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Oscillations of a Rotating Shaft with Symmetrical Nonlinear Spring Characteristics*

By Toshio YAMAMOTO**, Yukio ISHIDA***, and Junzo KAWASUMI****

A rotating shaft system composed of a circular disc, an elastic shaft and two bearings is usually subject to some gyroscopic effect and has four natural frequencies $p_1 \sim p_4$ which correspond to four modes of whirling motion. Of these four, the first two are of forward precession ($p_1 > p_2 > 0$) and the other two are of backward precession ($0 > p_3 > p_4$).

We consider a vertical shaft system, in which a shaft is supported by a double-row self-aligning ball bearing at the upper end and by a single-row deep groove ball bearing at the lower end. If the two center lines of the upper and lower bearings are well aligned and if the shaft is situated at the middle of the angular clearance of the lower bearing, the elastic restoring force of the shaft has symmetrical nonlinear spring characteristics. For this rotating shaft system, summed-and-differential harmonic oscillations of the types $[2p_2 - p_3]$, $[2p_2 - p_4]$, $[p_2 - 2p_3]$, $[p_2 - 2p_4]$, and $[p_2 - p_3 - p_4]$, and a subharmonic oscillation of order $1/3$ of the type $[3p_2]$ can occur.

In this paper, these nonlinear forced whirling oscillations are studied experimentally. Furthermore, it is pointed out that the polar coordinates are suitable for representing the nonlinear restoring force characteristics in these whirling oscillations.

1. Introduction

When a single-row deep groove ball bearing is used for one of the two bearings of a rotating shaft system, the equilibrium position of the shaft center line deviates from the middle position of the "angular clearance"⁽¹⁾ of the ball bearing provided that the center lines of both bearing pedestals are not in alignment. Accordingly, unsymmetrical nonlinear characteristics (represented by even order terms in the polynomials for the restoring forces) appear more strongly than symmetrical nonlinear characteristics (represented by odd order terms) in the restoring forces of the shaft. Whirling oscillations, such as subharmonic oscillations of order $1/2$, summed-and-differential harmonic oscillations of the type $[p_i \pm p_j]$ (p_i and p_j are the natural frequencies of the system), etc., take place in such system. One of the authors has already reported experimental^{(1),(2),(3)} and theoretical^{(4),(5)} researches concerning these oscillations. Since the directional nonuniformity of shaft stiffness (the non-uniformity of linear terms in the polynomials representing restoring forces for different directions) and unsymmetrical nonlinear characteristics appear simulta-

neously, forced oscillations of synchronous backward precession take place concurrently with nonlinear forced oscillations. These oscillations have been also reported by one of the authors⁽⁶⁾.

Through our experiments, it can be pointed out that unsymmetrical nonlinear characteristics are likely to appear more strongly than symmetrical nonlinear characteristics when a single-row deep groove ball bearing is employed as one of the two bearings. However, if a vertical shaft system is assembled carefully to align both center lines of the upper and lower bearing pedestals, the shaft center line is located at the center of the angular clearance. Hence symmetrical nonlinear characteristics appear more strongly than unsymmetrical nonlinear characteristics.

In this paper, it is shown experimentally that various whirling motions due to symmetrical nonlinear spring characteristics, i.e. subharmonic oscillations of order $1/3$ and summed-and-differential harmonic oscillations of the types $[2p_i \pm p_j]$ and $[p_i \pm p_j \pm p_k]$, take place in a rotating shaft system whose bearing pedestals are aligned fairly well. These oscillations have not been reported in the previous papers^{(1)~(3)}. There are some reports^{(7)~(11)} by one of the authors and other researchers about various forced oscillations due to symmetrical nonlinear characteristics in rectilinear systems, but there exists no report about such oscillations of whirling motion in a rotating shaft system. Furthermore, it is pointed out in this paper that, when symmetrical nonlinear characteristics exist somewhat strongly, the following results are obtained: The

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** Professor, Faculty of Engineering, Nagoya University, Chikusa-ku, Nagoya.

*** Postgraduate, Faculty of Engineering, Nagoya University.

**** Nippondenso Co., Ltd. Kariya city, Aichi Pref.

$$\left. \begin{aligned} \omega_1 &= (1/3)\omega, & \omega_2 &= (1/3)\omega, \\ \omega_3 &= -(1/3)\omega, & \omega_4 &= -(1/3)\omega \end{aligned} \right\} \dots\dots\dots(4)$$

appear, respectively. In the neighborhoods of ω given by

$$|2p_i \pm p_j| = \omega \quad (i \neq j, \quad i, j = 1, 2, 3, 4) \dots\dots(5)$$

the summed-and-differential harmonic oscillations take place, i.e., two oscillations of the frequencies ω_i and ω_j having the relationships

$$\omega_i \pm p_i, \quad \omega_j \pm p_j, \quad |2\omega_i \pm \omega_j| = \omega \dots\dots\dots(6)$$

appear simultaneously. And in the neighborhoods of

$$|p_i \pm p_j \pm p_k| = \omega \quad (i \neq j \neq k, \quad i, j, k = 1, 2, 3, 4) \dots\dots(7)$$

The summed-and-differential harmonic oscillations consisting of three oscillations of the frequencies ω_i , ω_j , and ω_k having the relationships

$$\omega_i \pm p_i, \quad \omega_j \pm p_j, \quad \omega_k \pm p_k, \quad |\omega_i \pm \omega_j \pm \omega_k| = \omega \dots\dots(8)$$

appear. The relationship $p_1 > 2\omega$ is derived from (ii) because $I_p/I \approx 2$ holds in our experimental apparatus. It is easily seen from this and the relationship (iii) that there is no angular velocity ω which satisfies Eqs.(3), (5), and (7) containing p_1 . And similarly to the case of a rectilinear system^(9,11), it can be proved that, among summed-and-differential harmonic oscillations of whirling motion, only those of the summed type respecting absolute values of natural frequencies can take place in a rotating shaft system. Consequently, subharmonic oscillations which are expected

to occur in our experimental apparatus are limited to those which appear in the regions near the angular velocities:

$$3p_2 = \omega, \quad -3p_3 = \omega, \quad -3p_4 = \omega \dots\dots\dots(9)$$

$$\left. \begin{aligned} p_2 - 2p_4 &= \omega, & 2p_2 - p_4 &= \omega, & -p_3 - 2p_4 &= \omega \\ 2p_2 - p_3 &= \omega, & -2p_3 - p_4 &= \omega, & p_2 - 2p_3 &= \omega \end{aligned} \right\} \dots\dots\dots(10 \text{ a})$$

$$p_2 - p_3 - p_4 = \omega \dots\dots\dots(10 \text{ b})$$

These angular velocities are indicated by the abscissas of the intersection points E , F_2 , F_1 , $G_1 \sim G_6$, and G'_1 in Fig.2. We represent these angular velocities by the symbols ω_3 , $\omega'_{1/3}$, $\omega_{-1/3}$, $\omega_{2/4}$, $\omega_{2/4}$, $\omega_{3/4}$, $\omega_{2/3}$, $\omega_{3/4}$, $\omega_{2/3}$, and $\omega_{2/4}$, respectively.

Most forced oscillations due to unsymmetrical nonlinear characteristics are caused by the second power terms of coordinates. Among these oscillations, those which are expected to occur are limited to those which appear in the regions near the angular velocities

$$2p_2 = \omega, \quad -2p_3 = \omega, \quad -2p_4 = \omega \dots\dots\dots(11)$$

$$p_2 - p_4 = \omega, \quad -p_3 - p_4 = \omega, \quad p_2 - p_3 = \omega \dots\dots(12)$$

for the same reason as before. These angular velocities are given by C , D_2 , D_1 , and $H_1 \sim H_3$ in Fig.2. We represent these by $\omega_{1/2}$, $\omega'_{1/2}$, $\omega_{-1/2}$, $\omega_{2/4}$, $\omega_{3/4}$, and $\omega_{2/3}$.

3. Experimental results

(I) Spring characteristics The experimental apparatus was assembled carefully to attain good alignment of center lines of the upper and the lower bearing pedestals. Its spring characteristics are shown in

Fig.3 Natural frequencies when $\omega=0$

Fig.4 Spring characteristics (I)

Fig.5 Spring characteristics (II)

Figs.3 and 4.

In Fig.3, it was examined whether the rotating directional nonuniformity of shaft stiffness exists as in the previous paper⁽¹³⁾ or not. When $\omega=0$, the natural frequencies p_{01} and p_{02} in the directions of the maximum and minimum stiffnesses, respectively, are measured for various angular position θ (Fig.1) of the disk. The former and the latter are indicated by the symbols $\bullet$ and $\circ$, respectively, and the frequencies which can not be distinguished are shown by the symbol $\circ$. From this figure, it is found that the difference between p_{01} and p_{02} is little and does not change with θ . Accordingly, it can be concluded that the rotating nonuniformity of shaft stiffness does not exist.

The relationships between the moment exerted on the disk and the inclination angle of the disk for various directions denoted by φ are shown in Fig.4, when the angular position of the disk is kept at $\theta=0$. In opposition to Fig.5 in the previous paper⁽¹⁾, symmetrical nonlinear characteristics appear more strongly than unsymmetrical nonlinear characteristics. Since all the characteristic curves in Fig.4 take a similar shape, it can be seen that this symmetrical nonlinear restoring force characteristics are directionally uniform (isotropic).

Spring characteristics shown in Fig.5 which were measured after the reassembly differ a little from those in Figs.3 and 4. Symmetrical nonlinear characteristics still appear strongly in Fig.5, but the shape of characteristic curve changes with the direction φ , that is, they are directionally nonuniform (anisotropic). In other words, when the restoring force characteristics, for example, in the x - and y -directions are

represented by $\alpha x + \beta_x x^3$ and $\alpha y + \beta_y y^3$, respectively, the coefficients β_x and β_y are equal in Fig.4 and unequal in Fig.5.

Figures 4 and 5 show the characteristic curves of hard spring type. This is understood because nonlinear characteristics

Fig.7 Frequencies of the summed-and-differential harmonic oscillation of the type $[2p_2-p_3]$

$\omega=3000\text{rpm}$, $\omega:\omega_2:\omega_3=37:17:-3$

Fig.8 Waves of the summed-and-differential harmonic oscillation of the type $[2p_2-p_3]$

Fig.6 Oscillations occurred in a system with isotropic symmetrical nonlinear spring characteristics

are caused by the angular clearance of the single-row deep groove ball bearing.

(II) Forced oscillations of whirling motion which are influenced by symmetrical nonlinear characteristics

(II-a) Oscillations whose shapes of resonance curves are influenced Among the oscillations occurring in the system with characteristics shown in Fig.4, only those related to symmetrical nonlinear characteristics are shown in Fig.6. The oscillations in the neighborhoods of ω_{b2} , ω_c , and ω_{b1} (Fig.2) can occur also in the linear systems.

The oscillation in the neighborhood of the major critical speed ω_c where $\omega=p_2$ holds has a typical resonance curve of a hard spring type, because of symmetrical nonlinear characteristics. It was accompanied by a jump and a hysteresis phenomena like that of rectilinear systems. Though the upper branch of the resonance curve extends further to the higher speed region, the experiment was stopped at about 1350rpm for safety. Hereafter, if the upper branch of a resonance curve ends with no arrow indicating jump at its highest edge, it implies that the experiment is stopped for the same reason.

In the neighborhoods of ω_{b2} and ω_{b1} , where $\omega=-p_3$ and $\omega=-p_4$ hold, respectively, forced oscillations of synchronous backward precession of the frequency $-\omega$ appear due to the directional nonuniformity of shaft stiffness. Their amplitudes are

Fig.9 Frequencies of the summed-and-differential harmonic oscillation of the type $[2p_2-p_4]$

proportional to this difference⁽¹⁴⁾, which is caused by the directional nonuniformity of stiffness of pedestals⁽¹⁴⁾ and by unsymmetrical nonlinear characteristics due to angular clearances of ball bearings⁽⁶⁾. Therefore, comparing with the previous paper⁽⁶⁾ for a system where unsymmetrical nonlinear characteristics appeared strongly, the amplitudes of such oscillations in Fig.6 are considerably small because the unsymmetrical nonlinear characteristics appear weakly in this case. The shape of the resonance curves in the paper⁽⁶⁾ were the same as those of a linear system because symmetrical nonlinear characteristics were weak, but those in this paper take the shape of a hard spring type because they are strong. It is similar to the case of the major critical speed.

In the case of Fig.5 having anisotropic nonlinear characteristics, almost the same results as those in Fig.6 are obtained for ω_c , ω_{b1} , and ω_{b2} .

(II-b) Oscillations whose occurrence is influenced In Fig.6, the summed-and-differential harmonic oscillations in the neighborhoods of ω_{23} and ω_{24} , where $\omega=2p_2-p_3$ and $\omega=2p_2-p_4$ hold, respectively, do not appear if there is no symmetrical nonlinear characteristics.

In the neighborhood of ω_{23} in Fig.6, two oscillations of forward and backward precessions, whose frequencies are $\omega_2 (\approx p_2)$ and $\omega_3 (\approx p_3)$, respectively, occur simultaneously in the wide range of $\omega > 2640$ rpm.

The frequencies ω_2 and ω_3 of the summed-and-differential harmonic oscillation $[2p_2-p_3]$ are shown in Fig.7. The points representing the values of $\omega_2 (\approx p_2)$ and ω_3

$\omega=4582$ rpm, $\omega:\omega_2:\omega_3=16:5:-6$

Fig.10 Waves of the summed-and-differential harmonic oscillation of the type $[2p_2-p_4]$

Fig.11 Oscillations occurring only in a system with anisotropic symmetrical nonlinear spring characteristics

($\approx p_3$) obtained in experiments are almost on two straight lines passing through the origin.

The vibratory waves of the oscillation of the type $[2p_2-p_3]$ are shown in Fig. 8. White vertical lines in the figure are rotation marks recorded at each rotation of the shaft by using the small paper P shown in Fig.1. The superimposed oscillation of a small amplitude is the harmonic component of the frequency ω , and it appears in all the vibratory waves shown in this paper. The waves have a period corresponding to the distance between two points marked A. In the interval AA, the

Fig.12 Frequencies of the summed-and-differential harmonic oscillation of the type $[p_2-2p_3]$

Fig.13 Frequencies of the summed-and-differential harmonic oscillation of the type $[p_2-2p_4]$

shaft rotates 37 times, the rapid oscillation swings 17 times, and the slow oscillation swings 3 times. Furthermore, from the comparison of the phases in the x - and y -directions, it can be seen that the rapid one is a forward precession and the slow one is a backward precession. We represent a forward precession by the sign plus and a backward precession by the sign minus. Thus we find $\omega:\omega_2:\omega_3=37:17:(-3)$ and $\omega=2\omega_2-\omega_3$.

In the neighborhood of ω_{24} in Fig.6, the summed-and-differential harmonic oscillations $[2p_2-p_4]$ occur in the range of $\omega > 4500$ rpm. The frequencies ω_2 and ω_4 are shown in Fig.9, which are almost on two straight lines passing through the origin. The vibratory waves are shown in Fig.10 where $\omega:\omega_2:\omega_4=16:5:(-6)$ and $\omega=2\omega_2-\omega_4$ hold.

The shapes of both oscillations of the

$\omega=4860$ rpm, $\omega:\omega_2:\omega_4=54:16:-19$
Fig.15 Waves of the summed-and-differential harmonic oscillation of the type $[p_2-2p_4]$

Fig.16 Frequencies of the summed-and-differential harmonic oscillation of the type $[p_2-p_3-p_4]$

$\omega=1878$ rpm, $\omega:\omega_2:\omega_3=40:26:-7$
Fig.14 Waves of the summed-and-differential harmonic oscillation of the type $[p_2-2p_3]$

types $[2p_2-p_3]$ and $[2p_2-p_4]$ are those of a hard spring type, as seen in Fig.6.

In the range of $\omega > 2600\text{rpm}$ in Fig.6, small harmonic oscillation of frequency ω (symbol o) appear as well as summed-and-differential harmonic oscillation (symbol ϕ). Which of the two will take place depends on the initial conditions.

The above-mentioned nonlinear forced oscillations are obtained in the system having the isotropic symmetrical nonlinear characteristics shown in Fig.4. In the system with anisotropic symmetrical nonlinear characteristics of Fig.5, the following oscillations occur in addition to the oscillations of the types $[2p_2-p_3]$ and $[2p_2-p_4]$ shown in Fig.6: The subharmonic oscillation of order 1/3 of the type $[3p_2]$, the summed-and-differential harmonic oscillations of the types $[p_2-2p_3]$ and $[p_2-2p_4]$, and the summed-and-differential harmonic oscillation of the type $[p_2-p_3-p_4]$ in which three oscillations appear simultaneously. These oscillations shown in Fig.11 are due to anisotropic symmetrical nonlinear characteristics, and all the resonance curves are those of a hard spring type.

In the neighborhoods of the angular velocities ω_{233} and ω_{244} , where $\omega = p_2-2p_3$ and $\omega = p_2-2p_4$ hold, respectively, the summed-and-differential harmonic oscillations of the types $[p_2-2p_3]$ and $[p_2-2p_4]$ occur. The former appears in the range of $\omega \approx 1860 \sim 1940\text{rpm}$, the latter in the range of $\omega \approx 4750 \sim 4930\text{rpm}$, and these ranges are comparatively narrow. The frequencies of these oscillations are shown in Figs.12 and 13, which are almost on two straight lines passing through the origin. The vibratory waves are shown in Figs.14 and 15, where we find the relationships $\omega:\omega_2:\omega_3=40:26:(-7)$, $\omega=\omega_2-2\omega_3$ and $\omega:\omega_2:\omega_4=54:16:(-19)$, $\omega=\omega_2-2\omega_4$.

The summed-and-differential harmonic oscillation of the type $[p_2-p_3-p_4]$ shown

in Fig.11 occur in the neighborhood of the angular velocity ω_{234} where $\omega = p_2-p_3-p_4$ holds. It consists of three oscillations with the frequencies $\omega_2(\approx p_2)$, $\omega_3(\approx p_3)$, and $\omega_4(\approx p_4)$. This oscillation appears in somewhat narrow range of $\omega \approx 3265 \sim 3320\text{rpm}$. Its frequencies and vibratory waves are shown in Figs.16 and 17, respectively. Their general features are similar to those of a rectilinear system reported in the paper⁽¹¹⁾. In the interval AA in Fig.17, the long wave with the frequency ω_3 oscillates 7 times in a backward precession, the short wave of the frequency ω_2 with large amplitude oscillates 43 times, and it can be found from the beating change of the wave with the frequency ω_2 that the wave with the frequency ω_4 is also present and oscillates 50 times in a backward precession. Since the shaft rotates 100 times in this interval, the relationship $43-(-7)-(-50)=100$, that is, $\omega = \omega_2-\omega_3-\omega_4$ holds.

It is known from Figs.6 and 11 that the ranges where the oscillations of the types $[p_2-2p_3]$, $[p_2-2p_4]$, and $[p_2-p_3-p_4]$ due to anisotropic symmetrical nonlinear characteristics occur are fairly narrow when compared with the ranges of the oscillations of the types $[2p_2-p_3]$ and $[2p_2-p_4]$ due to isotropic symmetrical nonlinear

Fig.18 Waves of the subharmonic oscillation of the type $[3p_2]$

Fig.17 Waves of the summed-and-differential harmonic oscillation of the type $[p_2-p_3-p_4]$

Table 1 Experimental results about the occurrence of various nonlinear forced oscillations caused by symmetrical nonlinear spring characteristics

Kinds of oscillations Spring characteristics	Subharmonic oscillations of order 1/3			Summed-and-differential harmonic oscillations						
				Two oscillations						Three
	$3p_2$	$-3p_3$	$-3p_4$	$2p_2-p_3$	$2p_2-p_4$	p_2-2p_3	p_2-2p_4	$-2p_3-p_4$	$-p_3-2p_4$	$p_2-p_3-p_4$
B-I	X	X	X	O	O	X	X	X	X	X
B-II	O	X	X	O	O	O	O	X	X	O

B-I: Isotropic symmetrical nonlinear spring characteristics(Fig.4), B-II: Anisotropic symmetrical nonlinear spring characteristics(Fig.5), O:appearance, X:non-appearance

characteristics.

The subharmonic oscillation of order 1/3 of forward precession in the neighborhood of ω_3 where $\omega=3p_2$ holds has a resonance curve of a hard spring type like that of a rectilinear system, as shown in Fig.11. It occurs in the comparatively wide range of $\omega>4120\text{rpm}$. Its vibratory waves are shown in Fig.18. The small wave in this figure is the harmonic component of frequency ω as mentioned previously.

The oscillations of the types $[2p_2-p_3]$ and $[2p_2-p_4]$ in Fig.6, occurring in the system with isotropic symmetrical nonlinear characteristics, are also found to appear in the system with anisotropic symmetrical nonlinear characteristics. Their general features in the latter system are almost the same as those in the former.

Through many experiments performed in a wide range of ω , no other oscillation due to symmetrical nonlinear characteristics than those mentioned in (II-b) occurred.

The experimental results about the occurrence of the oscillations of Eqs.(9) and (10) in the previous chapter are shown in Table 1. Considerations about Table 1 are given in the next chapter.

(III) Forced oscillations of whirling motion which are influenced by unsymmetrical nonlinear characteristics

(III-a) Oscillations whose occurrence is

influenced Various features of these oscillations for the spring characteristics of Fig.4 are the same as those for Fig.5. In other words, the existence of anisotropy of symmetrical nonlinear characteristics does not affect them.

Resonance curves of the subharmonic oscillation of order 1/2 of the type $[2p_2]$ and of summed-and-differential harmonic oscillations of the types $[p_2-p_3]$, $[p_2-p_4]$ are shown in Fig.19. Because of strong symmetrical nonlinear characteristics, their resonance curves become a hard spring type and are bent strongly. The subharmonic oscillation appears in the wide range of $\omega\approx 2480\sim 2900\text{rpm}$. The oscillations $[p_2-p_3]$ and $[p_2-p_4]$ appear also in the wide ranges of $\omega\approx 1500\sim 1650\text{rpm}$ and $\omega\approx 2950\sim 3290\text{rpm}$, respectively.

The experimental apparatus of the previous papers⁽¹⁾⁽²⁾⁽³⁾ are similar to that of Fig.1. Accordingly, by comparing the results of these previous papers with those in Fig.19, we can ascertain the effects of the magnitudes of unsymmetrical and symmetrical nonlinear characteristics to various oscillations caused by unsymmetrical nonlinear characteristics.

The experimental results concerning the appearance of the oscillations of Eqs. (11) and (12) which are expected to occur are summarized in Table 2. From this, we

Fig.19 Oscillations due to unsymmetrical nonlinear spring characteristics

Table 2 Experimental results about the appearance of various forced oscillations caused by unsymmetrical nonlinear spring characteristics

Kinds of oscillations Spring characteristics		Subharmonic oscillations of order 1/2			Summed-and-differential harmonic oscillations		
		$2p_2$	$-2p_3$	$-2p_4$	p_2-p_3	p_2-p_4	$-p_3-p_4$
A	Appearance or non-appearance	0	Δ	Δ	0	0	Δ
	Width of the range of appearance(rpm)	25, 40, 150	20, 30	75	100	50, 100, 200	50, 100
	Amplitude	large	small	small	large	large	small
B-I	Appearance or non-appearance	0	x	x	0	0	x
B-II	Width of the range of appearance(rpm)	420			150,200	340	

A: The case of the previous reports⁽¹⁾⁽²⁾⁽³⁾, having strong unsymmetrical nonlinear spring characteristics. B-I: The case of Fig.4 having strong symmetrical nonlinear spring characteristics(isotropic). B-II: The case of Fig.5 having strong symmetrical nonlinear spring characteristics(anisotropic). 0:appearance, X:non-appearance, Δ:non-appearance or appearance with small amplitude depending on assembly

can obtain the following conclusions: The oscillations with large amplitudes in the previous papers also appear in this experiment. Those with small amplitudes in the previous papers, however, can not occur in this experiment because of small unsymmetrical nonlinear characteristics. The resonance curves of the oscillations in this experiment are bent strongly toward the higher frequency regions and hence they appears in the wide ranges, because of stronger symmetrical nonlinear characteristics than those in the previous papers.

4. Representations of nonlinear spring characteristics suitable for analysing whirling oscillations and equations of motion of the system

The nonlinear spring characteristics of a rotating shaft system are distributed two-dimensionally in the x, y -plane (Fig.1). Therefore it is rational to express them in a polar coordinate system, and such representation is different from that used in a rectilinear system.

The experimental apparatus in this paper has four degrees of freedom, where a deflection and an inclination of the disk couple with each other. As the simplest case, we consider an inclination motion of a system in which the disk is mounted at the middle position between the upper and lower bearing pedestals. In this case, a deflection and an inclination do not couple with each other. Accordingly, for the inclination motion, this system has two degrees of freedom, in which whirling motions take place. The deflection motion is not whirling and we do not treat such motion here. The configuration of this system can be determined by the components in the x - and y -directions θ_x and θ_y of the inclination angle of the disk.

We deal with the most simple symmetrical nonlinear characteristics represented by polynomials up to the third power terms of coordinates θ_x and θ_y . Consequently, the potential energy of the system is expressed as follows:

$$V = (1/2)\delta(\theta_x^2 + \theta_y^2) + \beta_{40}\theta_x^4 + \beta_{31}\theta_x^3\theta_y + \beta_{22}\theta_x^2\theta_y^2 + \beta_{13}\theta_x\theta_y^3 + \beta_{04}\theta_y^4 \quad (13)$$

where δ is the spring constant of the shaft and β_{ij} is the coefficients of the nonlinear terms. Instead of the coordinates θ_x and θ_y , we use the polar coordinates (θ, φ) given by

$$\theta_x = \theta \cos \varphi, \quad \theta_y = \theta \sin \varphi \quad (14)$$

Then Eq.(13) for the potential energy is transformed into

$$V = (1/2)\delta\theta^2 + (\beta^{(0)} + \beta_c^{(2)} \cos 2\varphi + \beta_s^{(2)} \sin 2\varphi + \beta_c^{(4)} \cos 4\varphi + \beta_s^{(4)} \sin 4\varphi)\theta^4 \quad (15)$$

where

$$\left. \begin{aligned} \beta^{(0)} &= \frac{1}{8} [3(\beta_{40} + \beta_{04}) + \beta_{22}], \quad \beta_c^{(2)} = \frac{1}{2} (\beta_{40} - \beta_{04}), \\ \beta_s^{(2)} &= \frac{1}{4} (\beta_{31} + \beta_{13}), \quad \beta_c^{(4)} = \frac{1}{8} (\beta_{40} + \beta_{04} - \beta_{22}), \end{aligned} \right\}$$

$$\beta_s^{(4)} = \frac{1}{8} (\beta_{31} - \beta_{13}) \quad (16)$$

In the right-hand side of Eq.(15), the first term corresponds to a linear term in spring characteristics, the second term is an isotropic nonlinear term, the third and fourth terms are anisotropic nonlinear terms changing their magnitudes twice while the angle φ changes from 0 to 2π , and the fifth and sixth terms are anisotropic nonlinear terms changing four times. Using

$$\beta^{(2)} = \sqrt{\beta_c^{(2)2} + \beta_s^{(2)2}}, \quad \beta^{(4)} = \sqrt{\beta_c^{(4)2} + \beta_s^{(4)2}} \quad (17)$$

and rotating the coordinate axis suitably, we can express Eq.(15) in terms of only $\beta^{(2)}$ instead of $\beta_c^{(2)}$ and $\beta_s^{(2)}$, or in terms of only $\beta^{(4)}$ instead of $\beta_c^{(4)}$ and $\beta_s^{(4)}$.

Let the polar and diametral moments of inertia of the disk be I_p and I , respectively, time be t , the damping coefficient be c , and the dynamic unbalance be τ . By introducing the dimensionless quantities:

$$\left. \begin{aligned} I_p/I &= i_p, \quad t\sqrt{\delta/I} = t', \quad \omega\sqrt{I/\delta} = \omega', \\ c/\sqrt{\delta I} &= c', \quad \beta_{ij}/\delta = \beta_{ij}' \end{aligned} \right\} \quad (18)$$

and by omitting the primes, we obtain the following equations of motion expressed by dimensionless quantities:

$$\left. \begin{aligned} \ddot{\theta}_x + i_p \omega \dot{\theta}_y + c \dot{\theta}_x + \theta_x + \varphi_x &= (i_p - 1) \tau \omega^2 \cos \omega t \\ \ddot{\theta}_y - i_p \omega \dot{\theta}_x + c \dot{\theta}_y + \theta_y + \varphi_y &= (i_p - 1) \tau \omega^2 \sin \omega t \end{aligned} \right\} \quad (19)$$

where the nonlinear terms φ_x and φ_y are given by

$$\partial V / \partial \theta_x = \theta_x + \varphi_x, \quad \partial V / \partial \theta_y = \theta_y + \varphi_y \quad (20)$$

The system has two natural frequencies; one is for a forward precession $p_f \{=(i_p \omega + \sqrt{i_p^2 \omega^2 + 4})/2\}$ and the other is for a backward precession $p_b \{=(i_p \omega - \sqrt{i_p^2 \omega^2 + 4})/2\}$. By putting

$$\theta_x = X_1 + X_2, \quad \theta_y = -\frac{1}{p_f} \dot{X}_1 - \frac{1}{p_b} \dot{X}_2 \quad (21)$$

we obtain the equations of motion expressed by the normal coordinates X_1 and X_2 as follows⁽⁴⁾:

$$\left. \begin{aligned} \ddot{X}_1 + p_f^2 X_1 &= \frac{p_f^2}{1 + p_f^2} \left\{ \left(\frac{p_f}{\omega} + 1 \right) P \cos \omega t \right. \\ &\quad \left. + c(-\dot{\theta}_x + p_f \theta_y) + (-\varphi_x + p_f \Phi_y) \right\} \\ \ddot{X}_2 + p_b^2 X_2 &= \frac{p_b^2}{1 + p_b^2} \left\{ \left(\frac{p_b}{\omega} + 1 \right) P \cos \omega t \right. \\ &\quad \left. + c(-\dot{\theta}_x + p_b \theta_y) + (-\varphi_x + p_b \Phi_y) \right\} \end{aligned} \right\} \quad (22)$$

where

$$P = (i_p - 1) \tau \omega^2, \quad \Phi_y = \int \varphi_y dt \quad (23)$$

From Eq.(22), the resonance curves for the harmonic oscillation of frequency ω in the neighborhood of the major critical speed ω_c are obtained as

$$\frac{1}{4} \{ (2 - i_p)(\omega^2 - p_f^2) - 8\beta^{(0)} R^2 \}^2 R^2 + \frac{c^2}{1 - i_p} R^2 = P^2 \quad (24)$$

where R is the amplitude. The above

equation shows that only the coefficient $\beta^{(0)}$ of the isotropic symmetrical nonlinear term is concerned with this oscillation. And by putting $P=0$ and $c=0$ in Eq.(24), we obtain the equation of backbone curve

$$R^2 = (2 - i_p)(\omega^2 - p_f^2)/8\beta^{(0)} \quad (25)$$

which shows that the sign of $\beta^{(0)}$ determines whether the resonance curve is that of hard or soft spring type and its magnitude determines the degree of bend of the resonance curve.

Next, we consider the oscillation of synchronous backward precession of the frequency $-\omega$ in the neighborhood of the critical speed ω_b . The backbone curve of this resonance curve is given by the following equation containing only $\beta^{(0)}$:

$$R^2 = [(1 + i_p)\omega^2 - 1]/4\beta^{(0)} \quad (26)$$

In general, the equation of the resonance curve is a complex one containing $\beta^{(0)}$, $\beta^{(2)}$, and $\beta^{(4)}$. When the difference of shaft stiffness between the x - and y -directions is denoted by 2Δ and the mean shaft stiffness by 1, the shaft stiffness in the x - and y -directions are represented by $1+\Delta$ and $1-\Delta$, respectively. In the case where only $\beta^{(0)}$ exists, the resonance curve is given by

$$\left. \begin{aligned} R^2 [(1 + i_p)\omega^2 - 1] - 4\beta^{(0)}(2F^2 + R^2)^2 \\ + c^2\omega^2 R^2 = \Delta^2 F^2 \\ F = P/(1 + i_p\omega^2 - \omega^2) \end{aligned} \right\} \quad (27)$$

From Eq.(22), the resonance curves and the backbone curves of subharmonic oscillations of order 1/3 and of summed-and-differential harmonic oscillations are obtained. By using the coefficients of Eq.(16) in the expression of these equations, we obtain the following results:

- (i) All equations of backbone curves contain only $\beta^{(0)}$. Thus, the type (hard or soft) and the degree of bend of resonance curves are determined by the sign and the magnitude of $\beta^{(0)}$, respectively.
- (ii) The summed-and-differential harmonic oscillation of the type $[2p_f - p_b]$ can take place when $\beta^{(0)}$ exists.
- (iii) The summed-and-differential harmonic oscillation of the type $[p_f - 2p_b]$ and the subharmonic oscillation of order 1/3 of the type $[3p_f]$ can take place when both $\beta^{(0)}$ and $\beta^{(2)}$ exist. That is, not only the isotropic component but also the anisotropic component which changes twice while φ changes by 2π is necessary for their appearance.
- (iv) The subharmonic oscillation of order 1/3 of the type $[-3p_b]$ needs both $\beta^{(0)}$ and $\beta^{(4)}$ for its appearance.

It is understood from the above conclusions that only the type $[2p_f - p_b]$ occurs in the system of spring characteristics of Fig.4, the types $[p_f - 2p_b]$ and $[3p_f]$ occur in addition to the type $[2p_f - p_b]$ in the system of Fig.5, and the type $[-3p_b]$ cannot be expected to occur in such an experimental apparatus because it is difficult for such a system to have the component $\beta^{(4)}$.

The above conclusions can be extended to a system with four degrees of freedom. Because p_f corresponds to p_2 and p_b corresponds

to p_3 and p_4 , the results shown in Table 1 can be easily understood.

We will discuss theoretically various nonlinear forced oscillations in connection with β_{ij} and $\beta^{(n)}$ in a following paper.

5. Conclusions

(1) When a shaft is supported by single-row deep groove ball bearings, nonlinear spring characteristics appear in a rotating shaft system. Whether unsymmetrical or symmetrical nonlinear characteristics appear depends on the degree of shift of the shaft center line from the center of an "angular clearance" of a ball bearing.

(2) Being different from the case of a rectilinear system, nonlinear spring characteristics should be considered as two-dimensional distribution in a rotating shaft system.

(3) Symmetrical nonlinear characteristics represented by the third powers of coordinates can be classified into the isotropic component and the anisotropic components which change their magnitudes 2 and 4 times while the shaft whirls once.

(4) In the system having only the isotropic symmetrical nonlinear characteristics, the summed-and-differential harmonic oscillations of the types $[2p_2 - p_3]$ and $[2p_2 - p_4]$ occur.

(5) In the system having both the isotropic and anisotropic components of symmetrical nonlinear characteristics, the subharmonic oscillation of order 1/3 of the type $[3p_2]$ and the summed-and-differential harmonic oscillations of the types $[p_2 - 2p_3]$, $[p_2 - 2p_4]$, $[p_2 - p_3 - p_4]$ can occur as well as the types $[2p_2 - p_3]$, $[2p_2 - p_4]$.

(6) The subharmonic oscillations of order 1/3 of the types $[-3p_3]$, $[-3p_4]$ and the summed-and-differential harmonic oscillations of the types $[-2p_3 - p_4]$, $[-p_3 - 2p_4]$ do not occur in the experiments performed in this paper.

(7) In the experiments of the above conclusions (4) and (5), jump phenomena and hysteresis phenomena, which are induced by the symmetrical nonlinear characteristics, take place in the neighborhoods of the major critical speed and the critical speeds of synchronous backward precession.

(8) Among oscillations due to unsymmetrical nonlinear characteristics, the subharmonic oscillation of order 1/2 of the type $[2p_2]$ and the summed-and-differential harmonic oscillations of the types $[p_2 - p_3]$, $[p_2 - p_4]$ occur in the experiments in this paper. The resonance curves of these oscillations bend severely to the higher speed side because of the strong symmetrical nonlinear characteristics.

(9) By utilizing the polar coordinate representation of nonlinear spring characteristics, it can be made clear what kinds of nonlinear forced oscillations will occur.

References

- (1) Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.21, No.111

- (1955-11), p.853.
Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.9, No.1 (1957-5), p.33.
- (2) Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.22, No.115 (1956-3), p.172.
Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.9, No.1 (1957-5), p.43.
- (3) Yamamoto, T., *Bull. JSME*, Vol.3, No. 12 (1960-10), p.397.
- (4) Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.22, No.123 (1956-11), p.868.
Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.9, No.1 (1957-5), p.53.
- (5) Yamamoto, T., *Bull. JSME*, Vol.4, No. 13 (1961-2), p.51.
- (6) Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.22, No.115 (1956-3), p.167.
Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.9, No.1 (1957-5), p.71.
- (7) Nishihara, T., Sawaragi, Y., and Okada, Y., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.17, No.68 (1951), p.26.
- (8) Nishino, K., *J. Japan Soc. Appl. Mech.* (in Japanese), Vol.3, No.18 (1950-8), p.121.
- (9) Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.27, No.182 (1961-10), p.1676.
Yamamoto, T. and Hayashi, S., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.18, No.2 (1966-11), p.124.
- (10) Yamamoto, T. and Hayashi, S., *Bull. JSME*, Vol.6 No.23 (1963-8), p.420.
- (11) Hayashi, S., *Trans., Japan Soc. Mech. Engrs.* (in Japanese), Vol.32, No.234 (1966-2), p.219.
Yamamoto, T. and Hayashi, S., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.18, No.2 (1966-11), p.139.
- (12) Yamamoto, T., *Jour. Japan Soc. Mech. Engrs.* (in Japanese), Vol.66, No.531 (1963-4), p.528.
- (13) Yamamoto, T. and Ishida, Y., *Bull. JSME*, Vol.17, No.103 (1974-1), p.59.
- (14) Shimoyama, Y. and Yamamoto, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.20, No.91 (1954-3), p.215.
Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol. 6, No.2 (1954-10), p.119.