Numerical Method for Time-Dependent Two-Dimensional Viscous Flows (Part 1, Fundamental Method)*

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Although the Navier-Stokes equations have been solved directly by using the finite-difference method, the calculation requires in general a considerable amount of computer time. In the present paper, a numerical method for time-dependent two-dimensional laminar flows is proposed with the aim of reducing the time spent and increasing the accuracy. In the method a convective-difference scheme with interpolation formula is used and the same iterative calculations as in the method of characteristics are applied. The accuracy of the solutions is investigated theoretically from the truncation errors, and the stability criteria are derived from the inherited errors. A comparison of such accuracy and also some numerical examples of the present and the other available methods show that the second approximation of the present method is the most suitable.

1. Introduction

The numerical solutions of the Navier-Stokes equations by finite difference methods have been used as a powerful means for unsteady two-dimensional laminar flows. A number of methods have been proposed to date and successfully applied to various kinds of problems. However, since such calculations require considerably long computer time, their applications are restricted to simple flow problems at relatively low Reynolds numbers. This restriction will be removed as the computer becomes larger and faster, and also mitigated to a certain extent by means of the improvement of numerical methods. To this end one numerical method is proposed in the present paper. Although the method will be further favourable by using non-uniform grids, in Part I we employ the most standard square grid. The detail of the method, the accuracy and the stability of solution, and a comparison of the method with the existing one are presented in the following.

Nomenclature

\[
\begin{align*}
d/dt & = \partial/\partial t + \mathbf{v} \cdot \nabla \\
\partial/dt & = \partial/\partial t - \mathbf{v} \cdot \nabla \\
h & : \text{grid space} \\
N & : 4\delta t/h^2 \\
t & : \text{time} \\
\mathbf{v} & : \text{velocity} = (u,v)
\end{align*}
\]

\(X,Y : \text{see Eqs. (7) and (11)}\)
\(x,y : \text{Cartesian coordinates} \)
\(\Delta : \text{Laplacian} = \partial^2/\partial x^2 + \partial^2/\partial y^2 \)
\(\delta t : \text{time increment} \)
\(\mathbf{v} : \text{kinematic viscosity} \)
\(\psi : \text{stream function} \)
\(\omega : \text{vorticity} \)
\(\phi : \text{see Eq. (13)} \)

Subscripts

\(x,y,t : \text{derivatives with respect to } x, y, \text{ and } t, \text{ respectively} \)
\(i,j : \text{i-th column and j-th row} \)
\(0,1 : \text{see Fig. 1} \)

Superscripts

\(n : \text{n-th time step} \)
\(1,2,\ldots,N : \text{first, second, and N-th (N \geq 2) approximation, respectively} \)

2. Governing equations and the available methods of solution

At the beginning, let us review in brief the existing numerical methods available for unsteady laminar flows. They belong to either of the two approaches, i.e. the \(p-v\) method in which the equations

\[\frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{v} \cdot \nabla \mathbf{v} \]

\[\Delta p = -(u^2)_{xx} - (uv)_{xy} - (v^2)_{yy} \]

are treated, or the \(\psi-\omega\) method in which

\[d\omega/dt = \mathbf{v} \cdot \nabla \omega \]

\[\Delta \omega = -\omega \]

are treated, or the \(\psi-\omega\) method in which

\[\omega = u_x - v_y \]

\[\phi_x = -v, \quad \phi_y = u \]

The \(p-v\) method\(^{[8]}(2)\) is applicable even to the problems with free surface and of three-dimensional flow. However in the method it is necessary to take care of the condition of continuity, whereas in the \(\psi-\omega\) method...

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this condition is satisfied automatically.

The initial value problems on the Helmholtz equation (1) can be solved by some of the following explicit and implicit methods. The explicit method contains Fromm's method(2) using the second-order DuFort-Frankel scheme, and the convective difference methods using the first-order upstream difference(13), the second-order five-point and nine-point formulas(5,14), the fourth-order formulas(15), etc. The higher-order convective-difference methods are usually employed for the advection without viscous term. In the above methods it is important to attain the good stability of solutions, and to select the time increment  at such that the Courant stability condition can be satisfied.

\[ N^C = 4v \delta t / \lambda < 1 \] \hspace{1cm} (5)

\[ |X| + |Y| = (|u| + |v|) \delta t / h < 1 \] \hspace{1cm} (6)

The implicit method contains the Crank-Nicholson difference scheme(16), the second-order backward time-difference scheme(17), the higher-order convective-difference scheme(18) and Pearson's ADI scheme(19). And the time increment  can be taken larger than in the explicit method, although the computer time per time step increases.

In the method proposed here is used the existing convective-difference scheme using interpolation formulas and the same iterative calculation as in the method of characteristics(10,20) is applied in order to increase the accuracy.

3. Numerical method

3.1 Calculation of vorticity \( \omega \)

Now we consider a fluid element lying on a grid point \((x_i, y_i)\) at time  \( \eta \). Denoting its position at time  \( \eta \) by \((x_o, y_o)\) (Fig 1),

\[ \frac{1}{h}(x_o-x_i) = \frac{1}{h} \int_{t_a}^{t_1} u(x, y, t) \, dt, \]

\[ \frac{1}{h}(y_o-y_i) = \frac{1}{h} \int_{t_a}^{t_1} v(x, y, t) \, dt, \]

where the path of integration is taken along the fluid path line. Integrating Eq. (1) along the path line,

\[ \omega_i = \omega_o + \nu \int_{t_a}^{t_1} \Delta \omega \, dt \]

where the values of  \( \omega_o \) are determined from the following Lagrangian interpolation polynomials.

\[ X^{(N)} = - \frac{1}{2} (\omega^{(N)} + \omega^{(N+1)}) \delta t / h \]

\[ Y^{(N)} = - \frac{1}{2} (\omega^{(N)} + \omega^{(N+1)}) \delta t / h \]

\[ \omega^{(N)} = \omega^{(N)} + \frac{1}{2} \nu[(\Delta \omega)^{(N)} + (\Delta \omega)^{(N+1)}] \delta t \]

The computational procedure is that first  \( X^{(N)} \) and  \( Y^{(N)} \) are determined by interpolating  \( \omega^{(N)} \) and  \( \omega^{(N+1)} \) using  \( x^{(N)} \) and  \( y^{(N)} \), and then the  \( N \)-th approximation  \( \omega^{(N)} \) is determined by interpolating  \( \omega^{(N)} \) and  \( \omega^{(N+1)} \) using  \( x^{(N)} \) and  \( y^{(N)} \).

3.2 Boundary conditions

Let  \( \psi(y) \) be a smooth function for  \( y = 0 \), and put  \( \psi(0) = \psi_0, \psi(1) = \psi_1, \) and  \( \psi(2h) = \psi_2 \)
The quadratic form through these points is
\[ \mathcal{V}(y) = \frac{1}{2} \left( -3 \mathcal{V}_y + 4 \mathcal{V}_e - \mathcal{V}_z \right) \frac{y}{h} + \frac{1}{2} \left( \mathcal{V}_y - 2 \mathcal{V}_e + \mathcal{V}_z \right) \left( \frac{y}{h} \right)^2 \]
Integrating with respect to \( y \)
\[ \int_0^h \mathcal{V}(y) \, dy = \frac{h^2}{12} \left( 5 \mathcal{V}_y + 8 \mathcal{V}_e - \mathcal{V}_z \right) \]
If a rigid wall parallel to \( x \)-axis moves in its own plane with speed \( u_b \), then the relation
\[ u_b - u_i = \int_0^h \omega \, dy \]
holds in general due to \( |v_t| \ll |u_y| \). Therefore using Eq. (16), at the lower boundary points
\[ \omega_e = \frac{1}{5} \left( -3 \omega_i + \omega_r + 12 (u_b - u_i)/h \right) \]
which is the wall boundary condition for no-slip flow.

3.3 Computational procedure
The Crout method is used here to solve the usual difference equations for stream function \( \psi \)
\[ \psi_{i,j+1}^{(k^2)} = -\omega_{i,j} + O(h^2) \] \hspace{1cm} (18)
The velocity components are evaluated from
\[ u_{i,j} = (\psi_{i,j+1}^{(k^2)} - \psi_{i,j-1}^{(k^2)})/2h + O(h^2) \] \hspace{1cm} (19)
\[ v_{i,j} = (\psi_{i+1,j}^{(k^2)} - \psi_{i-1,j}^{(k^2)})/2h + O(h^2) \]
and the values of \( \Delta \omega \) from
\[ \Delta \omega_{i,j} = \omega_{i,j+1} - \omega_{i,j-1} - (\omega_{i+1,j} - \omega_{i-1,j})/\delta t \]
\[ + u_{i,j} \left( \omega_{i,j+1} - \omega_{i,j-1} - (\omega_{i+1,j} - \omega_{i-1,j})/2h \right) \] \hspace{1cm} (20)

4.1 Errors in the first approximation \( \omega_i^{(1)} \)
The error \( e^{(1)} \) in \( \omega_i^{(1)} \) is defined as
\[ e^{(1)} = \omega_i^{(1)} - \omega_i \]
The truncation error caused by initial errors and \( T \) the truncation number of difference equation. Round-off error is neglected as sufficiently small. By the calculation using Taylor expansion (see Appendix), we get

4.2 Errors in the \( n \)-th approximation \( \omega_i^{(N)} \) \((N \geq 2)\)
We define the error as
\[ e^{(N)} = \omega_i^{(N)} - \omega_i \] \hspace{1cm} (24)
Calculating the truncation error in Eq. (15) by Taylor expansions (see Appendix), we get
\[ T_i^{(N)} = \frac{h^4}{31} \left( X^{(N+1)}(1 - X^{(N+2)}) \omega_{i+1}^{(N+1)} + Y^{(N+1)}(1 - Y^{(N+2)}) \omega_{i+2}^{(N+1)} \right) \]
\[ + \frac{\delta t}{12} \left( - \frac{d}{dt} \omega_i + v \cdot \frac{d}{dt} \nabla \omega \right) \] \hspace{1cm} (25)
Each term on the right-hand side is interpreted in the same way as in Eq. (22).
From Eqs. (10), (13) and (15) we obtain
\[ \omega_{i+1}^{(N)} = \frac{1}{8} \left( \sum_{r_i,j} c_{r_i,j} \omega_{r_i,j}^{(N)} + \sum_{r_i,j} c_{r_i,j} \omega_{r_i,j}^{(N)} \right) - T_i^{(N)} \] \hspace{1cm} (26)
\[ \omega_{i+2}^{(N)} = \frac{1}{8} \left( \sum_{r_i,j} c_{r_i,j} \omega_{r_i,j}^{(N)} + \sum_{r_i,j} c_{r_i,j} \omega_{r_i,j}^{(N)} \right) + \omega_{i+1}^{(N)} \] \hspace{1cm} (27)
Considering \( e^{(N-1)} = e^{(N)} \), from Eqs. (24), (26) and (27)
\[ e_i^{(N)} = \frac{2}{2 + N} \left[ \sum_{j \neq i} c_{r_{i,j}} e_i^{(N)} + \frac{N}{8} \left( e_i + e_i + e_i^{(N)} + e_i + e_i^{(N)} \right) + e_i + e_i^{(N)} + e_i^{(N)} + e_i + e_i^{(N)} + e_i + e_i^{(N)} \right] \]

Similar relations are derived for \( e_i^{(N)}, \ldots \). Substituting these relations into Eq. (28), and neglecting the smaller terms, we get

\[ e_i^{(N)} = \sum_{j \neq i} c_{r_{i,j}} e_i^{(N)} \left[ \frac{2 - N}{2 + N} e_i^{(N)} + \frac{N}{2 + N} \right] \left( e_i^{(N)} + e_i^{(N)} \right) + \frac{2}{2 + N} \left( 1 + \frac{N}{2 + N} \right) T_i^{(N)} \]

provided that \( c_i c_i c_i c_i = e_i e_i e_i e_i \), and \( e_i^{(N)} = e_i e_i \), \( T_i^{(N)} = T_i \), on account of the insignificant change of variables among the neighboring grid points. The first term on the righthand side in Eq. (29) becomes the inherited error, and the second term the truncation.

4.3 Errors in \( \omega \) of the existing method

We take Fromm's method as a typical existing explicit method, and consider the corresponding difference form for vorticity

\[ \omega_i^{(N)} = \frac{1}{1 + N} \left( \omega_i^{(N)} + X(\omega_{i+1}^{(N)} - \omega_{i-1}^{(N)}) + Y(\omega_{i+1}^{(N)} - \omega_{i-1}^{(N)}) \right) \]

Errors involved in this form are

\[ T_i^{(N)} = \frac{2h^4}{3!} (X\omega_{xx} + Y\omega_{yy}) + N^4 \left( \omega_{xxx} + \omega_{yyy} \right) - \frac{2}{3} \delta t \omega_{xx} + N^4 \delta t^2 \omega_{xx} + \ldots \]

\[ L_i^{(N)} = \frac{1}{1 + N} \left( 1 - N \right) \frac{N}{2} + \frac{1}{2} + X \left( \frac{N}{2} + X \right) \left( \omega_{i+1}^{(N)} + \omega_{i-1}^{(N)} \right) \frac{N}{2} - Y \right) e_{i,i}^{(N)} \]

4.4 Comparison of truncation errors

The truncation errors obtained, i.e., Eqs. (22), (29) and (31), indicate that they consist of three terms of

\[ \frac{1}{3!} h^4 \omega_{XX}, \frac{1}{4!} h^4 \omega_{YY}, \delta t \]

So we can examine the accuracy of the methods by means of the order of estimation of each term. Truncation errors principally depend on \( N, X, Y \). In the following we examine them within the region of \( N, |X|, |Y| \leq 1 \) where stable solution is ensured. The first terms have the coefficient of \( h^4 \) \( \omega_{XX}/3! \) as shown in Fig. 2. The first and \( N \)-th approximations of the present method using interpolation formulas take smaller values than Fromm's method using central-differences, especially when \( |X| \) is close to unity. The second terms are the order of \( O(h^2) \) and smaller than the first terms whose order is \( O(h^4) \) in general. Values of the coefficient of \( h^4 (\omega_{XX} + \omega_{YY})/4! \) are shown in Fig. 3. The last terms are \( O(\delta t^2) \) for both the first approximation and Fromm's method, and \( O(\delta t^3) \) for the \( N \)-th approximation. This fact and the inner derivative on the path-line along which the vorticities are transported and substantially preserved, indicate that the last term of the \( N \)-th approximation is very small. Furthermore the term of the steady state is equal to zero in Fromm's method, which is not the case in the first and \( N \)-th approximations. From the above considerations it can be expected that the first and \( N \)-th approximations are more accurate than Fromm's method for relatively large values of \( \delta t \), and that the second approximation becomes more advantageous in case of the flow seriously depending on time.

![Fig. 2 Values of the coefficient of (1/3!)h^4 \omega_{XX}](image2)

![Fig. 3 Values of the coefficient of (1/4!)h^4 (\omega_{XX} + \omega_{YY})](image3)
4.5 Stability of solution

The inherited errors in the first and $N$-th approximations of vorticity $\omega$ are given in Eqs. (22) and (29), respectively, and have the contour maps of the coefficient of $e_{0}^{n}$ and $e_{\omega}^{n}$ shown in Figs. 4 and 5. The error $\epsilon_{n}$ at time $t^{n+1}$ depends on its initial distribution at time $t^{n}$ and on the values of the coefficients. The numerical stability concerned with the error has been verified in general by assuming for the errors the plane sinusoidal waves and performing a Fourier analysis. It is known that the case of sufficiently large wave-length (i.e., uniform error distribution), where Eq. (12) for $N = 0$ is neutrally stable for all the values of $x$ and $y$, is the most severe condition. But in the case of irregular error distribution, local instabilities often appear in the above case. It is easy to verify in the case of uniform error the numerical stability of Eq. (12) for finite $N$ and of Eq. (15) for all values of $x$ and $y$, but difficult to estimate the stability in an arbitrary error distribution. Here stability criteria are deduced from the fundamental idea that the stable solution is obtained when the initial errors on each grid point propagate without amplifying, i.e., in the region where the coefficients do not exceed unity (see Figs. 4 and 5).

$$|X|, |Y| \leq 1 - N/4 \quad (33-a)$$

in the first approximation with four-point formula,

$$|X|, |Y| \leq \frac{1}{2}(\sqrt{9-2N}-1) \quad (33-b)$$

in the first approximation with nine-point formula,

$$|X|, |Y| \leq 1 \quad (33-c)$$

in the $N$-th approximation. Fig. 6 shows the relation among the maximum value $\left(\frac{\partial t}{h}\right)_{max}$, $h/v$, and max $|\omega|$, $|\omega|$, which satisfy (5) and (33) simultaneously. Choosing $k$ such that $\Delta t < \left(\frac{\partial t}{h}\right)_{max}$, we can expect the numerically stable solutions.

5. Numerical examples

Two numerical examples are solved here by the first and the second approximations of the present method and by Fromm's method to demonstrate their accuracy and computer time. A set of linear equations for $\psi$ are computed by the Crout method to exclude the errors of iteration. For the boundary conditions in the calculations two existing formulas (34-a),

$$\omega_{B} = \frac{1}{h^{2}}(\phi_{B} - \phi_{B} + u_{u} b) \quad (34-a)$$

and Eq. (17) are used.

- First approximation with nine-point formula,
- First approximation with four-point formula,
- $N$-th approximation.
Computations were run on NEAC 2200 at the Computer Center of the Tohoku University.

5.1 Recirculating flow in a square cavity

At first we consider a starting flow problem in the square cavity (whose size $L=1$) for which the fluid is initially at rest for $t \leq 0$ and begins to circulate together with the upper side wall motion (whose speed $U=1$) for $t > 0$ (see Fig.7). Since the treatment of the singularities at two points in the upper corners is a difficult pending problem, it seems satisfactory to put $\omega = 2/h$ at these point without paying special attention according to the previous works\(^{(18)}\). Now we set $Re = UL/\nu = 100$, $h = 1/8$, $1/12$, $1/18$, and $\delta t = 0.1$, $0.05$. The calculated streamlines at $t = 1$ (upper wall has just moved for distance $L$) and $t = 20$ (the recirculating flow has approached almost steady state) are shown in Fig.7. It is found that the accuracy of the solutions increases for small values of $h$, but is not affected very much by $\delta t$, and that the accuracy of the first approximation and Fromm's method is equivalent, and the second approximation is better than them, i.e. the second approximation with $h = 1/8$ is comparable to the first approximation with $h = 1/18$. The time variations of $\psi$ at $x = 0.625$ and $y = 0.75$, where $\psi$ is approximately maximum, are shown in Fig.8. This shows that the second approximation with $h = 1/18$ at large $t$ approaches the results of steady solutions by Burggraf with $h = 1/40^{(27)}$. Stable solutions are obtained when the stability criteria derived in Sec. 4.5 are satisfied. The computer time required per time step is summarized in Table 1. The time for each method increases substantially in proportion to $1/h^2$. Thus the computer time for the second approximation can be reduced remarkably as compared with the other method. The above results by the first and second approximations were determined using the four-point formula (9) and they almost coincide with those of nine-point formula. The third approximation gives about the same results as the second approximation. The influence of the boundary formulas on the problem is small.

Fig.7 Streamlines in a square cavity

--- First approximation
--- Second approximation
--- Fromm's method

Fig.8 Time-variations of $\psi$ at $x = 0.625$ and $y = 0.75$

Table 1 Comparison of computer time required per time step (milli sec)

<table>
<thead>
<tr>
<th></th>
<th>$h$ (Number of interior grid point)</th>
<th>First approximation</th>
<th>Second approximation</th>
<th>Fromm's method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square cavity</td>
<td>$1/8$ ( 49)</td>
<td>68</td>
<td>147</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>$1/12$ (121)</td>
<td>224</td>
<td>460</td>
<td>230</td>
</tr>
<tr>
<td></td>
<td>$1/18$ (289)</td>
<td>720</td>
<td>1510</td>
<td>736</td>
</tr>
<tr>
<td>Grooved channel</td>
<td>$1/4$ ( 72)</td>
<td>97</td>
<td>194</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td>$1/8$ (320)</td>
<td>671</td>
<td>1364</td>
<td>671</td>
</tr>
</tbody>
</table>
5.2 Flow through rectangular-grooved channel

We next consider an unsteady channel flow problem. Let rectangular grooves be cut on only one side wall periodically as shown in Fig.9, and let the fluid be at rest for $t < 0$, then flow with mean velocity $u = 1.0$ for $t \geq 0$. Since the problem also has the singularities at the edge of grooves, it shall be treated according to the previous works\(^{(34)}\), that is, the value of $\omega_y$ at the front edge $A$ is determined along only the $y$-direction, and the value of $\omega_y$ at the rear edge $B$ takes the average value between the $x$- and $y$-directions. The calculated streamlines at time $t = 3, 15$ in case of $Re = u_0/b = 100$, $h = 1/4(\delta t = 0.15)$ and $h = 1/8 (\delta t = 0.075)$ are shown in Fig.9, and the time-variations of the pressure coefficient $\lambda$ in Fig.10, where $\lambda = (b/\kappa)\delta p/(1/2)u^2$, $\delta p$ denoting the pressure drop per channel length $\kappa$. The four-point formula and the boundary formula (17) are used in the first and second approximations, while the boundary formulas (17) and (34a) are used in Fromm's method. The second approximation gives the most accurate solutions, the first approximation and Fromm's method giving less accurate ones in this order. It is shown from Figs.9 and 10 that the flow patterns of the first approximation are similar to the second approximation, and that the boundary formula (34a) gives different results from the other boundary formulas. In addition, for both the solutions using nine-point formula and Fromm's method instabilities appear near the rear corner and especially in Fromm's method an oscillation occurs over the region.

6. Conclusions

A numerical solution for time-dependent two-dimensional flow of incompressible viscous fluid has been proposed in this paper.

(1) In the present method, the Helmholtz vorticity equation is solved explicitly by the convective-difference scheme using Lagrangian interpolation polynomials and the same iteration as in the method of characteristics.
(2) The truncation errors have been obtained to compare the accuracy of the present and the existing methods, and the inherited errors to deduce stability criteria.

(3) Two numerical examples have been solved to demonstrate the accuracy of the solution and the computer time required. In the case of a square cavity flow problem using the same grid, the second approximation is considerably more accurate than the first approximation and Fromm's method. In the case of a grooved channel flow problem, the accuracy of the solutions become better in the following order: Fromm's method, the first approximation and the second approximation. Namely, the computer time for the second approximation can be reduced remarkably as compared with the existing method. The stable solutions are obtained when the stability criteria are satisfied.

**Acknowledgement**

In conclusion, the authors wish heartily to express their appreciation to Professor S. Fuchizawa for his guidance.

**Appendix**

[Derivation of $T^{(n)}_i$] Expanding $f$ into Taylor series around the point $(x_i, y_j, t^n)$,

$$
\int_{t^n}^{t^{n+1}} f(x, y, t) dt = \int_{t^n}^{t^{n+1}} \left[ f + (t-t^n)f_{t^n} + \frac{1}{2!} (t-t^n)^2 f_{t^n}^2 + \frac{1}{3!} (t-t^n)^3 f_{t^n}^3 + \ldots \right] dt
$$

where $f, f_t, f_{tt}, \ldots$ are abbreviated to $f, f_t, \ldots$, respectively. Substituting into the above expression

$$\mathbf{r} \equiv -\frac{1}{2!} \mathbf{v} \cdot (t-t^n)\mathbf{v} - \frac{1}{2} \left[ \frac{\partial t^2 - (t-t^n)^2}{\partial t} \right] \mathbf{v}_i + \frac{1}{2} \left[ (t-t^n)^2 \mathbf{v} \cdot \nabla \mathbf{v} + \ldots \right]
$$

and integrating it gives

$$
\int_{t^n}^{t^{n+1}} f(x, y, t) dt = \delta t \left[ f + \frac{1}{2!} \mathbf{v} \cdot \nabla f + \frac{1}{3!} \nabla^2 f \right] - \mathbf{v}_i \cdot \nabla f + \mathbf{v} \cdot \nabla f_i + \ldots 
$$

Rewriting (7) by using (35), and subtracting it from (11),

$$X = X^{(1)} \frac{\partial t}{\partial t} \mathbf{u} + \ldots, \quad Y = Y^{(1)} \frac{\partial t}{\partial t} \mathbf{u} + \ldots 
$$

Substituting (36) into the Taylor expansion of $f$,

$$f_{t^n+1} = f + h(X^{(1)} f_{x} + Y^{(1)} f_{y}) + \frac{1}{2!} h^2 (X^{(1)} f_{x} + Y^{(1)} f_{y}) + \frac{1}{3!} h^3 (X^{(1)} f_{x} + Y^{(1)} f_{y})
$$

and substituting the Taylor expansions of $f_{t^n+1}, f_{t^n+1}, \ldots$ into nine-point formula (10),

$$f_{t^n+1} = f + h(X^{(1)} f_{x} + Y^{(1)} f_{y}) + \frac{1}{2!} h^2 (X^{(1)} f_{x} + Y^{(1)} f_{y}) + \frac{1}{3!} h^3 (X^{(1)} f_{x} + Y^{(1)} f_{y})
$$

Taking the difference between (37) and (38), we get

$$f_{t^n} = f_{t^n} - \frac{1}{3!} h^4 (X^{(1)} f_{x} + Y^{(1)} f_{y}) - \frac{1}{2!} h^2 (X^{(1)} f_{x} + Y^{(1)} f_{y}) + \ldots
$$

On the other hand, since the relation

$$\triangle \omega = \frac{1}{h^2} \omega_{x+1,y} + \frac{2h^2}{4!} (\omega_{x+2,y} + \omega_{x+3,y}) + \ldots
$$

holds, using (35) we get

$$\int_{t^n}^{t^{n+1}} \triangle \omega(x, y, t) dt = \frac{\partial t}{\partial t} \omega_{x+1,y} + \frac{2h^2}{4!} (\omega_{x+2,y} + \omega_{x+3,y}) + \ldots
$$

Substituting (39) and (41) into (8), and using (12),

$$T^{(1)}_i = \omega_{1,i} - \omega_i = \frac{1}{3!} h^2 X^{(1)}(1-X^{(1)}) \omega_{xx} + Y^{(1)}(1-Y^{(1)}) \omega_{yy} + \frac{1}{2!} h^2 (\omega_{xx} + \omega_{yy}) + \ldots
$$

Substituting into the expression the relation

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \omega + \frac{\partial u}{\partial t} \omega_x + \frac{\partial v}{\partial t} \omega_y = \nu \frac{\partial^2 \omega}{\partial t^2}
$$

derived from (1), yields Eq. (22).

[Derivation of $T^{(n)}_i$] Substituting into (35) the relation

$$\frac{1}{2} \left( f_{t^n} + f_{t^n+1} \right) = f + \frac{1}{2!} \mathbf{v} \cdot \nabla f + \frac{1}{3!} \nabla^2 f (f_{t^n} + 2\mathbf{v} \cdot \nabla f_{t^n} + \ldots
$$

derived by Taylor expansion gives

$$\int_{t^n}^{t^{n+1}} f(x, y, t) dt = \delta t \left[ \frac{1}{2} \left( f_{t^n} + f_{t^n+1} \right) - \frac{1}{12} \mathbf{v} \cdot \nabla f_{t^n} + \ldots \right]
$$

143
Using (42), in the same way as in the first approximation,

$$X=X^{(N)} + \frac{\partial^3}{12h} \frac{d^3 u}{dt^3} + \cdots, \quad Y=Y^{(N)} + \frac{\partial^3}{12h} \frac{d^3 v}{dt^3} + \cdots$$

(43)

Using (43), and calculating in the same way,

$$f_s = f_s^{(N)} - \frac{1}{3!} h^3 X^{(N)} (1 - X^{(N)^2}) f_{ss} + Y^{(N)} (1 - Y^{(N)^2}) f_{sv} + \frac{1}{12} \partial^3 \frac{d^3 u}{dt^3} f_s + \frac{1}{12} \partial^3 \frac{d^3 v}{dt^3} f_s + \cdots$$

(44)

Substituting (40) into (42), we get

$$\int_{t_0}^{t_1} \Delta \omega(x, y, t) dt = \frac{\partial t}{2} \frac{h^2}{3!} \left[ X^{(N)} (1 - X^{(N)^2}) \Delta \omega_{ss} + Y^{(N)} (1 - Y^{(N)^2}) \Delta \omega_{sv} \right] + \frac{\partial t}{2} \frac{h^2}{12} \left[ \frac{d^3 u}{dt^3} \Delta \omega_s + \frac{d^3 v}{dt^3} \Delta \omega_v + \cdots \right]$$

(45)

Substituting (44) and (45) into (8), and using (15),

$$T^{(N)} = \omega^{(N)} - \omega_1 = \frac{h^3}{3!} \left[ X^{(N)} (1 - X^{(N)^2}) \left( \omega_{ss} + \frac{N h^2}{8} \Delta \omega_{ss} \right) \right. + Y^{(N)} (1 - Y^{(N)^2}) \left( \omega_{sv} + \frac{N h^2}{8} \Delta \omega_{sv} \right)] - \frac{\partial^3}{12} \frac{d^3 u}{dt^3} \left( \omega_s + \frac{N h^2}{8} \Delta \omega_s \right) + \left. \frac{\partial^3}{12} \frac{d^3 v}{dt^3} \left( \omega_v + \frac{N h^2}{8} \Delta \omega_v \right) \right] + \frac{2 h^2}{4!} \left[ \frac{d^3 u}{dt^3} \omega_s + \frac{d^3 v}{dt^3} \omega_v \right] + \frac{\nu}{12} \frac{d^3}{dt^3} \left( \frac{d^2 u}{dt^2} \omega_s + \frac{d^2 v}{dt^2} \omega_v \right)$$

The term of $N$ is to be neglected since $0 < N < 1$. Substituting into the expression the relation

$$\frac{d^2}{dt^2} \omega_s + \nu \frac{\partial}{\partial t} \nabla \omega_s + \frac{d^3 u}{dt^3} \omega_s + \frac{d^3 v}{dt^3} \omega_v = \nu \frac{d^3}{dt^3} \Delta \omega$$

derived from (1), yields Eq. (25).

References

(11) Chan, R.K.-C., et al., Proc. 2nd Int. Conf. in Numerical Methods in Fluid Dy-