621.25 : 534.113

Balancing of a Flexible Rotor*

(The Fourth Report, Some Experiments on a Seven-disked Flexible Rotor/Bearing System)

By Shuzo MIWA**, Takafumi NAKAI***,
Ichiro MIMURA****, Yuichi MINAMI*****

The conditional equation of balance of a flexible rotor and the possibility of a new balancing procedure by aid of a mode separation network have previously been proposed by one of the authors. This paper deals with some experiments to verify the theory, where a variety of unbalance distributions is artificially given to a seven-disked flexible rotor which is supported by two flexible bearings and is balanced based on our theory. The bearing vibration together with dynamic shaft deflection after balancing is minimized over a wide range of rotational speeds as high as its third critical speed. Thus the theory is experimentally proved to be valid for balancing a flexible rotor.

1. Introduction

The conditional equation for the balance of a flexible rotor supported by two bearings, flexible or rigid, has been previously presented by one of the authors⁽¹⁾.

As second step of the research, the possibility of balancing a flexible rotor for unknown distribution of initial unbalances was investigated theoretically, introducing a new balancing procedure by aid of a 'mode separation network' through which the state of balance was to be achieved of the rotor only from the information on the bearing vibrations.⁽¹⁾ This paper intends to

make proof of these theories by experiments. Experiments like this have hitherto been given by, for instance, Parkinson and his team⁽³⁾ or Kawai and Iwatsubo⁽⁴⁾. But their definitions on 'balance of a flexible rotor' differ slightly from that of the authors and, in addition to this, their method of balancing a rotor requires the measurement of shaft vibration at separate axial positions other than bearing station, which would be almost unfit for practical use in rotating machines. This paper deals with two experiments on balancing a flexible rotor; one is for proving the conditional equation of balance and the other for verifying the balancing procedure by means of a mode separation network.

Weight of each disk	0.36 kg	Model No. of ball bearing	11200
Weight of bearing block (left)	1.69 kg	Supporting stiffness of bearing	
Weight of bearing block (right)	1.79 kg	(for each bearing block)	4.5 kg/mm

Fig. 1 Test-rotor/bearing system

* Received 7th April, 1972

** Associate Professor, Aoyama-Gakuin University, Tokyo

*** Lecturer, Aoyama-Gakuin University, Tokyo

**** Design Engineer, Fuji Electric Co. Ltd., Kawasaki

***** Research Fellow, Akashi Seisakusho, Ltd., Tokyo

2. Analysis on the vibration and balancing of the test rotor/bearing system

In previous paper⁽¹⁾⁽²⁾, a flexible rotor was considered to be a system of distributed mass along the shaft axis (continuous system), but here a rotor whose masses are concentrated at certain axial stations of the shaft axis (discrete system) is treated for simplicity of analysis and experiments.

The difference between the two systems, continuous and discrete, does not require any substantial change of the theory.

The rotor/bearing system applied in the experiments is illustrated in Fig. 1. The flexible rotor consists of a uniform slender shaft carrying seven disks of the same mass and size at equally spaced axial stations as shown in the Figure (in our theory, however, the mass distribution and the axial arrangement of seven disks can be arbitrary). Both ends of the shaft are supported by ball bearings of self aligning type, so that end condition of the shaft is regarded as simply supported, i.e. pinned-pinned ends to the bearing box. The bearing box is further suspended by two parallel leaf springs from the bed, and they constitute an inverted pendulum system which is free to vibrate in the horizontal plane. Consequently the end condition of the rotor (the shaft with seven disks) is 'pinned-pinned' for the shaft vibration in vertical plane and is 'flexibly supported' for vibration in horizontal plane.

There being no coupled motion between two components of shaft vibration in vertical and horizontal planes, only the latter, i.e. vibration in horizontal plane, is considered in the following experiments as representative of the vibration of the rotor on 'generally flexible' bearings. One end of the rotor is connected to a small universal joint shaft for drive, but the overhanging part of the rotor is designed as small as possible. Since the shaft vibration due to unbalance forces is very small compared with the distance between two bearings, the system is considered to be linear. Further, the following assumptions are practically recognized in our rotor/bearing system.

- (1) There is no gyroscopic effect of disk motion, no influence of gravity, no internal nor external damping.
- (2) The bending stiffness of the shaft is isotropic and the stiffness of leaf springs for supporting bearing boxes is constant as for the vibration in the horizontal plane.
- (3) The shaft is supported with pinned-pinned condition to the bearing boxes, which are supported flexibly on the base.
- (4) Mass of shaft itself is ignored compared with those of disks and bearing boxes, hence the rotor/bearing system constitutes a discrete system of nine degrees of freedom.
- (5) A half of the mass of universal joint shaft is considered to be concentrated at the end of the rotor shaft, and no other influence does not come to the rotor from the drive shaft.

The nomenclature used in this paper

are as follows:

- 0-xyz : Co-ordinates rotating with the rotor at angular velocity ω . The z-axis coincides with the shaft axis at $\omega = 0$.
- 0-XYZ : Spatially fixed co-ordinates, Z coincides with the shaft axis z.
- M_i : Concentrated mass at $z = z_i$ ($i=0, 1, 2, \dots, 8$).
- u_i : Initial unbalance at $z = z_i$ ($i=1, 2, \dots, 7$).
- U_i : Compensating unbalance at $z = z_i$ ($i = 1, 2, \dots, 7$).
- u_{ix} : Component of rotating vector $u_i e^{j\omega t}$
- $\phi_n(z)$, ω_n : N-th characteristic function of shaft deflection, and n-th critical speed of the system.
Note: $\phi_n(z)$ is defined and enumerated only at $z = z_i$ ($i = 1, 2, \dots, 8$).
- α_{ij} : Influence factor to the shaft deflection at z_i when unit of transverse load is applied at z_j .
- K : The stiffness of bearing support, or bearing stiffness as it is called (per left or right bearing).

2.1 The critical speed and the characteristic function of the system

The equation of motion of the rotor/bearing system is expressed as for the movement of X-direction of the 0-XYZ system as:

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{08} \\ \alpha_{10} & \alpha_{11} & \dots & \alpha_{18} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{80} & \alpha_{81} & \dots & \alpha_{88} \end{bmatrix} \begin{bmatrix} -M_0 \ddot{X}_0 \\ -M_1 \ddot{X}_1 + \omega^2 u_{1x} \\ \vdots \\ -M_7 \ddot{X}_7 + \omega^2 u_{7x} \\ -M_8 \ddot{X}_8 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_8 \end{bmatrix} \quad \dots \dots (1)$$

Considering that the exciting forces are induced by the X-components of unbalances $u_{ix} = \hat{u}_{ix} e^{j\omega t}$ (here $\hat{u}_{ix}$ is the X-component of u_i when $t = 0$), the trial solution of X_i may reasonably be assumed as $X_i = \bar{X}_i e^{j\omega t}$ for steady motion of mass M_i . Substituting this trial solution into eq.(1) and re-arranging gives

$$\begin{bmatrix} 1 - \omega^2 M_0 \alpha_{00} & -\omega^2 M_1 \alpha_{01} & \dots & -\omega^2 M_8 \alpha_{08} \\ -\omega^2 M_0 \alpha_{10} & 1 - \omega^2 M_1 \alpha_{11} & \dots & -\omega^2 M_8 \alpha_{18} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega^2 M_0 \alpha_{80} & -\omega^2 M_1 \alpha_{81} & \dots & 1 - \omega^2 M_8 \alpha_{88} \end{bmatrix} \begin{bmatrix} \bar{X}_0 \\ \bar{X}_1 \\ \vdots \\ \bar{X}_8 \end{bmatrix} = \omega^2 \begin{bmatrix} \sum_{j=1}^7 \alpha_{0j} \hat{u}_{jx} \\ \sum_{j=1}^7 \alpha_{1j} \hat{u}_{jx} \\ \vdots \\ \sum_{j=1}^7 \alpha_{8j} \hat{u}_{jx} \end{bmatrix} \quad \dots \dots (2)$$

Influence factor α_{ij} can be calculated by the theory of strength of material for end conditions given in assumption (3) of previous section as

$$\alpha_{ij} = \frac{\rho^3}{8EI} \left[\xi_i (1 - \xi_j) \{ 1 - (1 - \xi_j)^2 W(1 - \xi_j) - \xi_j^2 \} \right. \\ \left. + (\xi_i - \xi_j)^3 W(\xi_i - \xi_j) \right] - \xi_i \frac{1 - 2\xi_j}{K} + \frac{1 - \xi_j}{K} \quad \dots \dots (3)$$

Where $\xi_i = z_i/L$, $\xi_j = z_j/L$ and $W(\xi)$ is unit weighting function defined as

$$W(\xi) = \begin{cases} 1 & \text{when } \xi \geq 0 \\ 0 & \text{when } \xi < 0 \end{cases}$$

When RHS of eq.(2) is equal to zero, eq.(2) gives the equation of free vibration, which is a linear homogeneous algebraic equation in $\bar{X}_i$ ($i = 0, 1, 2, \dots, 8$). This equation has a solution other than the trivial one, $\bar{X}_i = 0$ ($i = 0, 1, 2, \dots, 8$), only if the determinant $\Delta(\omega)$ of the coefficients matrix in eq. (2) is zero. Therefore

$$\Delta(\omega) = \begin{vmatrix} 1 - \omega^2 M_0 \alpha_{00} & -\omega^2 M_1 \alpha_{01} & \dots & -\omega^2 M_8 \alpha_{08} \\ -\omega^2 M_0 \alpha_{10} & 1 - \omega^2 M_1 \alpha_{11} & \dots & -\omega^2 M_8 \alpha_{18} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega^2 M_0 \alpha_{80} & -\omega^2 M_1 \alpha_{81} & \dots & 1 - \omega^2 M_8 \alpha_{88} \end{vmatrix} = 0 \quad (4)$$

, which is a frequency equation. Substituting ω_j of eq.(3) into eq.(4) and solving eq. (4) for ω gives nine distinct roots of $\omega = \omega_n$ ($n = 1, 2, \dots, 9$), which are n -th critical speed of the system.

The amplitude ratio of the principal modes can be obtained by re-arrangement of eq. (2) assuming that RHS of eq.(2) is zero, i.e.

$$\begin{bmatrix} 1 - \omega^2 M_0 \alpha_{00} & -\omega^2 M_1 \alpha_{01} & \dots & -\omega^2 M_8 \alpha_{08} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega^2 M_0 \alpha_{80} & -\omega^2 M_1 \alpha_{81} & \dots & 1 - \omega^2 M_8 \alpha_{88} \end{bmatrix} \begin{bmatrix} \bar{X}_0 / \bar{X}_8 \\ \bar{X}_1 / \bar{X}_8 \\ \vdots \\ \bar{X}_7 / \bar{X}_8 \end{bmatrix} = \omega^2 M_8 \begin{bmatrix} \alpha_{08} \\ \alpha_{18} \\ \vdots \\ \alpha_{78} \end{bmatrix} \quad (5)$$

By substituting the value of each of ω_n ($n = 1, 2, \dots, 9$) into eq.(5) we obtain the solution vector $(\bar{X}_0 / \bar{X}_8, \dots, \bar{X}_7 / \bar{X}_8)_{\omega = \omega_n}$ for each principal mode of $n = 1, 2, \dots, 9$. This solution vector implies the amplitude ratio of masses M_i ($i = 0, 1, 2, \dots, 8$) to M when the system vibrates under the n -th critical speed $\omega = \omega_n$. The solution vector is, therefore, no other than the mode of vibration at $\omega = \omega_n$, so that we express the amplitude ratio as

$$(\bar{X}_i / \bar{X}_8)_{\omega = \omega_n} = \phi_n(z_i) \quad (6)$$

Referring to the case of a continuous mass

system, we call the solution vector for $\omega = \omega_n$ as n -th characteristic function for the discrete mass system. This $\phi_n(z)$ can therefore, be defined only at $z = z_i$ ($i = 0, 1, 2, \dots, 8$). For the end conditions shown in Fig. 1, the characteristic function defined in eq.(6) has the orthogonality relation as follows:

$$\sum_{i=0}^8 M_i \phi_m(z_i) \phi_n(z_i) = \begin{cases} 0 & (m \neq n) \\ \neq 0 & (m = n) \end{cases} \quad (7)$$

The numerical values of ω_n and $\phi_n(z)$ calculated for the system of Fig. 1 will be shown later in section 3.2.

2.2 The conditional equation of balance

The conditional equation of balance for a flexible rotor consisting of continuous masses has been presented in the previous paper¹. Applying this to the rotor of discrete masses as shown in Fig. 1, a set of compensating unbalances U_i to be added at $z = z_i$ for initial unbalances u_j ($j = 1, 2, \dots, 7$) at $z = z_j$ should satisfy the following equation as

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{h+2} \\ \phi_1(z_1) & \phi_1(z_2) & \dots & \phi_1(z_{h+2}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_h(z_1) & \phi_h(z_2) & \dots & \phi_h(z_{h+2}) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{h+2} \end{bmatrix} = - \begin{bmatrix} \sum_{j=1}^7 u_j \\ \sum_{j=1}^7 u_j z_j \\ \sum_{j=1}^7 u_j \phi_1(z_j) \\ \vdots \\ \sum_{j=1}^7 u_j \phi_h(z_j) \end{bmatrix} \quad (8)$$

where h is the maximum order of the mode to be balanced. Hence, the correction planes as many as $i = h + 2$ are generally necessary and sufficient.

Unbalances u_j and U_i are vectors in a plane perpendicular to shaft axis (vectors in xy -plane fixed to the rotor), so that components of these unbalances projected on x - or y -axis should also satisfy the relation of eq. (8). For simplicity, all of unbalances, u_j and U_i are placed on the rotor at the same angular position, i.e. in an xz -plane, in the following experiments. Thus u_j and U_i are regarded to be scalar, but this does not injure the generality of the theory.

Unbalance distributions of two typical types are concerned in the experiments as shown in Table 1. They are (i) uniformly distributed

Table 1. Typical distribution of initial unbalances and necessary corrections for them*

		Unit: gram						
Disk No.		1	2	3	4	5	6	7
Uniformly distributed unbalances	Initial	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	Correction	0.9	1.45	-	2.3	-	1.45	0.9
Triangularly distributed unbalances	Initial	0.0	0.4	0.8	1.2	1.6	2.0	2.4
	Correction	0.0	0.6	-	2.7	-	2.9	2.2

* Five-plane-balancing (balancing up to the third mode) is considered. Correction weights are calculated values. Radius where unbalance masses are to be attached: 33 mm

buted unbalances and (ii) triangularly distributed ones. In RHS of eq.(8), the column matrix of u_j , the top element is a component proportional to unbalance forces, the second is one proportional to moment of unbalance forces, and the third through bottom are ones proportional to the first, second, ..., up to n -th dynamic shaft deflection due to unbalance forces. As the system shown later in Fig. 5 is nearly symmetrical as for mid-span, it is estimated that the uniformly distributed unbalances contain no modal component of even-numbered order, whereas the triangularly distributed ones have every order of modal components.

In the experiments we assume the maximum operating speed of the rotor lies between the critical speeds of ω_3 and ω_4 , so that the rotor is to be balanced up to the third modal component of shaft deflection. The number of correction planes shall, therefore, be $h+2=3+2=5$. The selection of five correction planes from seven disks is arbitrary, but we select them symmetrically at z_i of $i=1,2,4,6$ and 7 . Correction unbalances U_i necessary to balance two types of initial unbalances can be calculated from eq.(8) if $\phi_n(z_i)$ is known, numerical results of which are tabulated in Table 1. It is quite natural that different sets of corrections are obtained even for the same initial unbalances depending on the selection of correction planes.

2.3 Balancing procedure by means of mode separation network

Detail of the method has been given in the literature⁽²⁾, so this is briefly reviewed here for the system shown in Fig. 1. (a) At first, balance the rotor at low speed where any significant shaft deflection does not yet occur. Through this low speed balance, or dynamic balance, the first and second elements from top of RHS of eq.(8) vanish. Correction unbalances further needed after low speed balance being U_i , eq.(8) becomes

$$T \cdot \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{h+2} \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ u^1 \\ u^2 \\ \vdots \\ u^h \end{bmatrix} \quad \dots\dots (8a)$$

where T is the square matrix appearing in LHS of eq.(8) and $u^n = \sum_{j=1}^n u_j \phi_n(z_j)$. It should be noted that u_j as a component of u^n is a composite unbalance existing originally in the rotor plus correction unbalance applied for low speed balancing.

(b) According to eq.(21) of the previous paper⁽¹⁾ the amplitude of bearing vibration X_0 or X_3 at any speed ω can be expressed as a function of u^n of eq.(8a) or u_j (in case where u_j lies at any angular position as for j , X_0 and X_3 will be complex amplitude) as follows:

$$\left. \begin{aligned} X_i &= \sum_{n=1}^q A_n(\omega) \frac{u^n}{N_n} \cdot \phi_n(z_i), \quad (i=0,3) \\ A_n(\omega) &= (\omega/\omega_n)^2 / [1 - (\omega/\omega_n)^2] \\ N_n &= \sum_{j=0}^g M_j \phi_n^2(z_j) \end{aligned} \right\} \quad \dots\dots (9)$$

In general, $A_m(\omega) \ll A_n(\omega)$ holds for any m of $m > n$, so that bearing vibration may be well approximated for speed $\omega = \Omega_1 < \omega_1$ as

$$X_i = A_1(\Omega_1) \frac{u^1}{N_1} \phi_1(z_i) + A_2(\Omega_1) \frac{u^2}{N_2} \phi_2(z_i) \quad (i=0,3) \quad \dots\dots (9a)$$

The second term of RHS of eq.(9a) should not generally be neglected, because it cannot always be concluded for unknown magnitudes of u^1 and u^2 that $A_2(\Omega_1)u^2 \ll A_1(\Omega_1)u^1$ even when $A_2(\Omega_1) \ll A_1(\Omega_1)$.

(c) Pick up the bearing vibration as electric signal, e.g. electromotive force, and the following operation is performed by an electric computing network, such that

$$\begin{aligned} X_c(\Omega_1) &= X_0(\Omega_1) + \gamma X_3(\Omega_1) \\ &= \frac{A_1(\Omega_1)}{N_1} u^1 \{ \phi_1(z_0) + \gamma \phi_1(z_3) \} \\ &\quad + \frac{A_2(\Omega_1)}{N_2} u^2 \{ \phi_2(z_0) + \gamma \phi_2(z_3) \} \end{aligned} \quad \dots\dots (10)$$

Adjusting γ of eq.(10) and setting it as $\gamma = \gamma_1 = -\phi_2(z_0)/\phi_2(z_3)$ makes the second term of eq.(10) null irrespective of the magnitude of u^2 . This means that the output signal $X_c(\Omega_1)$ of the network has the same phase angle of u^1 and its magnitude is proportional to that of u^1 . Consequently, to measure output X_c is to know the third row element of RHS of eq.(8a).

(d) Calculate a set of correction unbalances $\{U_i\}_{i=1}^h$ from eq.(8a) by assuming every element except u^1 in RHS of eq.(8a) to be null. This set of correction unbalances compensates only the first modal component of shaft deflection u^1 without disturbing the state of 'low speed balance' as well as any modal components of order m other than n . Apply these correction unbalances on the rotor and balance it for the modal component u^1 of initial unbalances.

(e) The bearing vibration at speed of $\omega = \Omega_2 < \omega_2$ after balancing u^1 becomes

$$X_i(\Omega_2) = \frac{A_2(\Omega_2)}{N_2} u^2 \phi_2(z_i) + \frac{A_3(\Omega_2)}{N_3} u^3 \phi_3(z_i) \quad (i=0,3) \quad \dots\dots (9b)$$

Like step (c), adjust the factor γ of the network equal to $\gamma = \gamma_2 = -\phi_3(z_0)/\phi_3(z_3)$, then output of the network becomes proportional to u^2 independent of higher component u^3 . Calculate the compensating unbalance $\{U_i\}_{i=2}^h$ in similar manner to step (d) and attach it on the rotor to balance out the effect of u^2 . (f) Similar procedures are repeated until compensation of u^n reaches u^h for desired order h .

An example of the network to perform the operation of eq.(10) and how to adjust the factor γ for the purpose have been presented precisely in the literature⁽²⁾.

Fig. 2 Schematic diagram of the whole system

3. Experiments

3.1 Apparatus

A block diagram of the apparatus used in the experiments is shown in Fig. 2. The main part of the apparatus including the test rotor with its bearings is photographically shown in Fig. 3.

Detail of the rotor and its bearings has been given in section 1. One end of the rotor is connected to the drive shaft by a small universal joint shaft and can be rotated at any speed between 250 and 5,000 rpm. The vibrations of left and right bearings are detected by electromagnetic pickups and induced electric signals, the phase and the magnitude of which correspond to those of bearing vibrations, are led into the mode separation network and are finally indicated on the meter. The indicator head including electrical networks and both meters to indicate the phase and amount of the signal is one of a micro-balancer manufactured by Akashi Seisakusho, Ltd. The over-all characteristics of the measuring system is flat over the speed range of 1,500 to 5,000 rpm, and the accuracy of meter indication is $\pm 5^\circ$ for phase indication and $\pm 10\%$ for amount indication. Bearing vibration as small as 0.05μ in amplitude can be detected by the equipment as its maximum sensitivity. For speeds lower than 1,500 rpm, the accuracy of indication is maintained at the same level as above by use of a correction table, while minimum

Fig. 3 Main parts of the apparatus

detectable bearing vibration decreases in this speed range and bearing vibration of 1μ can be detected at 400 rpm. But we consider this sensitivity is still enough for our purpose.

Angle datum mark attached to the drive shaft supplies a synchronous signal to count the rotational speed of the rotor, and through the datum mark the phase angle indicated on the meter is referred to the angular position of unbalance.

Dynamic shaft deflection (transverse vibration of shaft) on the other hand, is picked up by a non-contact type vibrometer, designed and manufactured by us. Four detector heads of vibrometer are arranged at four different stations along the shaft axis, so that four signals of shaft vibra-

tion are simultaneously picked up, amplified and finally recorded on the recording sheet of electromagnetic oscillograph. Phase reference signal produced by angle datum mark is also given to the recording sheet at the same time. Sensitivity of detecting shaft vibration depends on the setting and calibration of the detector and amplifier. In normal operation, indicator movement as much as 20 mm is obtained for 100μ of shaft displacement. To ascertain the sensitivity, we calibrated the apparatus every time before and after the test run. Since the plane separation network of dynamic balancing machine serves well as a modal separation network, we made use of the former contained in the indicator head of the micro-balancer for the purpose.

3.2 Measurement of critical speeds and characteristic functions

To utilize the conditional equation of balance, eq.(8), the characteristic function of shaft deflection $\phi_n(z)$ shall at first be determined. $\phi_n(z)$ can be calculated and enumerated by an electronic digital computer for a given rotor/bearing system, but our experiments will be in vain if the characteristic function thus calculated differs greatly from that of the rotor at hand. In this connection, characteristic functions together with critical speeds are further determined experimentally. Moreover, experimental determination of the characteristic function is of great importance for the actual rotating machines where the bearing stiffness is difficult to evaluate. One of our experiments given in section 3.4 bears out this standpoint, and all dimensions necessary to balance the rotor will be determined experimentally, i.e. by the way which is applicable for rotating machines at site. (i) Measurement of critical speed ω_n : Add the triangularly distributed unbalances (see Table 1) to the rotor. Since these unbalances contain every order of modal component, odd and even, the rotor comes into resonance at speeds near n -th critical speed of $n = 1, 2, \dots$. Observing the growth of shaft deflection with varying rotational speed gives critical speeds, which are shown in Table 2 comparing with that obtained by calculation. Measured values of ω_1 to ω_3 are in good coincidence with calculated ones. Fourth critical speed ω_4 can not be measured, because the base of the apparatus is not so firm as to bear severe vibration which may occur at speed range near ω_4 . Al-

though we are concerning only with the vibration of the rotor in horizontal plane, the rotor itself vibrates also in vertical plane, where the rotor has an alternative set of critical speeds owing to different end conditions ---the rotor is supported elastically in horizontal plane, whereas it has pinned-pinned ends in vertical plane as denoted in section 1. Therefore, we have to be aware of alternative critical speeds for vibration in vertical plane, the measured values of which are also shown in Table 1.

Calculation of ω_n and $\phi_n(z)$ for rotor/bearing system of Fig. 1 is performed by use of an electronic digital computer of IBM-system 7040/1401 in the university.

(ii) Measurement of $\phi_n(z)$: When an unbalanced rotor is rotated at speed near n -th critical speed ω_n , the shaft vibration of n -th modal component is exaggerated, on which basis $\phi_n(z)$ is determined experimentally by measuring the shaft deflection at that time.

There is, however, an alternative method to determine $\phi_n(z)$, which is developed by the authors. According to our method, $\phi_n(z)$ can be obtained by only measuring the bearing vibrations for trial unbalance placed at

Fig. 4 Characteristic functions (mode shapes) of shaft deflection

Table 2. Critical speeds ω_n of the system

No. of mode	Flexibly supported (horizontal)			Simply supported (vertical)		
	calculated	measured	error*	calculated	measured	error*
1	883	900	+ 1.9	1 090	1 100	+ 0.9
2	1 388	1 350	- 2.7	4 370	-	-
3	1 815	1 760	- 3.0	-	-	-
4	4 701	-	-	-	-	-

* Error = (measured / calculated) - 1

axial station $z_1, z_2, \dots$, in turn. Theory of the method will later be shown in appendix.

The results obtained by experiment are illustrated in Fig. 4. As clearly seen in the Figure, measured values are in good coincidence with calculated ones.

Fig. 5 Balance effect for equally distributed initial unbalances

Fig. 6 Balance effect for equally distributed initial unbalances (shaft deflection before and after balancing)

- * Influence of initial bent is compensated beforehand.
- ** Phase angle of initial unbalance is regarded as plus side.

3.3 Experiment to verify conditional equation of balance

To verify the conditional equation of balance proposed by the author,⁶⁾ artificial unbalances u_j ($j = 1, 2, \dots, 7$) of known magnitude are given to the rotor and the shaft vibration caused by these unbalances

is measured. Next, correction unbalances calculated by conditional equation of balance, eq. (8), are given to the rotor and the shaft vibration after correction is again measured for various speeds, from which the effect of balance is observed. For the experiment two types of artificial unbalances u_j of (i) uniform distribution and (ii) triangular distribution (see section 2.2) are applied to the rotor. The shaft deflection up to its third

mode is to be eliminated (i.e. $h = 3$), so that correction planes as many as $h + 2 = 5$ are necessary and sufficient. Correction unbalances U_j for two types of initial unbalances (i) and (ii) are calculated by eq. (8) and they are tabulated in Table 1.

(i) Balancing for initial unbalances distributed uniformly: The shaft of the rotor has an initial bent of the shape of $\delta = \delta_0 \sin(\pi z / \ell)$ inherently, and it is observed that the shaft is deflected in a plane including the shaft axis. Since it seems almost impossible to straighten the shaft owing to its slenderness, the following procedures of the experiment are applied to avoid the influence of initial bent: (a) Firstly, the rotor is rotated without giving any artificial unbalances and the responses of shaft vibration together with bearing vibration are measured for various speeds of ω . These responses correspond to the effect of initial bent of the shaft. (b) Next, artificial unbalances u_j (initial unbalances) are given to the rotor at the same angular position of initial bent δ and the rotor is again rotated at speed ω , where vibrations of shaft and bearings are measured. When the measurement of vibration is impossible owing to excess vibration near or at the critical speeds, unbalances u_j are removed and applied again to the anti-phase of initial bent. (c) Correction unbalances U_j calculated by eq. (8) for unbalances u_j are further given to the rotor and the vibration responses are measured for various values of ω . (d) All data acquired by step (b) are corrected vectorially by that of (a). The data after correction are reasonably regarded as the vibration response for initial unbalance u_j given to the ideal rotor having no initial bent.

Fig. 7 Balance effect for triangularly distributed unbalances

Fig. 8 Balance effect for triangularly distributed initial unbalances (shaft deflection before and after balancing)

- * Influence of initial bent is compensated beforehand.
- ** Phase angle of initial unbalance is regarded as plus side.

Table 3. Sets of trial unbalances for each mode

Disk No.	1	2	3	4	5	6	7
For first mode (contains neither dynamic unbalances nor 2nd modal component)	-	-1.0	-	2.0	-	1.0	-
	1.0	-2.0	-	-	-	4.0	-3.0
	-	1.0	-2.13	-	2.38	-1.25	-
For second mode (contains neither dynamic unbalance nor 1st modal component)	1.0	-1.5	-	-	-	1.5	-1.0
	-	1.0	-2.0	-	2.0	-1.0	-
	1.0	-	-3.0	-	3.0	-	-1.0
For third mode (does not contain any of dynamic unbalance, 1st and 2nd modal component)	-1.0	-	3.24	-4.58	2.34	-	-
	-24.4	38.5	-	-22.4	-	7.3	1.0
	-	-5.30	10.86	-12.36	7.80	-1.00	-

* Numerical values are mass ratio to be attached on peripheral surface (at radius 33mm) of each disk. Masses are to be added in-phase or out-of-phase, according to plus or minus signs of the figures.

(e) The same correction is applied to all data acquired by step (c). The data after correction correspond to the response for the co-existence of unbalances u_j and U_i , from which balance effect by U_i is evaluated and the propriety of the conditional equation of balance is verified.

The responses of bearing vibration before and after balancing are shown in Fig. 5. In Fig. 6 the dynamic shaft deflection is illustrated for various speeds. Since the distributed unbalances have only modal components of odd numbers, bearing vibration as well as the shaft deflection increases greatly near the critical speeds of ω_1 , ω_3 as seen in Figures.

Significantly good results are recognized as for the effect of balancing, which features the reduction of bearing vibration together with the shaft deflection for a wide range of speeds

(ii) Balancing for triangularly distributed initial unbalances: The bearing vibration and the dynamic shaft deflection before and after balancing are illustrated in Figs. 7 and 8. In this case also the

effect of balancing is successfully recognized. Influence of the initial bent of the shaft has been excluded from the data in the same manner as before.

From a series of experiments shown above it can be concluded that such a small vibration at both bearings after balancing signifies small dynamic forces acting at bearings, because the rotor is supported by flexible bearings in the experiments. This means that a small magnitude of dynamic force will be transmitted to the foundation through the bearing pedestals. Therefore, for speed range of ω smaller than ω_h , eq. (8) guarantees the rotor such state that (a) dynamic force acting at bearings and (b) the modal components of shaft deflection of the order of 1, 2, ..., h are eliminated.

3.4 Experiment for modal balancing by means of mode separation network

This experiment is to realize the possibility of balancing a rotor only by measuring vibration at speeds not so close to the critical speeds. Thus the procedure is applicable to balancing an actual rotating machine. The principle of the procedure has been given in section 2.3. In the experiment, artificial unbalances u_i (initial unbalances) distributed triangularly are initially applied to the rotor in order to contain all modal components for $n = 1, 2, 3, \dots$. For simplicity, initial unbalances are given to the rotor at same angular position. Mode separation network used is a plane separation network of dynamic balancing apparatus. We are intending to balance the rotor up to the second mode of the shaft deflection, which requires 4-plane-balancing, so that it is necessary to set the separation factor γ for γ_1 and γ_3 , in the network.

To set two factors the following experimental procedures are applied (γ can also be set by calculation in case where the modal shapes of shaft deflection are known.)

- Balance the rotor dynamically at 400 rpm through two-plane-balancing.
- Rotate the rotor at 800 rpm (beneath the first critical speed) after placing a set of four trial unbalances which has only the second modal component and does not contain any components of dynamic unbalance nor modal component of the first mode (see Table 3). Set the cal-

ibration factor γ of mode separation network by mixing two outputs of left and right bearing vibrations.

(c) After balancing the rotor for the first modal component of initial unbalance, run the rotor having the trial unbalance for the third modal component (see Table 3) at 1300 rpm (beneath the second critical speed).

As preliminary stage of the experiment, the effect of mode separation network is checked. The results are shown in Figs. 9 and 10. In Fig. 9, the rotor has initial unbalances distributed triangularly, which causes the magnification of bearing vibrations at each of the critical speeds, ω_1 , ω_2 and ω_3 , while output of the network calibrated to eliminate the effect of the second modal component gives the response as if there were no second modal component in it. To make sure the elimination of the second modal component, two alternative sets of unbalances, each of which has only the second modal component of different magnitude, are given separately to the rotor and the output of the network is observed for each of two states of unbalances. Through these checking runs, it is ascertained that the mode separation network functions quite satisfactorily as we expected, for the influence of newly given unbalances is not recognized in the output at all.

Fig. 9 Effect of eliminating 2nd modal component of unbalance by use of mode separation network

Fig. 10 effect of eliminating 3rd modal component of unbalance by use of mode separation network

Fig. 11 Balance effect of modal balancing by use of mode separation network (balancing of triangularly distributed initial unbalance up to its second modal component)

Fig. 12 Balance effect of modal balancing by use of mode separation network (balancing of triangularly distributed initial unbalance up to its second mode)

- * Unbalances due to initial bent of the shaft are also balanced in the test.
- ** Phase angle of initial unbalances is regarded as plus side.

The same can be said also for Fig. 10, but elimination of the component of the third mode seems to be imperfect. This comes from inaccurate application of trial unbalances on the rotor. In general, great accuracy is required in applying trial unbalances especially for higher modes. It should be noted in Fig. 10 that elimination of the third modal component accompanies the elimination of the first mode to some extent. This originates from the approximate symmetry of our rotor/bearing system, where the forms of $\phi_2(z_i)$ and $\phi_3(z_i)$ are both nearly symmetrical as for mid-span so that elimination factor for the third mode γ_3 becomes nearly equal to γ_1 for the first mode.

After the effect of mode separation network has been examined, the rotor with triangular initial unbalance is balanced up to its second modal component of unbalance by the following procedure.

- (a) Bring the rotor into the state of dynamic balance at speed of $\omega \ll \omega_1$ by performing two-plane-balancing.
- (b) Run the rotor at 800 rpm (beneath the first critical speed) and measure u' of eq. (8a) or eq. (10) by reading the meter indication, output of the network, from which the second modal component has already been eliminated. Balance the rotor for u' by applying correction unbalances in four planes. How to make four corrections to pre-determined correction planes is shown in Table 3.
- (c) Run the rotor again at 1300 rpm (beneath the second critical speed) and measure the second modal component of initial unbalance u'' by the output of the network eliminated for the third mode. Balance the rotor in four-plane-correction according to the distribution given in Table 3. In steps (b) and (c) above, phase angle and magnitude of the individual modal component u^m can be easily and accurately measured by aid of a mode separation network, without suffering any disturbance from modal component u^n of other than m ($n \neq m$).

The results obtained in the experiment are shown in Figs. 11 and 12. Fig. 11 is for left and right bearing vibration before and after balancing and Fig. 12 is for the response of shaft deflection for initial unbalance and that after correction. As seen in these Figures, the rotor is satisfactorily balanced for the modal components of up to second order. Considerable bearing vibration and shaft deflection remain at speed near the third critical speed ω_3 even after balancing, but it is quite natural because the rotor is balanced only for the first and second modal components of unbalances.

4. Conclusions

From experiments on a seven-disked flexible rotor on two flexible bearings the following are concluded:

- (1) The critical speeds and the characteristic function of the rotor/bearing system measured experimentally are in good coincidence with those calculated.
- (2) Mode shapes of characteristic function can be experimentally obtained by only measuring the bearing vibration caused by trial unbalances (see Appendix).
- (3) The rotor can be well balanced by the correction masses obtained by eq.(8) for a variety of unbalance distributions. The bearing vibration together with the dynamic shaft deflection is minimized for a wide range of rotating speeds, thus the validity of conditional equation of balance, eq.(8), is ascertained.
- (4) Mode separation network for eliminating a certain particular modal component of unbalances makes modal balancing easy and accurate. The test rotor is balanced successfully by aid of the network.

Acknowledgement

The authors wish to thank sincerely Mr. H. Sei and Mr. E. Masuyama, who are once their colleagues of the laboratory, for their help in performing the experiments.

APPENDIX:

The experimental determination of $\phi_n(z)$

The explanation is given for a rotor with axially distributing masses (continuous system), but it is applied also to the discrete system after slight modification.

For a rotor/bearing system having a set of orthogonal functions $\{\phi_n(z)\}$ as for its shaft deflection, dynamic shaft deflection $r(z)$ is produced by unbalance $u(z)$ as seen in the literature⁽¹⁾ as

$$r(z) = \sum_{n=1}^{\infty} A_n(\omega) \epsilon_n \phi_n(z) \quad \dots (11)$$

where

$$A_n(\omega) = \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2}$$

: Magnification factor for

$$\epsilon_n = \frac{\int_0^l u(z) \phi_n(z) dz}{\int_0^l \mu \cdot \phi_n^2(z) dz}$$

: N-th modal mass eccentricity,
 μ as rotor mass per unit of shaft length

By substituting $z = 0$ or l in eq.(11), the left or right bearing vibration is obtained. In the following procedures, however, merely the measurement of bearing vibration $r_A = r(0)$ or $r_B = r(l)$ is necessary. (a) The m-th characteristic function $\phi_m(z)$ is to be found.

Choose the test speed at $\omega = \Omega_m$ in the vicinity of m-th critical speed ω_m , where magnification factor $A_m(\Omega_m)$ is extremely smaller than $A_m(\Omega_m)$. Hence the vibration (displacement) at bearing A is derived from eq.(11) as

$$r_A = A_m(\Omega_m) \epsilon_m \phi_m(0) \quad \dots (12)$$

, provided ϵ_m has a certain finite value (the same is true for another bearing B). (b) After a trial unbalance U_T of known magnitude is given at $z = z_i$, run the rotor again at speed Ω_m and measure the bearing vibration r'_{Ai} . Here, the modal mass eccentricity shall be

$$\begin{aligned} \epsilon'_{mi} &= \frac{\int_0^l u \cdot \phi_m dz + U_T \phi_m(z_i)}{\int_0^l \mu \cdot \phi_m^2 dz} \\ &= \epsilon_m + \frac{U_T}{\int_0^l \mu \cdot \phi_m^2 dz} \cdot \phi_m(z_i) \quad \dots (13) \end{aligned}$$

so that

$$\begin{aligned} r'_{Ai} &= A_m(\Omega_m) \epsilon'_{mi} \phi_m(0) \\ &= r_A + A_m(\Omega_m) \frac{U_T \phi_m(0)}{\int_0^l \mu \cdot \phi_m^2 dz} \cdot \phi_m(z_i) \quad \dots (14) \end{aligned}$$

(c) Calculate $\Delta r_{Ai} = r'_{Ai} - r_A$ from the data obtained by steps (a) and (b). According to eq.(14), Δr_{Ai} has the same phase angle as U_T and its magnitude is

$$\Delta r_{Ai} = \lambda_A \cdot \phi_m(z_i) \quad \dots (15)$$

where λ_A is a factor appearing in the second term of RHS of eq.(14), multiplier of $\phi_m(z_i)$, and is a system constant independent of axial station z_i of unbalance U_T .

(d) Remove U_T at z_i , and apply it again at various values of z for $0 \leq z \leq l$. Repeat steps (b) and (c) and obtain Δr_{Ai} for various $z = z_i (i=1, 2, \dots)$. Clearly the magnitudes of Δr_{Ai} are proportional to that of $\phi_m(z_i)$. In the problem of balancing it is unnecessary to normalize $\phi_m(z_i)$, but it may be practically recommended to normalize by taking the ratio of Δr_{Ai} to the maximum value of $(\Delta r_{Ai})_{\max}$ as

$$\frac{\Delta r_{Ai}}{(\Delta r_{Ai})_{\max}} = \frac{\phi_m(z_i)}{[\phi_m(z_i)]_{\max}} = \varphi(z_i), \quad (i=1, 2, \dots)$$

, which can be satisfactorily regarded as the m-th characteristic function of the system.

Note: Where a perfectly balanced rotor is available, the measured values of r_{Ai} are directly proportional to $\phi_m(z_i)$. For a flexible rotor, however, it is almost impossible to prepare such a rotor, so that the procedures given above are generally recommended for practical use.

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