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An Extension of Love's Solution for the Axisymmetric Problems of Elasticity*

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This paper is concerned with an extension of Love's solution for the axially symmetric problems in the theory of elasticity. The principal results are as follows.

(1) Love's solution is generalized such that it may be possible to apply it to the problems accompanied with both torsional deformations and body-forces, because Love's solution in its usual form is inapplicable to the foregoing cases. (2) The generalized Love's solution has the completeness of solution for the axially symmetric problems. The completeness of Love's solution in its usual form is only assured for the cases of torsion-free deformations with no body-force but in z-direction. (3) The torsional solution obtained from the generalized Love's stress function is equivalent to the solutions of other known methods. (4) An attempt is made to investigate a general functional form of Love's stress function which can be applied to any practical problems. (5) The stress function for body-force problems is investigated, and examples of its application are shown.

1. Introduction

The problems in the theory of elasticity have been attacked mainly by means of stress functions in various ways.⁽¹⁾⁽²⁾⁽³⁾ When an elastic solid is in the state of symmetrical strain, the method of approach presented by A.E.H. Love⁽⁴⁾ is most widely used and many problems have been investigated by using it. However, it seems that the completeness of Love's solution as a method of solution of axisymmetric elasticity problems remains unknown. Evidently, Love's solution is inapplicable to a torsion problem and it is not clear whether it can be applied to the problems with body-force.

H.M. Westergaard⁽⁵⁾ pointed out that in the state of axial symmetry Love's solution coincides with a solution derived from the component of z-direction of Galerkin's vector stress function when expressed in terms of cylindrical co-ordinates. It follows from this result that Love's solution can be applied to the problems with body-force acting only in z-direction, but H.M. Westergaard did not make any mention of the problems concerning the components of other directions. T. Mogami⁽⁶⁾ investigated the relation between Love's solution and the solution for two dimensional problems of elasticity. G.D.L. Penha & B. Child⁽⁷⁾ approached Love's solution from the general form of the stress tensor. However, these studies do

not allude to the problems accompanied with either torsional deformation or body-force. In the same case as above, Love's solution had been extended to the problems of an anisotropic elastic medium by R.A. Eubanks & E. Sternberg,⁽⁸⁾ and others. R.D. Mindlin⁽⁹⁾ showed some interesting connections between Galerkin's and Neuber-Papkovich's stress functions, etc. by using Helmholtz's theorem in rectangular co-ordinates, but these connections are not valid generally when transformed to cylindrical co-ordinates.

It seems that an essential meaning of the technique employed in Mindlin's treatise is to split the displacement vector into vectors having simpler properties in order to transform the coupled equilibrium equations of elastic solids into the uncoupled ones. In this paper, a complete solution is presented in terms of three stress functions satisfying the above uncoupled equations derived by applying Mindlin's technique to the axially symmetric equilibrium equations expressed in cylindrical co-ordinates. With this aim, we shall generalize Love's solution such that it may be possible to apply it to the problems with torsional deformations and body-forces, and investigate the completeness of the solution and the possibility of reduction of a stress function. In addition, we shall also discuss the relation between the torsional solution obtained from generalized Love's solution and the solutions of other known methods, the general forms of stress functions applicable to any practical problems, and the stress function for body-force problems.

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2. An extension of Love's solution

When the displacement vector of an elastic body is independent of angle θ in cylindrical co-ordinates (r, θ, z) , the deformations are generally said to be axially symmetrical. As for these deformations, it is well known that the displacement components u_j satisfy the following simultaneous partial differential equations expressed in cylindrical co-ordinates,

$$\mu(\nabla^2 - \frac{1}{r^2})u_j + \frac{\mu}{r^2}[0, 0, u_3] + \frac{\mu}{1-2\nu} \text{grad}_j \text{div } \mathbf{U} + F_j = 0 \quad (1)$$

where u_j ($j=1, 2, 3$) are components of the displacement vector $\mathbf{U}$ in r -, θ - and z -directions. μ and ν are the shear modulus of elasticity and Poisson's ratio respectively. F_j ($j=1, 2, 3$) are j -components of the body-force vector $\mathbf{F}$ and other undefined notations are those used customarily. In what follows, u_j and F_j will be simply called displacements and body-force.

By virtue of the Helmholtz theorem, any displacements u_j satisfying the fundamental equation (1) may be given by the following expression:

$$u_j = \text{grad}_j A(r, z) + \text{rot}_j B(r, z) \quad (2)$$

where $A(r, z)$ is a scalar potential and $B(r, z)$ is a vector potential such that $\text{div } \mathbf{B} = 0$. Substitution of u_j of Eq.(2) into Eq.(1) yields the fundamental equations in terms of the potential functions A and B .

$$\begin{aligned} (\nabla^2 - \frac{1}{r^2}) \{ \frac{2(1-\nu)}{1-2\nu} \frac{\partial A}{\partial r} - \frac{\partial B_2}{\partial z} \} &= -\frac{F_1}{\mu}, \\ (\nabla^2 - \frac{1}{r^2}) (\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial r}) &= -\frac{F_2}{\mu}, \\ \nabla^2 \{ \frac{2(1-\nu)}{1-2\nu} \frac{\partial A}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r B_2) \} &= -\frac{F_3}{\mu} \end{aligned} \quad (3)$$

where B_j are j -components of the vector potential $\mathbf{B}$.

It may be admissible to represent A and B as follows:

$$A = \frac{1-2\nu}{2\mu} \text{div } \phi, \quad B = -\frac{1-\nu}{\mu} \text{rot } \phi \quad (4)$$

because the functions A and B are any independent functions defined above in Eq.(2). Substituting Eq.(4) into Eq.(3), the simultaneous equations (3) are transformed into three uncoupled equations expressed by the unknown functions ϕ_j ,

$$\text{that is,} \quad (\nabla^2 - \frac{1}{r^2})^2 \phi_j = -\frac{F_j}{1-\nu}, \quad (j=1, 2),$$

$$\nabla^4 \phi_3 = -\frac{F_3}{1-\nu}, \quad (5)$$

Consequently, it is concluded that the displacements u_j satisfying the fundamental equation (1) may be represented as

$$u_j = \frac{1-\nu}{\mu} \{ (\nabla^2 - \frac{1}{r^2}) \phi_j + \frac{1}{r^2} [0, 0, \phi_3] \} - \frac{1}{2\mu} \text{grad}_j \text{div } \phi \quad (6)$$

without impairing the generality. By the decomposition of u_j the fundamental equation (1) is transformed into the uncoupled equation (5) in terms of the unknown functions (stress functions) ϕ_j . It can be recognized that the solution u_j shown by Eq.(6) is an extension of Love's solution and is applicable to any problems with both torsional deformations and body-forces. These facts will be shown by the following results. Since u_j of Eq.(6) may be classified according to the components of ϕ_j , we have

(i) ϕ_1 ; $u_2 = 0$,

$$u_1 = \frac{1-\nu}{\mu} (\nabla^2 - \frac{1}{r^2}) \phi_1 - \frac{1}{2\mu} \frac{\partial}{\partial r} \{ \frac{1}{r} \frac{\partial}{\partial r} (r \phi_1) \},$$

$$u_3 = -\frac{1}{2\mu} \frac{\partial}{\partial z} \{ \frac{1}{r} \frac{\partial}{\partial r} (r \phi_1) \}. \quad (7)$$

(ii) ϕ_2 ; $u_1 = u_3 = 0$,

$$u_2 = \frac{1-\nu}{\mu} (\nabla^2 - \frac{1}{r^2}) \phi_2. \quad (8)$$

(iii) ϕ_3 ; $u_2 = 0$,

$$u_1 = -\frac{1}{2\mu} \frac{\partial^2}{\partial r \partial z} \phi_3, \quad u_3 = \frac{1}{2\mu} \{ 2(1-\nu) \nabla^2 \phi_3 - \frac{\partial^2}{\partial z^2} \phi_3 \}. \quad (9)$$

It is easily seen that u_j shown in Eq.(9) coincides with Love's solution and ϕ_3 corresponds to Love's stress function $\chi(r, z)$ because of $2\mu = E/(1+\nu)$.

It is obvious from Eq.(8) that the stress function ϕ_2 represents torsional deformations. On the relation between the stress function ϕ_2 and stress functions which have been used up to now for torsion problems, we shall discuss in a later chapter. The stress function ϕ_1 presents a new method for solutions, and it provides a useful technique for the problems with body-force acting in r -direction. The relation between the solutions obtained from the stress function ϕ_1 and ϕ_3 will be investigated in the following chapter.

Now, as is well known, the vector stress function $G(x, y, z)$ shown by Galerkin in rectangular co-ordinates (x, y, z) satisfies also a biharmonic equation. Therefore, we shall explain the connection between

Eq.(6) and Galerkin's solution ⁽¹⁰⁾

$$2\mu U = 2(1-\nu)\nabla^2 G - \text{grad div } G, \quad (10)$$

where

$$\nabla^4 G(x,y,z) = -\frac{F}{1-\nu} \quad (11)$$

For any vector V , the following relation holds generally:

$$V_x + iV_y = (V_1 + V_2)e^{i\theta}, \quad V_z = V_3 \quad (12)$$

where V_x, V_y, V_z are components of vector V in rectangular co-ordinates and V_j ($j = 1, 2, 3$) are components in cylindrical co-ordinates, respectively. No useful result is obtained in general by transforming Eqs.(10) and (11) to cylindrical co-ordinates. However, when U, G and F are axisymmetric, by employing the relation of Eq.(12), Eqs.(10) and (11) are transformed to the following form in terms of cylindrical co-ordinates.

$$2\mu u_j = 2(1-\nu)\left\{(\nabla^2 - \frac{1}{r^2})G_j + \frac{1}{r^2}[0, 0, G_3]\right\} - \text{grad div } G \quad (13)$$

where

$$\begin{aligned} (\nabla^2 - \frac{1}{r^2})^2 G_j(r,z) &= -\frac{F_j(r,z)}{1-\nu}, \\ \nabla^4 G_3(r,z) &= -\frac{F_3(r,z)}{1-\nu}. \end{aligned} \quad (j = 1, 2) \quad (14)$$

From these results, we can easily see that the expressions (13), (14) and Eqs.(5), (6) have equality with each other and include Westergaard's interpretation for body-force F_3 in z -direction. From Eqs. (5), (6) we can also show Neuber-Papkovich's solution in axial symmetry, which has been shown by A.I. Lur'e ⁽¹¹⁾ for the case with no body-force and by R. Furuhashi ⁽¹²⁾ for the case with body-forces.

3. Completeness of the solution

In the preceding chapter, we get an extended Love's solution for axially symmetric elastic problems which consists of three stress functions ϕ_j satisfying Eq.(5). Apparently this solution has completeness for axially symmetric problems accompanied with both torsional deformations and body-forces, because the method of obtaining the extended Love's solution is to transform the fundamental equation (1) in terms of u_j to the equations expressed by the unknown functions (stress function) ϕ_j , decomposing the displacement vector U satisfying the fundamental equation (1) into vectors having simpler properties than the vector U without loss of generality. Accordingly, Eq.(5), which the stress function must satisfy, repre-

sents the fundamental equation transformed into the expression, in reality.

The completeness of Love's solution in its usual form is assured for the problems of torsion-free deformations with no body-force but in z -direction. This fact can be shown as follows. In the case above mentioned, Eq.(3) is expressed by

$$\begin{aligned} (\nabla^2 - \frac{1}{r^2})\left\{\frac{2(1-\nu)}{1-2\nu}\frac{\partial A}{\partial r} - \frac{\partial B_2}{\partial z}\right\} &= 0, \\ \nabla^2\left\{\frac{2(1-\nu)}{1-2\nu}\frac{\partial A}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rB_2)\right\} &= -\frac{F_3}{\mu} \end{aligned} \quad (15)$$

because of $u_2 = 0, F_1 = F_2 = 0$.

Now, let us introduce such a new scalar potential function $\chi(r,z)$ as

$$A = \frac{1-2\nu}{2\mu}\frac{\partial}{\partial z}\chi, \quad B_2 = -\frac{1-\nu}{\mu}\frac{\partial}{\partial r}\chi \quad (16)$$

Substituting Eq.(16) into Eq.(15), the first expression of Eq.(15) is always satisfied and the second one becomes an inhomogeneous biharmonic equation expressed by an unknown function χ as follows.

$$\nabla^4 \chi(r,z) = -\frac{F_3}{1-\nu}. \quad (17)$$

At the same time, the displacements u_j may be expressed by

$$u_j = \frac{1-\nu}{\mu}\nabla^2[0, 0, \chi] - \frac{1}{2\mu}\text{grad}_j \frac{\partial}{\partial z}\chi \quad (18)$$

These expressions exactly accord with Love's solution in its usual form since $2\mu(1+\nu) = E$.

As to the stress functions ϕ_1 and ϕ_2 , if we repeat similar considerations as above for the problems accompanied with body-force acting either in r - or θ -direction alone, the following facts are easily observed. The completeness of the solution employing the stress function ϕ_1 alone is assured for the case of torsion-free deformations with no body-force but in r -direction, and the stress function ϕ_2 gives a complete solution for torsion problems. These results imply that in the case of no body-force the solution $\{u_j | u_j = u_j(\phi_1)\}$ which can be represented by the stress function ϕ_1 coincides with the solution $\{u_j | u_j = u_j(\phi_3)\}$ which can be represented by ϕ_3 . Accordingly, if the body-force is absent, we may omit a stress function either ϕ_1 or ϕ_3 from among ϕ_j . A proof of the result has appeared in an author's paper ⁽²⁶⁾ on elastodynamics.

4. Stress function for torsion problems

In chapter 2, we have established a solution for torsion problems, which is the solution expressed by the stress function ϕ_2 . Let us examine the relation between the torsional solution by ϕ_2 and the solutions of other known methods which were presented by Neuber-Papkovich, ⁽¹¹⁾ Michell, ⁽¹³⁾ Boussinesq ⁽¹⁾ and others ⁽¹⁴⁾ in axial symmetry. Table 1 lists the above mentioned solutions in which $S(r,z), N(r,z), M(r,z), \psi(r,z)$ are stress functions, and

it has been assumed that the body-force is absent.

Here we shall provide the following definition. The notation u_A represents the solutions u_j which can be obtained from a stress function A, that is $u_A = \{u_j \mid u_j = u_j(A)\}$. Similarly, $u_B = \{u_j \mid u_j = u_j(B)\}$ for a stress function B. For the case $u_A = u_B$, we say that the solutions of the stress functions A and B are equivalent, whereas for the case $u_A \supset u_B$ or $u_A \subset u_B$, we say that the solution obtained from the stress function A (B) includes the solution by the stress function B (A).

(i) The extended Love's stress function ϕ_2 and Neuber-Papkovich's stress function N. The relation

$$(\nabla^2 - \frac{1}{r^2})\phi_2 = 2N \quad (19)$$

holds always. Obviously, this means that $u_{\phi_2} = u_N$. Therefore the stress functions ϕ_2 and N are equivalent.

(ii) N and Boussinesq's stress function S. The following relation holds for an arbitrary function $f(r, z)$ of r and z:

$$(\nabla^2 - \frac{1}{r^2})\frac{\partial}{\partial r}f = \frac{\partial}{\partial r}\nabla^2 f \quad (20)$$

In the left-hand member of this equation, if we assume that $\partial f / \partial r = 2(1-\nu)N$, then $\nabla^2 f = C(z)$ is obtained from the right-hand member, where $C(z)$ is an arbitrary function of z alone. Therefore, we have $f \supset S$ since $\nabla^2 S = 0$. However, it is easily seen that $\partial f / \partial r = \partial S / \partial r$ because the function f may be expressed by $f = f_1(r, z) + f_2(z)$, where $\nabla^2 f_1 = 0$ and $\nabla^2 f_2 = C$. Consequently, the relation

$$2(1-\nu)N = \frac{\partial}{\partial r}S \quad (21)$$

holds, and then we have $u_N = u_S$.

(iii) N and Michell's stress function M. The following relation holds for an arbitrary function $f(r, z)$:

$$(\nabla^2 - \frac{1}{r^2})\frac{1}{r^2}\frac{\partial}{\partial r}f = \frac{1}{r^2}\frac{\partial}{\partial r}(\nabla^2 - \frac{4}{r}\frac{\partial}{\partial r})f \quad (22)$$

In the left-hand member of this equation, if we assume that

$$\frac{1}{r^2}\frac{\partial f}{\partial r} = -4(1-\nu)\frac{\partial N}{\partial z}$$

then

$$(\nabla^2 - \frac{4}{r}\frac{\partial}{\partial r})f = C(z)$$

is obtained from the right-hand member. Therefore, $f \supset M$. However, for the similar reason to that in the case (ii), we have

$$4(1-\nu)\frac{\partial}{\partial z}N = -\frac{1}{r^2}\frac{\partial}{\partial r}M \quad (23)$$

From Eq.(23), we see that $u_N = u_M$.

(iv) N and ψ . The following relation holds for an arbitrary function $f(r, z)$:

$$(\nabla^2 - \frac{1}{r^2})(rf) = r(\nabla^2 + \frac{2}{r}\frac{\partial}{\partial r})f \quad (24)$$

In the left-hand member, if we assume that $rf = -4(1-\nu)N$, then $f = \psi$ is obtained from the right-hand member. Consequently, the relation

$$4(1-\nu)N = -r\psi \quad (25)$$

holds, then we have $u_N = u_\psi$.

From these results, it is obviously concluded that the torsional solution obtained from the extended Love's stress function is equivalent to the solution of other known methods by stress functions.

5. An investigation of the functional form of the stress function ϕ_j

In the stress analysis applying not only Love's solution but also the solutions employing a stress function, as is well known, it is required to find such functions as satisfy certain boundary conditions from among the solutions of the partial differential equation for the stress function. In general, it is necessary to find as many stress functions mentioned above as the boundary conditions of a problem. Hence, in view of the above fact, let us seek a general functional form of the

Table 1 Solutions for axially symmetric torsion problems

Solution	$2\mu u_2$	$\tau_{r\theta}$	$\tau_{z\theta}$	Condition required for stress function
Boussinesq	$-2\frac{\partial S}{\partial r}$	$-(2\frac{\partial}{\partial r^2} + \frac{\partial}{\partial z^2})S$	$-\frac{\partial S}{\partial r\partial z}$	$\nabla^2 S = 0$
Neuber-Papkovich	$-4(1-\nu)N$	$-2(1-\nu)r\frac{\partial}{\partial r}(\frac{N}{r})$	$-2(1-\nu)\frac{\partial N}{\partial z}$	$(\nabla^2 - \frac{1}{r^2})N = 0$
Michell		$-\frac{1}{r^2}\frac{\partial M}{\partial z}$	$\frac{1}{r^2}\frac{\partial M}{\partial r}$	$(\nabla^2 - \frac{4}{r}\frac{\partial}{\partial r})M = 0$
Reference(14)	$r\psi$	$r\frac{\partial \psi}{\partial r}$	$r\frac{\partial \psi}{\partial z}$	$(\nabla^2 + \frac{2}{r}\frac{\partial}{\partial r})\psi = 0$
Love (extended)	$2(1-\nu) \cdot (\nabla^2 - \frac{1}{r^2})\phi_2$	$(1-\nu) \cdot r\frac{\partial}{\partial r}\{\frac{1}{r}(\nabla^2 - \frac{1}{r^2})\phi_2\}$	$(1-\nu) \cdot (\nabla^2 - \frac{1}{r^2})\frac{\partial \phi_2}{\partial z}$	$(\nabla^2 - \frac{1}{r^2})^2\phi_2 = 0$

extended Love's stress function ϕ_j by the method of separation of variables, which is applicable to any practical problems. It seems as pointed out by Shibahara & Oda⁽¹⁵⁾ that the stress functions for problems solved up to now are usually presented for particular cases, and only a few of them synthesize the general form of the stress function.

The stress function for body-force problems will be investigated in the next chapter; here we consider the case with no body-force. In this case, it is easily seen from Eq.(5) that the extended Love's stress function ϕ_j must satisfy the following equations.

$$(\nabla^2 - \frac{1}{r^2})^2 \phi_j(r, z) = 0, \quad (j = 1, 2) \quad (26)$$

$$\nabla^4 \phi_3(r, z) = 0. \quad (27)$$

5.1 Stress function ϕ_3

In Eq.(27), let us simply put ϕ instead of ϕ_3 , and write

$$\nabla^2 \phi = \Phi(r, z) \quad (28)$$

then Eq.(27) becomes

$$\nabla^2 \Phi(r, z) = 0. \quad (29)$$

Therefore, if we find Φ satisfying Eq.(29), the stress function ϕ can be obtained from Eq.(28) by substitution of Φ into Eq.(28).

As the solution of Eq.(29) in the form of separated variables, the following functions have been already known.⁽¹⁶⁾

$$\Phi_1 = \{a_1 J_0(x) + a_2 Y_0(x)\} (b_1 e^{\alpha z} + b_2 e^{-\alpha z}),$$

$$\Phi_2 = \{a_3 I_0(x) + a_4 K_0(x)\} (b_3 \sin \alpha z + b_4 \cos \alpha z),$$

$$\Phi_3 = \{a_5 + a_6 \log r\} (b_5 z + b_6). \quad (30)$$

where $J_n(x)$, $Y_n(x)$, $I_n(x)$, $K_n(x)$ are Bessel functions of the order n ; $x = \alpha r$; and a , b , α are arbitrary constants. The solutions of Eq.(28) may be expressed by the sum of ϕ_g satisfying the equation $\nabla^2 \phi_g = 0$ and the particular solution ϕ_p of the equation $\nabla^2 \phi_p = \Phi$, but only the term ϕ_p gives independent new functions because the term ϕ_g coincides with the functions of Eq.(30). As the term ϕ_p , we can obtain the following functions:

$$\begin{aligned} \phi_{p1} &= \frac{r}{2\alpha} \{a_1' J_1(x) + a_2' Y_1(x)\} (b_1' e^{\alpha z} + b_2' e^{-\alpha z}) \\ &\quad + \frac{z}{2\alpha} \{a_1'' J_1(x) + a_2'' Y_1(x)\} (b_1'' e^{\alpha z} + b_2'' e^{-\alpha z}), \\ \phi_{p2} &= \frac{r}{2\alpha} \{a_3' I_1(x) - a_4' K_1(x)\} (b_3' \sin \alpha z + b_4' \cos \alpha z) \\ &\quad - \frac{z}{2\alpha} \{a_3'' I_1(x) + a_4'' K_1(x)\} (b_3'' \cos \alpha z - b_4'' \sin \alpha z), \\ \phi_{p3} &= \frac{1}{4} r^2 \{a_5' + a_6' (\log r - 1)\} (b_5' z + b_6') \\ &\quad + \frac{1}{2} z^2 (a_5'' + a_6'' \log r) (\frac{1}{3} b_5'' z + b_6''). \end{aligned} \quad (31)$$

From these results, we can conclude that the following expression is permitted for Love's stress function ϕ_3 .

$$\begin{aligned} \phi_3 &= [a_1 J_0(x) + a_2 Y_0(x) + \frac{r}{2\alpha} \{a_1' J_1(x) + a_2' Y_1(x)\}] (b_1 e^{\alpha z} + b_2 e^{-\alpha z}) \\ &\quad + \frac{z}{2\alpha} \{a_1'' J_0(x) + a_2'' Y_0(x)\} (b_1' e^{\alpha z} - b_2' e^{-\alpha z}) \\ &\quad + [a_3 I_0(x) + a_4 K_0(x) + \frac{r}{2\alpha} \{a_3' I_1(x) - a_4' K_1(x)\}] (b_3 \sin \alpha z + b_4 \cos \alpha z) \\ &\quad - \frac{z}{2\alpha} \{a_3'' I_0(x) + a_4'' K_0(x)\} (b_3' \cos \alpha z - b_4' \sin \alpha z) \\ &\quad + [a_5 + a_6 \log r + \frac{1}{4} r^2 \{a_5' + a_6' (\log r - 1)\}] (b_5 z + b_6) \\ &\quad + \frac{1}{2} (a_5'' + a_6'' \log r) (\frac{1}{3} b_5' z + b_6') z^2 + g_1(r, z). \end{aligned} \quad (32)$$

where $g_1(r, z)$ is a polynomial function of r and z satisfying Eq.(27).

5.2 Relation between the stress

function used up to now and the stress function shown above
There are many investigations^{(17)~(22)}

applying Love's solution and considerations of the functional form,^{(15), (23)} but it seems that Love's stress function χ used up to now is summarized as follows.

$$\begin{aligned} \chi = & \{A'I_0(x) + B'rI_1(x) + A''K_0(x) + B''rK_1(x)\}(C_1\sin\alpha z + C_2\cos\alpha z) \\ & + \{AJ_0(x) + BY_0(x)\}(C'\sinh\alpha z + D'z\cosh\alpha z + C''\cosh\alpha z + D''z\sinh\alpha z) \\ & + \frac{1}{6}A_0z + B_0z\log r + \frac{1}{2}C_0zr^2 \end{aligned} \quad (33)$$

where A, B, C, D are arbitrary constants.

It is easily seen that the stress function χ of Eq.(33) is equivalent to ϕ_3 shown by Eq.(32) if we put $a_1' = a_2' = a_3'' = a_4'' = a_5 = a_6' = a_6'' = b_6' = b_6'' = 0$ in Eq.(32), that is,

the stress function ϕ_3 shown in Eq.(32) contains that of Eq.(33). Therefore we can obtain the following stress function as the terms which have been unknown up to now:

$$\begin{aligned} \phi_3 = & \frac{r}{2\alpha} \{a_1'J_1(x) + a_2'Y_1(x)\}(b_1e^{\alpha z} + b_2e^{-\alpha z}) \\ & - \frac{z}{2\alpha} \{a_3''I_0(x) + a_4''K_0(x)\}(b_3'\cos\alpha z - b_4'\sin\alpha z) \\ & + [a_5 + a_6\log r + \frac{1}{4}r^2\{a_5' + a_6'(\log r - 1)\}]b_6 + \frac{1}{4}a_6'(\log r - 1)b_5zr^2 \\ & + \frac{1}{2}a_6''z^2(\frac{1}{3}b_5'z + b_6')\log r. \end{aligned} \quad (34)$$

5.3 Stress function ϕ_1 and ϕ_2

In Eq.(26), let us simply put ϕ instead of ϕ_j , and write

$$(\nabla^2 - \frac{1}{r^2})\phi(r, z) = \Phi(r, z) \quad (35)$$

then Eq.(26) becomes

$$(\nabla^2 - \frac{1}{r^2})\Phi(r, z) = 0 \quad (36)$$

As the solutions of Eq.(36) in the

form of separated variables, we can find the functions

$$\begin{aligned} \phi_1 = & \{a_7J_1(x) + a_8Y_1(x)\}(b_7e^{\alpha z} + b_8e^{-\alpha z}), \\ \phi_2 = & \{a_9I_1(x) + a_{10}K_1(x)\}(b_9\sin\alpha z + b_{10}\cos\alpha z), \\ \phi_3 = & (a_{11}r + a_{12}r^{-1})(b_{11}z + b_{12}). \end{aligned} \quad (37)$$

Considerations similar to those of the case of ϕ_3 yield

$$\begin{aligned} \phi_{p1} = & \frac{r}{2\alpha} \{a_7'J_2(x) + a_8'Y_2(x)\}(b_7'e^{\alpha z} + b_8'e^{-\alpha z}) \\ & + \frac{z}{2\alpha} \{a_7''J_1(x) + a_8''Y_1(x)\}(b_7''e^{\alpha z} - b_8''e^{-\alpha z}), \\ \phi_{p2} = & \frac{r}{2\alpha} \{a_9'I_2(x) - a_{10}'K_2(x)\}(b_9'\sin\alpha z - b_{10}'\cos\alpha z) \\ & - \frac{z}{2\alpha} \{a_9''I_1(x) + a_{10}''K_1(x)\}(b_9''\cos\alpha z - b_{10}''\sin\alpha z), \\ \phi_{p3} = & \frac{r}{4}(\frac{1}{2}a_{11}'r + 2a_{12}'\log r)(b_{11}'z + b_{12}') \\ & + \frac{1}{2}(a_{11}''r + a_{12}''r^{-1})(\frac{1}{3}b_{11}''z + b_{12}'')z^2. \end{aligned} \quad (38)$$

Therefore, the following expression is permitted for the stress function ϕ_j ($j=1,2$):

$$\begin{aligned} \phi_j = & [a_7J_1(x) + a_8Y_1(x) + \frac{r}{2\alpha} \{a_7'J_2(x) + a_8'Y_2(x)\}](b_7e^{\alpha z} + b_8e^{-\alpha z}) \\ & + \frac{z}{2\alpha} \{a_7''J_1(x) + a_8''Y_1(x)\}(b_7''e^{\alpha z} - b_8''e^{-\alpha z}) \\ & + [a_9I_1(x) + a_{10}K_1(x) + \frac{r}{2\alpha} \{a_9'I_2(x) - a_{10}'K_2(x)\}](b_9\sin\alpha z + b_{10}\cos\alpha z) \\ & - \frac{z}{2\alpha} \{a_9''I_1(x) + a_{10}''K_1(x)\}(b_9'\cos\alpha z - b_{10}'\sin\alpha z) \\ & + [a_{11}r + a_{12}r^{-1} + \frac{r}{4} \{ \frac{1}{2}a_{11}'r + 2a_{12}'\log r \}](b_{11}z + b_{12}) \\ & + \frac{1}{2}(a_{11}''r + a_{12}''r^{-1})(\frac{1}{3}b_{11}''z + b_{12}'')z^2 + g_2(r, z), \quad (j=1,2). \end{aligned} \quad (39)$$

where $g_2(r, z)$ is a polynomial function of r and z satisfying Eq.(26)

It is easily confirmed that $\{u_j | u_j = u_j(\phi_1)\} = \{u_j | u_j = u_j(\phi_3)\}$ if we add a few terms from the polynomial functions $g(r, z)$ to Eqs.(32) and (39).

6. Stress function for body-force problems

6.1 Stress function for body-force problems

In this chapter, a solution for body-force problems by means of stress functions is shown, provided that the body-forces act axially symmetrically about the z -axis and satisfy Dirichlet's conditions. If the

$$F_j = \frac{\text{Re}}{2\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} e^{-i\beta(z-\zeta)} d\zeta \int_0^{\infty} \alpha J_m(\alpha r) d\alpha \int_0^{\infty} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho \quad (41)$$

where Re denotes only a real part of the expression, and $m=1$ for $j=1, 2$, $m=0$ for $j=3$. For the body-forces F_j acting in other domains, we can show similar expressions to the above.

(i) Stress functions ϕ_j for the body-

$$\phi_j = -\frac{\text{Re}}{2\pi(1-\nu)} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} e^{-i\beta(z-\zeta)} d\zeta \int_0^{\infty} \frac{\alpha J_m(\alpha r)}{(\alpha^2 + \beta^2)^2} d\alpha \int_0^{\infty} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho. \quad (42)$$

If we give similar considerations to the above to the cases of body-forces acting in other domains, we can obtain the following results:

$$\phi_j = -\frac{2}{\pi(1-\nu)} \int_0^{\infty} \cos \beta z d\beta \int_0^{\infty} \cos \beta \zeta d\beta \int_0^{\infty} \frac{\alpha J_m(\alpha r)}{(\alpha^2 + \beta^2)^2} d\alpha \int_0^{\infty} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho. \quad (43)$$

(iii) Stress functions ϕ_j for the body-forces F_j acting in an infinite solid cylinder domain ($-\infty < z < \infty$, $0 < r < r_2$).

$$\phi_j = -\frac{\text{Re}}{\pi r_2^2(1-\nu)} \sum_{\alpha} \frac{J_m(\alpha r)}{[J_{m+1}(\alpha r_2)]^2} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \frac{e^{-i\beta(z-\zeta)}}{(\alpha^2 + \beta^2)^2} d\zeta \int_0^{r_2} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho. \quad (44)$$

where α are the roots of the equation

$$J_m(\alpha r_2) = 0. \quad (45)$$

(iv) Stress functions ϕ_j for the body-forces F_j acting in an infinite plate domain ($-\ell < z < \ell$, $0 < r < \infty$).

$$\phi_j = -\frac{\text{Re}}{2\ell(1-\nu)} \sum_{n=0}^{\infty} \epsilon_n \int_{-\ell}^{\ell} e^{-i\beta(z-\zeta)} d\zeta \int_0^{\infty} \frac{\alpha J_m(\alpha r)}{(\alpha^2 + \beta^2)^2} d\alpha \int_0^{\infty} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho. \quad (46)$$

where $\beta = n\pi/\ell$, and $\epsilon_0 = 1$ for $n=0$, $\epsilon_n = 2$ for $n=1, 2, 3, \dots$

(v) Stress functions ϕ_j for the body forces F_j acting in a finite solid circular cylinder domain ($-\ell < z < \ell$, $0 < r < r_2$).

$$\phi_j = -\frac{\text{Re}}{\ell r_2^2(1-\nu)} \sum_{n=0}^{\infty} \sum_{\alpha} \frac{\epsilon_n J_m(\alpha r)}{[J_{m+1}(\alpha r_2)]^2} \int_{-\ell}^{\ell} \frac{e^{-i\beta(z-\zeta)}}{(\alpha^2 + \beta^2)^2} d\zeta \int_0^{r_2} \rho F_j(\rho, \zeta) J_m(\alpha \rho) d\rho. \quad (47)$$

6.2 Examples of application

As examples of application employing the stress functions ϕ_j for the body-force problems, let us consider the problems of a concentrated force acting in an infinite

domain considered is indefinitely extended, we shall add the conditions at infinity,

$$\lim_{r \rightarrow \infty} |F_j| = O(r^{-\frac{1}{2}-h}),$$

$$\lim_{z \rightarrow \pm \infty} |F_j| = O(|z|^{-h}) \quad (40)$$

where $h > 1$. Then as is well known we can represent the body-forces $F_j(r, z)$ in the form of the Fourier integral for z and the Fourier-Bessel integral for r , and so on. For example, if we assume that the body-forces F_j act in an infinite domain, we have

forces F_j acting in an infinite domain ($-\infty < z < \infty$, $0 < r < \infty$). The stress functions ϕ_j are defined as the particular solutions of the inhomogeneous equations (5) after the substitution of Eq.(41) into Eq.(5). Therefore, we have

(ii) Stress functions ϕ_j for the body-forces F_j acting in a semi-infinite domain ($0 < z < \infty$, $0 < r < \infty$).

solid. Substitution of ϕ_j of Eq.(42) into Eqs.(7), (9) yields the formulas of displacements u_j . Namely,

(i) the displacements u_j of the stress function ϕ_1 for a body-force F_1 : $u_2 = 0$,

$$2\mu u_1 = \frac{\text{Re}}{2\pi(1-\nu)} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} e^{-i\beta(z-\zeta)} d\zeta \int_0^{\infty} \frac{\alpha J_1(\alpha r)}{(\alpha^2 + \beta^2)^2} \{2(1-\nu)(\alpha^2 + \beta^2) - \alpha^2\} d\alpha$$

$$\int_0^\infty \rho F_1(\rho, \zeta) J_1(\alpha \rho) d\rho, \\ 2\mu u_3 = -\frac{\text{Re}}{2\pi(1-\nu)} i \int_{-\infty}^\infty d\beta \int_{-\infty}^\infty \beta e^{-i\beta(z-\zeta)} d\zeta \int_0^\infty \frac{\alpha J_0(\alpha r)}{(\alpha^2 + \beta^2)^2} d\alpha \int_0^\infty \rho F_1(\rho, \zeta) J_1(\alpha \rho) d\rho. \quad (48)$$

(ii) the displacement u_2 of the stress function ϕ_2 for a body-force F_2 : $u_1 = u_3 = 0$,

$$2\mu u_2 = \frac{\text{Re}}{\pi} \int_{-\infty}^\infty d\beta \int_{-\infty}^\infty e^{-i\beta(z-\zeta)} d\zeta \int_0^\infty \frac{\alpha J_1(\alpha r)}{\alpha^2 + \beta^2} d\alpha \int_0^\infty \rho F_2(\rho, \zeta) J_1(\alpha \rho) d\rho. \quad (49)$$

(iii) the displacements u_j of the stress function ϕ_3 for a body-force F_3 : $u_2 = 0$,

$$2\mu u_1 = \frac{\text{Re}}{2\pi(1-\nu)} i \int_{-\infty}^\infty d\beta \int_{-\infty}^\infty \beta e^{-i\beta(z-\zeta)} d\zeta \int_0^\infty \frac{\alpha^2 J_1(\alpha r)}{\alpha^2 \beta^2} d\alpha \int_0^\infty \rho F_3(\rho, \zeta) J_0(\alpha \rho) d\rho, \\ 2\mu u_3 = \frac{\text{Re}}{2\pi(1-\nu)} \int_{-\infty}^\infty d\beta \int_{-\infty}^\infty e^{-i\beta(z-\zeta)} d\zeta \int_0^\infty \frac{\alpha J_0(\alpha r)}{(\alpha^2 + \beta^2)^2} \{2(1-\nu)(\alpha^2 + \beta^2) - \beta^2\} d\alpha \\ \int_0^\infty \rho F_3(\rho, \zeta) J_1(\alpha \rho) d\rho. \quad (50)$$

[Example I] A problem of a radial force

$$F_1 = \frac{1}{2\pi r^2} \delta(r) \delta(z) \quad (51)$$

acting at a point $[0,0,0]$ in an infinite solid as shown in Fig.1. Where $\delta(r)$, $\delta(z)$ denote Dirac delta function. Substituting Eq.(51) into Eq.(48), we can obtain

$$2\mu u_1 = \frac{1}{8\pi(1-\nu)} \left\{ \frac{3}{2} \frac{r^3}{(r^2 + z^2)^{5/2}} - \frac{2\nu r}{(r^2 + z^2)^{3/2}} \right\}, \\ 2\mu u_3 = \frac{1}{8\pi(1-\nu)} \left\{ \frac{3}{2} \frac{r^2 z}{(r^2 + z^2)^{5/2}} - \frac{z}{(r^2 + z^2)^{3/2}} \right\}. \quad (52)$$

Here, in the calculus of the infinite integrals, tables⁽²⁴⁾ of integrals were used.

[Example II] A problem of a torsional force acting on the circumference of a circle of radius r_1 with center at a point $[0,0,0]$ in an infinite solid as shown in Fig.2.

In this case, we may express the body-force F_2 by

$$F_2 = \frac{1}{2\pi r} \delta(r-r_1) \delta(z) \quad (53)$$

Substitution of Eq.(53) into Eq.(49) yields

$$2\mu u_2 = \frac{1}{2\pi^2} \frac{1}{(r_1 r)^{1/2}} Q_{1/2}\left(\frac{z^2 + r^2 + r_1^2}{2 r r_1}\right). \quad (54)$$

where $Q_{1/2}$ is a Legendre function of the

Fig.1 Radial force at a point

second kind.

[Example III] A problem of a simple concentrated force applied at a point $[0,0,0]$ in an indefinitely extended solid. In this case, we may express the body-force F_3 by $F_3 = \delta(r)\delta(z)/2\pi r$. Substitution of this F_3 into Eq.(50) yields

$$2\mu u_1 = \frac{1}{8\pi(1-\nu)} \frac{r z}{(r^2 + z^2)^{3/2}}, \\ 2\mu u_3 = \frac{1}{8\pi(1-\nu)} \left\{ \frac{3-4\nu}{(r^2 + z^2)^{1/2}} + \frac{z^2}{(r^2 + z^2)^{3/2}} \right\}. \quad (55)$$

It is easily seen that u_j of Eq.(55) agrees with the result given by the so-called Kelvin's solution, and u_j of Eqs.(52) and Eq.(54) satisfy the fundamental equation (1).

7. Concluding remarks

In this paper, we have been concerned with an extension of Love's solution for axially symmetric problems in elastostatics. The principal results are as follows:

(1) Love's solution has been generalized such that it may be possible to apply it to the problems accompanied with both torsional deformations and body-forces.

(2) The generalized Love's solution has completeness of solution for axially symmetric problems. The completeness of Love's solution in its usual form is

Fig.2 A torsional force acting on the circumference of a circle of radius r_1 .

assured for the case of torsion-free deformations with no body-force but in z -direction.

(3) The torsional solution obtained from the generalized Love's stress function is equivalent to the solutions of other known methods.

(4) An attempt is made to investigate the general functional form of Love's stress function, and it is shown that the general form proposed in the paper contains the functional form which has been used up to now.

(5) The stress function for body-force problems is investigated, and examples of its application have been shown. That is, the problems treated here are in the case that an indefinitely extended solid is subject to such forces as (i) a radial point force, (ii) a torsional force acting on the circumference of a circle of radius r_1 and (iii) a single point force acting in the z -direction.

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