\circ$, where this mode disappears and the lowest mode in the case $n=4$ is clearly observed.

In addition, it is seen from Fig.13 that the absolute values of the stresses tend to decrease nearly in proportion to the square of ρ_1 as the wall-thickness of the cylinder decreases.

$\rho_1=0.8$, $\rho=\rho_1$, $\nu=0.3$, $n=0\sim 40$, $m=1\sim 10$
Chain lines denote static solutions

Fig.12 Circumferential stress due to impulsive line loads with a step distribution in time (1)

$\rho = \rho_1$, $\theta = 0^\circ$, $\nu = 0.3$, $n = 0 \sim 40$, Chain lines denote static solutions

Fig.13 Circumferential stress due to impulsive line loads with a step distribution in time (2)

$\rho = \rho_1$, $\theta = 0^\circ$, $\nu = 0.3$, Chain lines denote static solutions

Fig.14 Radial displacement due to impulsive line loads with a step distribution in time

On the other hand, without reference to the wall-thickness, the displacement variations in time are more affected by the fundamental mode as shown in Figs.14.

The dynamic factor k evaluated from the circumferential stress on the inner surface behind loaded points is shown in Fig.8. The value of k in the case of line loads seems to be between those of the cases of $n=4$ and 6 for the distributed pressures.

5. Conclusions

In this paper, the problem of a cylinder subjected to diametrically opposite line loads is analyzed based on the dynamic theory of elasticity.

The following results are obtained.

- (1) This problem can be analyzed by dividing it into two auxiliary parts. One is a part of the uniform pressure which has been solved and the other a part of the pressure of trigonometric distribution along the circumferential boundary.
- (2) For the cylinder subjected to the impulsive pressure $p(t)\cos(n\theta)$, the stress in the case of $n=2$ is the largest.
- (3) In the cylinder subjected to a step loads along its outer generators, the maximum tensile stress appears on the lines of the inner surface corresponding to the loaded ones. And it oscillates mainly in the lowest mode in the case of $n=2$.
- (4) The maximum dynamic tensile stress tends to be twice as large as the corresponding static solution, when the wall-thickness of the cylinder becomes smaller.

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