532.543:521.65

A Study on a Disk Friction Pump

(1st report, Theoretical Analysis for Flow between Corotating Disks)*

By Susumu MURATA**, Yutaka MIYAKE*** and Yoshiyuki IEMOTO****

In order to obtain fundamental knowledge of performance and optimum dimensions of a disk friction pump, an analysis is made of the flow between two infinitely large corotating disks with throughflow supplied at the inner periphery of finite radius. The flow field is assumed to be either everywhere laminar or everywhere turbulent. The laminar case is treated by dividing the field into three regions; the region in the vicinity of the entrance periphery to which Görtler series is applied, the region far from the center axis for which analytical perturbation solution is used and the mid-region connecting the above two extreme regions for which finite difference computation is carried out. The solution gives a complete scheme of the flow field with satisfactory indications for the engineering purpose. Meanwhile, the turbulent case is treated by means of momentum equations with elaborated curvilinear coordinate axes, which leads to a physically acceptable flow pattern.

1. Introduction

The disk friction pump is a centrifugal one of extreme form. The rotor is composed of a row of disks which are assembled on a shaft with a constant small clearance. Hence, it belongs to a group of pumps of the smallest specific speed.

This type of pump was invented by Tesla⁽¹⁾ in the 1910's. However, it couldn't find wide application in those days. Since then almost 40 years passed without attracting any attention of engineers. In the 1950's its several advantages were noticed with development of industries, and a variety of applications have been tried as follows: pumps for pipelines transporting petroleum by virtue of higher efficiency than conventional ones for high viscous fluids⁽²⁾; fans for air-conditioners by virtue of their low noise performance which is attained because they have no separation of flow in the flow passage as the result of having no blade⁽³⁾; inducers in rocket⁽⁴⁾ and boiler feed pumps by virtue of high head⁽⁵⁾ and stable performance due to excellent anti-cavitation characteristic⁽⁶⁾. The extension of its application stimulated the investigations of the pump. Vannérus⁽⁷⁾ initiated the study of the performance which is a one-dimensional theory on the basis of Euler's momentum theory for centrifugal impeller. Including Vannérus' work, many one-dimensional theories were proposed^{(3),(8),(9)}. Hasinger⁽⁴⁾ and

Nendl⁽²⁾ improved the theory for performance of the pump assuming that the flow is laminar and that the velocity distributions of fluid in the peripheral and radial directions are both parabolic.

In the present paper is treated the flow between two corotating disks which make an elementary flow passage of the disk friction pump rotor, in order to obtain fundamental knowledge of performance and optimum dimensions of the pump.

The analysis is made of a flow between two infinitely large corotating disks with the throughflow supplied at the inner periphery of finite radius. The flow is assumed to be steady, incompressible, symmetric with respect to both the center axis and the midplane between two disks, and either everywhere laminar or everywhere turbulent. At the inlet periphery, the fluid is assumed to flow radially with uniform velocity, and the boundary-layers develop from the inner periphery on the disk surfaces.

As the laminar flow between two corotating disks is a fundamental one, many studies have been carried out. In 1952 one of the present authors, Murata⁽¹⁰⁾ first presented a regular perturbation solution whose first-order term represents the flow sufficiently far from the center axis. The first-order term is solved exactly, and the others are composed of the terms expanded in a series of Reynolds numbers. Later, Breiter⁽¹¹⁾ obtained the first-order term from linearized equations, and Peube⁽¹²⁾ gave an exact analytical form of the second-order term of Murata's solution. Furthermore, Matsch⁽¹³⁾ and Мисюра⁽¹⁴⁾ independently confirmed Murata's solution; however, their series solutions become less convergent as the rotational speed of disks becomes faster and/or the disk clearance becomes wider. In the small radial position, all the above-

* Received 12th August, 1974.

** Professor,
Faculty of Engineering, Osaka Univ.*** Associate Professor,
Faculty of Engineering, Osaka Univ.**** Assistant, Faculty of Engineering,
Fukui Univ., Fukui.

mentioned series solutions are not convergent. Hence, the boundary-layer equation which is valid as the governing equation of the flow now in consideration must be numerically solved in the region where these solutions are not valid. Breiter⁽¹¹⁾ gave a solution by means of finite difference method for that region. Boyd⁽¹⁵⁾ similarly employed the finite difference method and changed the governing equations to linear simultaneous ones; however, Boyd's solution is less accurate because it doesn't satisfy the equations of motion on the midplane between two disks. Byal'yi⁽¹⁶⁾ used a relaxation method on the basis of finite difference equations; however, this method is not economical and is not always convergent. Boyack⁽¹⁷⁾ estimated the performance of the pump on the basis of integral equations assuming a polynomial velocity distribution; however, the accuracy of this method is extremely low.

In the present paper the laminar flow is analyzed by dividing the flow field into three regions: for the region in the vicinity of the entrance periphery which is mathematically a singular point, Görtler series⁽¹⁸⁾ employed by Ishizawa⁽¹⁹⁾ to analyze the flow between two co-axial stationary disks is applied; for the region sufficiently far from the center axis, analytical perturbation solution which is available as mentioned above is utilized; and for the mid-region, which connects with the above two extreme regions, where the above two solutions don't converge rapidly, an implicit finite difference method is applied, in which the equation of continuity is used in the midplane between two disks on the contrary to Breiter who used the continuity equation on the disk walls. The reason for this modification is that in an explicit finite difference method, inevitable numerical errors are concentrated in one point, and instability in the computation occurs, though economical. The ensemble of the laminar solutions for three regions yields a complete scheme of the flow field with satisfactory indications for engineering purposes.

In the meanwhile, an entirely turbulent flow has been analyzed with momentum integral equations with assumed velocity distributions of the flow in the azimuthal and in the radial direction, and with assumed friction stresses on the disk wall^{(20),(21)}. In the present paper, three-dimensional turbulent boundary-layer is calculated on the basis of the momentum integral equations which are applied both for the inlet region where a potential-core exists and for the fully developed region where a potential-core disappears. A lengthy calculation is carried out using elaborately constructed orthogonal curvilinear coordinates in which one coordinate axis coincides with a streamline on the midplane between two disks. The solution leads to a physically acceptable flow pattern since the assumed velocity distribution is appropriate.

2. Theoretical analysis of laminar flow

2.1 Flow in the vicinity of the entrance periphery

The analysis of the boundary-layers growing from the entrance periphery along the disks will be performed in terms of Görtler series. In the cylindrical polar coordinate (r, θ, z) , where the z -axis coincides with the center axis and $z=0$ represents one disk wall, the equations of motion and the equation of continuity to which the boundary-layer approximation is applied are written:

$$\left. \begin{aligned} u(\partial u/\partial r) + w(\partial u/\partial z) - v^2/r \\ = -(1/\rho)(dp/dr) + \nu(\partial^2 u/\partial z^2) \\ u(\partial v/\partial r) + w(\partial v/\partial z) + uv/r \\ = \nu(\partial^2 v/\partial z^2) \end{aligned} \right\} \dots\dots\dots(1)$$

$$\partial(ur)/\partial r + \partial(wr)/\partial z = 0 \dots\dots\dots(2)$$

where u, v, w are, respectively, the velocity components in the radial, the tangential and the axial direction, and p is the static pressure, ρ the density of fluids, ν the kinetic viscosity of fluids. The alternative form of Eq.(2) is written:

$$Q = 4\pi r \int_0^{h/2} u dz \dots\dots\dots(3)$$

where Q is the flow rate through the disk clearance h . The boundary conditions are

$$r=r_i : u=U_i, \quad v=w=0 \dots\dots\dots(4)$$

$$r>r_i \left\{ \begin{aligned} z=0 : u=w=0, \quad v=r\omega \\ z=h/2 : u=U_p(r), \quad v=w=0 \end{aligned} \right\} \dots\dots\dots(5)$$

where r_i is the radius of entrance periphery of disks, ω the angular velocity, and U_p the flow velocity in the potential-core.

The dimensionless coordinates are introduced in place of the dimensional ones as follows:

$$\left. \begin{aligned} \zeta = \frac{16\pi^2\nu}{Q^2} \int_{r_i}^r U_p r^2 dr = \int_0^\xi U_* r_*^2 d\xi \\ \eta = \frac{U_p r z}{(2\nu \int_{r_i}^r U_p r^2 dr)^{1/2}} = \frac{4\pi}{Q} \frac{U_p r z}{(2\zeta)^{1/2}} \end{aligned} \right\} \dots\dots\dots(6)$$

where

$$\left. \begin{aligned} U_* = U_p/U_i, \quad r_* = r/r_i \\ \xi = (1/R_*) (r - r_i)/r_i \\ R_* = U_i h^2 / (4r_i \nu), \quad R_* = h^2 \omega / (4\nu) \end{aligned} \right\} \dots\dots\dots(7)$$

The stream function ψ defined as

$$u = (1/r)(\partial\psi/\partial z), \quad w = -(1/r)(\partial\psi/\partial r) \dots\dots\dots(8)$$

satisfies the continuity equation (2). Now is introduced the dimensionless stream function $F(\zeta, \eta)$ defined as follows:

$$\psi(r, z) = (Q/4\pi)(2\zeta)^{1/2} F(\zeta, \eta) \dots\dots\dots(9)$$

Then, the velocity components u and w become

$$\left. \begin{aligned} u &= U_p F_\eta \\ w &= -\frac{4\pi\nu}{Q} \frac{U_p r}{(2\zeta)^{1/2}} \left[F + 2\zeta F_\zeta \right. \\ &\quad \left. + \left\{ 2\zeta \frac{(U_p r)}{U_p} - 1 \right\} \eta F_\eta \right] \end{aligned} \right\} \dots\dots\dots (10)$$

where the subscripts ζ and η denote differentiation with respect to respective variables. As the counterpart, the tangential velocity v is represented by the dimensionless function $G(\zeta, \eta)$:

$$v = r\omega G(\zeta, \eta) \dots\dots\dots (11)$$

Substitution of Eqs.(10) and (11) into the equations of motion (1) yields

$$\left. \begin{aligned} U_* r_* &= \sum_{k=0}^{\infty} A_{k/2} \zeta^{k/2}, \quad F(\zeta, \eta) = \sum_{k=0}^{\infty} F_{k/2}(\eta) \zeta^{k/2}, \quad G(\zeta, \eta) = \sum_{k=0}^{\infty} G_{k/2}(\eta) \zeta^{k/2} \\ U_*(\zeta) &= \sum_{k=0}^{\infty} K_{k/2} \zeta^{k/2}, \quad r_*(\zeta) = \sum_{k=0}^{\infty} L_{k/2} \zeta^{k/2}, \quad \xi = \zeta \sum_{k=0}^{\infty} B_{k/2} \zeta^{k/2} \\ \zeta &= \xi \sum_{k=0}^{\infty} C_{k/2} \xi^{k/2}, \quad r_*(\xi) = \sum_{k=0}^{\infty} M_{k/2} \xi^{k/2}, \quad \lambda(\zeta) = \sum_{k=1}^{\infty} \gamma_{k/2} \zeta^{k/2} \end{aligned} \right\} \dots\dots\dots (15)$$

Substituting Eq.(15) into Eq.(12) and comparing terms in powers of $\zeta^{k/2}$, one obtains the following set of equations to which $F_{k/2}(\eta)$ and $G_{k/2}(\eta)$ are subjected:

$$\left. \begin{aligned} X_0 &= Y_0 = 0; \quad X_{1/2} = Y_{1/2} = 0; \\ X_{k/2} + \sum_{m=1}^k \sum_{l=0}^m \sum_{j=0}^l K_{l/2} K_{(j-l)/2} L_{(m-l)/2} X_{(k-m)/2} + 2(R_*^2/R_*) \sum_{j=0}^{k-2} G_{j/2} G_{(k-j-2)/2} &= 0 \\ Y_{k/2} + \sum_{m=1}^k \sum_{l=0}^m \sum_{j=0}^l K_{l/2} L_{(j-l)/2} L_{(m-l)/2} Y_{(k-m)/2} - 4R_* \sum_{j=0}^{k-2} F_{j/2} G_{(k-j-2)/2} &= 0 \quad (k=2, 3, 4, \dots) \end{aligned} \right\} \dots\dots\dots (16)$$

where

$$\left. \begin{aligned} X_0 &= F_0''' + F_0 F_0'', \quad Y_0 = G_0'' + F_0 G_0'; \\ X_{k/2} &= F_{k/2}''' + F_0 F_{k/2}'' - k F_0' F_{k/2}' + (k+1) F_0'' F_{k/2} - \gamma_{k/2} (F_0'^2 - 1) - \sum_{j=1}^{k-1} \sum_{i=0}^{k-j} \gamma_{j/2} F_{i/2}' F_{(k-i-j)/2}' \\ &\quad - F_0' \sum_{j=1}^{k-1} \gamma_{j/2} F_{(k-j)/2}' - \sum_{j=1}^{k-1} (k-j) F_{j/2}' F_{(k-j)/2}' + \sum_{j=1}^{k-1} (1+j) F_{j/2} F_{(k-j)/2}' \\ Y_{k/2} &= G_{k/2}'' + F_0 G_{k/2}' - k F_0' G_{k/2} - \sum_{j=1}^k (k-j) F_{j/2}' G_{(k-j)/2} + \sum_{j=1}^k (1+j) F_{j/2} G_{(k-j)/2}' \quad (k=1, 2, 3, \dots) \end{aligned} \right\} \dots\dots\dots (17)$$

where the primes denote differentiation with respect to η . The boundary conditions become

$$\left. \begin{aligned} F_0(0) &= F_0'(0) = 0, \quad F_0'(\infty) = 1 \\ G_0(0) &= 1, \quad G_0(\infty) = 0; \\ F_{k/2}(0) &= F_{k/2}'(0) = F_{k/2}'(\infty) = 0 \\ G_{k/2}(0) &= G_{k/2}(\infty) = 0 \quad (k=1, 2, 3, \dots) \end{aligned} \right\} \dots\dots\dots (18)$$

In the usual two-dimensional problems in which no cross flow component exists, $F_{k/2}$ are similar functions of η and can be calculated once and for all. However, in this problem $F_{k/2}$ and $G_{k/2}$ must be calculated specifying the combination of Ru and Rv . The set of ordinary differential equations (16) ~ (18) are computed by means of Runge-Kutta-Gill method, and the required velocity components are obtained using Eqs.(10) and (11).

2.2 Flow in the mid-region

In this region, the analysis can also be performed on the basis of boundary-layer equations. To calculate in dimensionless forms, the following quantities are defined:

$$\left. \begin{aligned} U_*^3 r_* \{ F_{\eta\eta} + F F_{\eta\eta} + \lambda(\zeta)(1 - F_*^2) \\ - 2\zeta(F_\eta F_{\eta\zeta} - F_{\eta\eta} F_\zeta) \} + 2(R_*^2/R_*) \zeta G^2 &= 0 \\ U_* r_*^3 \{ G_{\eta\eta} + (F + 2\zeta F_\zeta) G_\eta \\ - 2\zeta F_\eta G_\zeta \} - 4R_* \zeta F_\eta G &= 0 \end{aligned} \right\} \dots\dots\dots (12)$$

where $\lambda(\zeta)$ is a principal function:

$$\lambda(\zeta) = (2\zeta/U_*) (dU_*/d\zeta) \dots\dots\dots (13)$$

Similarly, the equation of continuity (3) becomes

$$U_* r_* = 1 + (2\zeta)^{1/2} \lim_{\eta \rightarrow \infty} [\eta - F(\zeta, \eta)] \dots\dots\dots (14)$$

To solve Eqs.(12) and (14), the quantities are expanded in a series of $\zeta^{1/2}$ as follows:

$$\left. \begin{aligned} \text{coordinates: } \alpha &= r/h, \quad \beta = z/h \\ \text{velocities: } u^* &= u/u_0, \quad v^* = v/v_0, \quad w^* = w/u_0 \\ \text{static pressure: } P &= p/\{(\rho/2)v_0^2\} \\ \text{Reynolds numbers: } R_1 &= u_0 h/\nu = 4(r_i/h)^2 R_* \\ R_2 &= v_0 h/\nu = 4R_* \end{aligned} \right\} \dots\dots\dots (19)$$

where $u_0 = Q/(2\pi h^2)$ and $v_0 = h\omega$. Then, the equations of motion become

$$\left. \begin{aligned} u^* (\partial u^* / \partial \alpha) + w^* (\partial u^* / \partial \beta) \\ - (R_2/R_1)^2 (v^{*2}/\alpha) &= -(1/2) \\ \times (R_2/R_1)^2 (dP/d\alpha) + (1/R_1) (\partial^2 u^* / \partial \beta^2) \\ u^* (\partial v^* / \partial \alpha) + w^* (\partial v^* / \partial \beta) \\ + u^* v^* / \alpha &= (1/R_1) (\partial^2 v^* / \partial \beta^2) \end{aligned} \right\} \dots\dots\dots (20)$$

and the equation of continuity is

$$\partial(u^* \alpha) / \partial \alpha + \partial(w^* \alpha) / \partial \beta = 0 \dots\dots\dots (21)$$

or

$$\alpha \int_0^{1/2} u^* d\beta = 1/2 \dots\dots\dots (22)$$

The boundary conditions are

$$\left. \begin{aligned} \beta = 0 : u^* &= w^* = 0, \quad v^* = \alpha \\ \beta = 1/2 : \partial u^* / \partial \beta &= \partial v^* / \partial \beta = w^* = 0 \end{aligned} \right\} \dots\dots\dots (23)$$

and from Eq.(21)

$$\beta = 1/2 : \partial^2 w^* / \partial \beta^2 = 0 \dots\dots\dots (24)$$

The flow can be computed continuously from one section to the next, in the downstream direction, since the differential equations (20) and (21) are parabolic. The mesh size for this finite difference scheme is not uniform in the whole field; in the β -direction the spacing is everywhere uniform, but in the α -direction the spacing becomes larger with an increased distance from the center axis. A grid point is designated by subscripts k and j , as in Fig.1. The flow at $\alpha = \alpha_{k-1}$ is given; the derivatives are replaced by the differential quotients, e.g.

$$\left. \begin{aligned} (\partial u^* / \partial \alpha)_{k,j} &= (u^*_{k,j} - u^*_{k-1,j}) / g_k \equiv \Delta u^*_{k,j} / g_k \\ (\partial u^* / \partial \beta)_{k,j} &= (u^*_{k,j+1} - u^*_{k,j-1}) / (2l) \\ (\partial^2 u^* / \partial \beta^2)_{k,j} &= (u^*_{k,j+1} + u^*_{k,j-1} - 2u^*_{k,j}) / l^2 \end{aligned} \right\} \dots (25)$$

A grid point k, j is represented by

$$\alpha_k = \alpha_{k-1} + g_k, \quad \beta_j = (j-1)l \quad (j=1, 2, 3, \dots, j_c) \dots (26)$$

When g_k are sufficiently small, all terms of higher order in Δ 's can be ignored in the equations. Equations (20) and (21) are then reduced to linear difference equations of $\Delta u^*_{k,j}$, $\Delta v^*_{k,j}$, $\Delta w^*_{k,j}$ and ΔP . The linear simultaneous equations are solved by elimination method and one obtains the required quantities. This numerical calculation is iterated until the values obtained by this method connect smoothly with the analytical perturbation solution from the infinite radial position. In the present paper, the length l is chosen as one twentieth of the distance from wall to midplane of the disks. Equation (22) provides checking of the numerical results. The dynamic pressure is determined by the average velocity excluding the axial velocity since the latter is sufficiently small at every radial position.

2.3 Flow in the region far from the center axis

The flow in this region can be calculated using the existing solution in the form of analytical perturbation series. This solution may be found in the reference (12).

2.4 Calculated results

The parameters which determine the flow are Reynolds number R_1 representing the flow rate through the disk clearance, Reynolds number R_2 representing the rotational speed of disks, and dimensionless entrance radius α_2 which is equal to r_2/h . The results are described in the following dimensionless quantities:

Fig.1 Grid used in the finite difference computation

$$\left. \begin{aligned} \text{radial velocity: } U &= u\alpha/u_0 \\ \text{tangential velocity: } V &= v/(v_0\alpha) \\ \text{axial velocity: } W &= w\alpha/u_0 \\ \text{static pressure: } P &= p/(\rho v_0^2/2) \\ \text{dynamic pressure: } P_d &= p_d/(\rho v_0^2/2) \end{aligned} \right\} \dots (27)$$

The static and the dynamic pressures are represented as zero at the entrance periphery. Figures 2 and 3 show the velocity distributions at some selected radii for $R_1=13500$, $R_2=17.5$, $\alpha_2=35$ and for $R_1=1640$, $R_2=17.3$, $\alpha_2=35$, respectively. Figures 4 and 5 show the static and the dynamic pressure distributions for these cases, respectively. The broken lines in the figures indicate the values of turbulent case, whose details will be mentioned in the following chapter. The

Fig.2 Widthwise distribution of velocities of laminar flow

Fig.3 Widthwise distribution of velocities of laminar flow

Görtler series solution is applied from α_2 to $\alpha=35.10$ for $R_1=13500$ and to $\alpha=35.01$ for $R_1=1640$. This series converges from inlet periphery to larger radial position as the

Fig.4 Static pressure variation in the radial direction

Fig.5 Dynamic pressure variation in the radial direction

Fig.6 Convergent limit of the analytical perturbation solution in the laminar case

flow rate is larger.

Figure 6 shows the change of radial velocity in the midplane $\beta=0.5$ with radius computed from the Görtler series - finite difference solution, and it may give an insight into the limit of validity of the analytical perturbation solution which is shown by broken and semi-broken lines in the figure. In the case of a smaller flow rate the analytical perturbation solution agrees well with the exact one except in the small region near the entrance, say, $\alpha=58$ in the example, and in the case of a larger flow rate the influence of the entrance on the flow extends to larger radial position and the perturbation solution doesn't converge yet at $\alpha=160$.

In the laminar case of large flow rate as shown in Fig.2, the radial flow velocity U in the potential-core increases because of the boundary-layer which develops from the entrance periphery, and correspondingly, the static pressure P decreases as shown in Fig.4 and the dynamic pressure P_d increases as shown in Fig.5. As the result, the total pressure decreases, which means that the rotating disks act as resistance instead of giving energy to the fluid. In this example, in the vicinity of $\alpha=40$, the total pressure recovers to the value at the entrance periphery, and the rotating disks begin to give energy to the fluid and act as a pump. The radial velocity U in the midplane between two disks continues to increase, and takes a maximum value at $\alpha=46.5$ as shown in Fig.3, but the static pressure P decreases monotonically from inlet periphery to this radial position. From this position the tangential velocity V increases by the entraining action of rotating disks, and P begins to increase. The distribution of U approaches the one in the infinite radial position where U is larger near the walls than near the midplane due to stronger centrifugal force near the walls. In the laminar case of a small flow rate as shown in Fig.3, the depression of P at the beginning is small and energy is given to the fluid almost from the entrance because the length of the inlet region is very short in the radial direction and the increase of U is small. The radial velocity U is larger near the disk walls than near the midplane, already at about $\alpha=40$. The peripheral velocity V becomes more and more uniform in the downstream region, and correspondingly, the centrifugal force exerted on the fluid becomes more and more uniformly distributed between the disks. Then, the action of the inertia force other than the centrifugal force manifests itself weakly, and U becomes smaller near the walls than near the midplane in the region of $\alpha=40 \sim 54$. With an increase of radial position, the inertia force decreases and the velocity distribution approaches that at infinity from the center axis where the centrifugal force and the pressure gradient equilibrate.

3. Theoretical analysis of turbulent flow

3.1 Coordinates and governing equations

Consider the orthogonal curvilinear coordinates (n, s, z) as illustrated in Fig. 7(a), (b); the z -axis coincides with the center axis; the s -axis, with streamline in the midplane between the disks; and the n -axis is perpendicular to both s - and z -axis. One disk wall is included in $z=0$, and the positive directions of the coordinates are determined such that n, s, z always constitute a right-handed coordinate system in this turn. The angle between the streamline in the midplane and the circumferential direction is denoted by λ , which is measured as positive when the main flow is outward and as negative when it is inward; and it is a function of only radius r . Then, the coordinates n, s are represented by the polar coordinates r, θ , as follows:

$$\left. \begin{aligned} n &= \pm \theta \pm \int_{r'}^r (\cot \lambda / r) dr \\ s &= \theta - \int_{r'}^r (\tan \lambda / r) dr \end{aligned} \right\} \dots\dots\dots (28)$$

where the upper and the lower of the double sign correspond to the outward and the inward flow, respectively; similarly the double signs in Eqs. (29) ~ (31), (41) follow this rule. The velocities of fluid in the n, s, z components are denoted by w_n, w_s, w_z , respectively, and the velocities on the midplane are represented by capital letters; then $w_n = w_s = 0$. The momentum integral equations can be obtained by integrating the equations of motion expressed in the above-mentioned coordinates from disk wall to outer edge of the boundary-layer. The resulting equations for the n - and s -axis are, respectively,

Fig. 7(a) Coordinate system for the outward flow

Fig. 7(b) Coordinate system for the inward flow

$$\begin{aligned} & \pm \frac{1}{r \sin \lambda} \frac{\partial}{\partial n} \int_0^\delta w_n^2 dz + \frac{1}{r \cos \lambda} \\ & \times \frac{\partial}{\partial s} \int_0^\delta w_n w_s dz + \left(\frac{\cos \lambda}{r} - \frac{d\lambda}{dr} \sin \lambda \right) \\ & \times \int_0^\delta (w_n^2 - w_s^2) dz - 2 \left(\frac{\sin \lambda}{r} + \frac{d\lambda}{dr} \cos \lambda \right) \\ & \times \int_0^\delta w_n w_s dz - 2\omega \int_0^\delta w_s dz \\ & = \mp \frac{\delta}{\rho} \frac{1}{r \sin \lambda} \frac{\partial p}{\partial n} + \delta r \omega^2 \cos \lambda - \frac{1}{\rho} (\tau_{nz})_{z=0} \end{aligned} \dots\dots\dots (29)$$

$$\begin{aligned} & \pm \frac{1}{r \sin \lambda} \frac{\partial}{\partial n} \int_0^\delta w_n w_s dz + \frac{1}{r \cos \lambda} \frac{\partial}{\partial s} \int_0^\delta w_s^2 dz \\ & - \frac{W_s^2}{r \cos \lambda} \frac{\partial \delta}{\partial s} + \left(\frac{\sin \lambda}{r} + \frac{d\lambda}{dr} \cos \lambda \right) \\ & \times \int_0^\delta (w_n^2 - w_s^2) dz + 2 \left(\frac{\cos \lambda}{r} - \frac{d\lambda}{dr} \sin \lambda \right) \\ & \times \int_0^\delta w_n w_s dz + 2\omega \int_0^\delta w_n dz \\ & = - \frac{\delta}{\rho} \frac{1}{r \cos \lambda} \frac{\partial p}{\partial s} - \delta r \omega^2 \sin \lambda - \frac{1}{\rho} (\tau_{sz})_{z=0} \end{aligned} \dots\dots\dots (30)$$

where τ_{nz} and τ_{sz} are the friction stresses in the n - and the s -direction, respectively, and δ is the boundary-layer thickness, $\delta \leq h/2$. The equations of motion which hold in the midplane between two disks are

$$\begin{aligned} & W_s^2 \left(-\frac{\cos \lambda}{r} + \frac{d\lambda}{dr} \sin \lambda \right) - 2\omega W_s \\ & = \mp \frac{1}{\rho} \frac{1}{r \sin \lambda} \frac{\partial p}{\partial n} + r \omega^2 \cos \lambda + \frac{1}{\rho} \left(\frac{\partial \tau_{nz}}{\partial z} \right)_{z=h/2} \end{aligned} \dots\dots\dots (31)$$

$$\begin{aligned} & \frac{W_s}{r \cos \lambda} \frac{\partial W_s}{\partial s} = - \frac{1}{\rho} \frac{1}{r \cos \lambda} \frac{\partial p}{\partial s} \\ & - r \omega^2 \sin \lambda + \frac{1}{\rho} \left(\frac{\partial \tau_{sz}}{\partial z} \right)_{z=h/2} \end{aligned} \dots\dots\dots (32)$$

These equations are employed to determine the static pressure. The equation of continuity is

$$\begin{aligned} u_0 h^2 &= 2r \left[\cos \lambda \int_0^\delta w_n dz \right. \\ & \left. - \sin \lambda \left\{ \int_0^\delta w_s dz + W_s \left(\frac{h}{2} - \delta \right) \right\} \right] \end{aligned} \dots\dots\dots (33)$$

3.2 Equations for calculation

In order to solve Eqs. (29) ~ (33), the velocity distributions, and the friction stresses on the wall and in the midplane between two disks must be given in advance. Therefore, the velocity distributions in the three-dimensional boundary-layer are assumed as follows:

$$\left. \begin{aligned} w_s &= W_s(z/\delta)^{1/7} \\ w_n &= \epsilon W_s(z/\delta)^{1/7} (1 - z/\delta)^2 \end{aligned} \right\} \dots\dots\dots (34)$$

where ϵ is a tangent of the angle between the direction of friction stress on the wall and that of the streamline in the midplane between two disks. According as $\epsilon > 0$ and $\epsilon < 0$, the velocity component w_n is $w_n < 0$ and $w_n > 0$, respectively, because $W_s < 0$. The various boundary-layer thicknesses which appear in Eqs. (29), (30) are given by the following equations:

$$\left. \begin{aligned} \int_0^\delta w_n dz &= c_1 \delta \varepsilon W_s, & c_1 &= 0.2598 \\ \int_0^\delta w_n^2 dz &= c_2 \delta \varepsilon^2 W_s^2, & c_2 &= 0.1097 \\ \int_0^\delta w_s dz &= c_3 \delta W_s, & c_3 &= 0.8750 \\ \int_0^\delta w_s^2 dz &= c_4 \delta W_s^2, & c_4 &= 0.7778 \\ \int_0^\delta w_n w_s dz &= c_5 \delta \varepsilon W_s^2, & c_5 &= 0.2071 \end{aligned} \right\} \dots (35)$$

It is assumed that

$$(\tau_{nz})_{z=0}/(\tau_{sz})_{z=0} = (\partial w_n/\partial z)_{z=0}/(\partial w_s/\partial z)_{z=0} = \varepsilon \dots (36)$$

as the relation between the above-mentioned two friction stresses on the disk wall, and that $(\tau_{sz})_{z=0}$ is equal to Blasius' friction stress on the wall for a turbulent boundary-layer along a flat plate;

$$(\tau_{sz})_{z=0} = 0.0225 \operatorname{sgn}(W_s) \rho W_s^2 (W_s \delta/\nu)^{-1/4} \dots (37)$$

where $\operatorname{sgn}(W_s) = -1$ since the direction of W_s is always the inverse of the positive s -direction. The gradient of friction stress

non-dimensional boundary-layer thickness: $\Delta = \delta/(h/2)$

non-dimensional velocity in the midplane between two disks: $u_c = W_s/u_0$ } .....(41)

Using above relations (35) ~ (38) and (40), one obtains the governing equations from Eqs. (29) and (30) for the inlet region where $\Delta < 1$ as follows:

$$\begin{aligned} & [2c_2 \varepsilon \cos \lambda - c_3 \sin \lambda - c_1 \varepsilon \cos \lambda (c_2 \varepsilon \cos \lambda - c_3 \sin \lambda) / \{(1-c_3) \sin \lambda + c_1 \varepsilon \cos \lambda\}] (d\varepsilon/d\alpha) \\ & + [(c_4 - 1 + c_2 \varepsilon^2) \sin \lambda - 2c_3 \varepsilon / \cos \lambda + \varepsilon (c_2 \varepsilon - c_3 \tan \lambda) (c_3 - 1 + c_1 \alpha u_c \varepsilon \cos \lambda) / (1 + \alpha u_c \sin \lambda) \\ & / \{(1-c_3) \sin \lambda + c_1 \varepsilon \cos \lambda\}] (d\lambda/d\alpha) = (c_2 \varepsilon^2 + c_4 - 1) \cos \lambda / \alpha + 2(R_2/R_1) \{c_3 - 1 + 2c_2 \varepsilon^2 \\ & - 2c_3 \varepsilon \tan \lambda - \varepsilon (c_2 \varepsilon - c_3 \tan \lambda) / (1 + \alpha u_c \sin \lambda)\} / u_c - 0.045 \operatorname{sgn}(u_c) \varepsilon (2/R_1)^{1/4} \mathcal{J}^{-5/4} |u_c|^{-1/4} \dots (42) \end{aligned}$$

$$\begin{aligned} & [c_5 - c_1 \{c_3 \varepsilon \cos \lambda + (1-c_4) \sin \lambda\} / \{(1-c_3) \sin \lambda + c_1 \varepsilon \cos \lambda\}] \cos \lambda (d\varepsilon/d\alpha) + [(c_2 \varepsilon^2 - c_4) \cos \lambda \\ & + (1-2c_4) \sin \lambda \tan \lambda + \{c_3 \varepsilon + (1-c_4) \tan \lambda\} (c_3 - 1 + c_1 \alpha u_c \varepsilon \cos \lambda) / (1 + \alpha u_c \sin \lambda) / \{(1-c_3) \sin \lambda \\ & + c_1 \varepsilon \cos \lambda\}] (d\lambda/d\alpha) = (1-c_4 - c_2 \varepsilon^2) \sin \lambda / \alpha + 2(R_2/R_1) [(2c_3 - c_1) \varepsilon + (1-2c_4) \tan \lambda \\ & - \{c_3 \varepsilon + (1-c_4) \tan \lambda\} / (1 + \alpha u_c \sin \lambda)] / u_c - 0.045 \operatorname{sgn}(u_c) (2/R_1)^{1/4} \mathcal{J}^{-5/4} |u_c|^{-1/4} \dots (43) \end{aligned}$$

$$(du_c/d\alpha) = u_c \tan \lambda (d\lambda/d\alpha) - u_c / \alpha - 2(R_2/R_1) / \cos \lambda \dots (44)$$

$$\Delta = (1 + \alpha u_c \sin \lambda) / [\alpha u_c \{(1-c_3) \sin \lambda + c_1 \varepsilon \cos \lambda\}] \dots (45)$$

$$P = \alpha^2 - (R_1/R_2)^2 u_c^2 + (\text{constant}) \dots (46)$$

The initial conditions for these equations are

$$\alpha = \alpha_i : u_{ci} = -\sqrt{(1/\alpha_i)^2 + (\alpha_i R_2/R_1)^2}, \quad \lambda_i = \tan^{-1}\{(R_1/R_2)/\alpha_i^2\}, \quad \Delta_i = \varepsilon_i = P_i = 0 \dots (47)$$

from the assumptions made for this analysis. It is, however, impossible to start computation with substitution of Eq. (47) into Eqs. (42) and (43) at $\alpha = \alpha_i$, because the leading edge is mathematically a singular point. Therefore, the new function f is introduced in the immediate neighborhood of the entrance $\alpha = \alpha_i$:

$$f = -\alpha u_c \sin \lambda - 1 \quad (f > 0, f \ll 1), \quad \varepsilon = k_1 f, \quad (d\lambda/d\alpha) = k_2 f^{-1/4} \dots (48)$$

Substituting Eq. (48) into Eqs. (42) and (43) and omitting the sufficiently small terms, one obtains

$$k_1 = \{(c_4 - 1)/(1 - c_4 + 2c_3)\} \sin \lambda_i \cos \lambda_i, \quad k_2 = -0.045 \operatorname{sgn}(u_c) (2/R_1)^{1/4} \{(1-c_3)^{5/4}/(1-c_4)\} \cos \lambda_i |u_{ci}|^{-1/4} \dots (49)$$

The third equation of Eq. (48) can't be analytically integrated yet. Then, the following variables at $\alpha = \alpha_i + \alpha^*$ are introduced:

$$u_c = u_{ci} + u_c^*, \quad \lambda = \lambda_i + \lambda^* \dots (50)$$

where the quantities with superscript * are infinitesimally small. Then the third equation of Eq. (48) becomes

$$(d\lambda^*/d\alpha^*) = k_2 (-\alpha^* u_{ci} \sin \lambda_i - \alpha_i u_c^* \sin \lambda_i - \alpha_i u_{ci} \lambda^* \cos \lambda_i)^{-1/4} \dots (51)$$

and from Eqs. (44) and (51), one obtains

$$u_c^* = (u_{ci} \tan \lambda_i) \lambda^* - \{u_{ci}/\alpha_i + 2(R_2/R_1)/\cos \lambda_i\} \alpha^* \dots (52)$$

Substitution of Eq. (52) into Eq. (51) and the use of Picard's successive approximation method yield

in the midplane between two disks is represented by

$$(\partial \tau_{nz}/\partial z)_{z=h/2} = (\partial \tau_{sz}/\partial z)_{z=h/2} = 0 \dots (38)$$

in the inlet region where $\delta = h/2$, and is assumed as

$$(\partial \tau_{nz}/\partial z)_{z=h/2} = k\varepsilon (\partial \tau_{sz}/\partial z)_{z=h/2} \dots (39)$$

in the fully developed region where $\delta = h/2$, and the coefficient k which is considered in the order of one is a value which must be determined by experiment; however, the value of k hardly influences the static and the dynamic pressures as will be shown later.

As the flow is assumed to be axisymmetric, the following relations are derived from Eq. (28):

$$\left. \begin{aligned} (\partial/\partial n) &= \pm r \sin \lambda \cos \lambda (d/dr) \\ (\partial/\partial s) &= -r \sin \lambda \cos \lambda (d/dr) \end{aligned} \right\} \dots (40)$$

Thus every variable becomes a function of only radius r , and the equations for numerical computation both in the outward and the inward flow are represented in the same expression. To calculate in non-dimensional forms, the following new variables are defined:

$$\lambda^* = \{(5/4)k_2\alpha^*\}^{4/5} (\sin \lambda_i \cos \lambda_i)^{1/4} \{1 - (1/5)\kappa + (5/48)\kappa^2 - (15/224)\kappa^3 + \dots\} \quad (53)$$

where $\kappa = 2(R_2/R_1)\alpha_i \tan \lambda_i \{(4/5) \sin \lambda_i \cos \lambda_i / k_2\}^{4/5} \alpha^{*1/5}$. The series in Eq.(53) converges rapidly. The quantities $\varepsilon, \lambda, u_c, \Delta, P$ are computed by Eqs.(52) and (53) in the immediate neighborhood of $\alpha = \alpha_i$, and then, by Eq.(48) in the region a little apart from $\alpha = \alpha_i$, and finally, the integration of Eqs.(42) ~ (46) by means of Runge-Kutta method for larger α is carried out until the boundary-layers on the disk surfaces coalesce.

For the fully developed region where $\Delta = 1$, the governing equations are

$$\begin{aligned} & [2c_2\varepsilon \cos \lambda - c_3 \sin \lambda - c_1\alpha u_c \cos \lambda \{2c_2\varepsilon^2 \cos \lambda - 2c_3\varepsilon \sin \lambda - k\varepsilon \sin \lambda \cos \lambda / (k\varepsilon \sin \lambda + \cos \lambda)\}] (d\varepsilon/d\alpha) \\ & + [(c_4 - c_2\varepsilon^2) \sin \lambda - 2c_3\varepsilon \cos \lambda - \sin \lambda \cos \lambda / (k\varepsilon \sin \lambda + \cos \lambda) + \alpha u_c (c_1\varepsilon \sin \lambda + c_3 \cos \lambda) \\ & \times \{2c_2\varepsilon^2 \cos \lambda - 2c_3\varepsilon \sin \lambda - k\varepsilon \sin \lambda \cos \lambda / (k\varepsilon \sin \lambda + \cos \lambda)\}] (d\lambda/d\alpha) = (c_4 - 1 + c_2\varepsilon^2) \cos \lambda / \alpha \\ & + 2(R_2/R_1) [c_3 - \cos \lambda / (k\varepsilon \sin \lambda + \cos \lambda)] / u_c - 0.045 \operatorname{sgn}(u_c) \varepsilon (2/R_1)^{1/4} |u_c|^{-1/4} \quad (54) \end{aligned}$$

$$\begin{aligned} & [c_5 - c_1\alpha u_c \{2c_5\varepsilon \sin \lambda - 2c_4 \sin \lambda + k\varepsilon \sin^2 \lambda / (k\varepsilon \sin \lambda + \cos \lambda)\}] \cos \lambda (d\varepsilon/d\alpha) + [(c_2\varepsilon^2 - c_4) \cos \lambda \\ & - 2c_5\varepsilon \sin \lambda + \sin^2 \lambda / (k\varepsilon \sin \lambda + \cos \lambda) + \alpha u_c (c_1\varepsilon \sin \lambda + c_3 \cos \lambda) \{2c_5\varepsilon \cos \lambda - 2c_4 \sin \lambda \\ & + k\varepsilon \sin^2 \lambda / (k\varepsilon \sin \lambda + \cos \lambda)\}] (d\lambda/d\alpha) = (1 - c_4 - c_2\varepsilon^2) \sin \lambda / \alpha + 2(R_2/R_1) \{-c_1\varepsilon \\ & + \sin \lambda / (k\varepsilon \sin \lambda + \cos \lambda)\} / u_c - 0.045 \operatorname{sgn}(u_c) (2/R_1)^{1/4} |u_c|^{-1/4} \quad (55) \end{aligned}$$

$$u_c = 1 / \{\alpha (c_1\varepsilon \cos \lambda - c_3 \sin \lambda)\} \quad (56)$$

$$\begin{aligned} (dP/d\alpha) &= 2\alpha + 4(R_1/R_2)u_c / (k\varepsilon \sin \lambda + \cos \lambda) + 2(R_1/R_2)^2 u_c^2 [(k\varepsilon \sin \lambda + \cos \lambda) / \alpha \\ &+ c_1 k \varepsilon \alpha u_c \sin \lambda \cos \lambda (d\varepsilon/d\alpha) - \{1 + k \varepsilon \alpha u_c (c_1\varepsilon \sin \lambda + c_3 \cos \lambda)\} \sin \lambda (d\lambda/d\alpha)] / (k\varepsilon \sin \lambda + \cos \lambda) \quad (57) \end{aligned}$$

The initial values of $\varepsilon, \lambda, u_c, P$ for these equations are supplied by the above-mentioned computation in the developing region.

For the region of sufficiently large values of α , one may deduce the perturbation solution:

$$\left. \begin{aligned} \tan \lambda &= t_0 + t_1 X + t_2 X^2 + \dots \\ \varepsilon &= \varepsilon_0 + \varepsilon_1 X + \varepsilon_2 X^2 + \dots \\ u_c &= \sqrt{1 + \tan^2 \lambda} / [\alpha \{(c_1\varepsilon_1 - c_3 t_1) X \\ &+ (c_1\varepsilon_2 - c_3 t_2) X^2 + \dots\}] \end{aligned} \right\} \quad (58)$$

where

$$\left. \begin{aligned} X &= [0.045/R_2]^4 \{R_1/(2\alpha)\}^{3/7} \\ t_0 &= \pm \sqrt{c_1(1-c_3)/k}/c_3, \quad \varepsilon_0 = (c_3/c_1)t_0 \\ t_1 &= (1+t_0)^{3/14} (2-c_3+c_3 k) \\ & / \{2k^{4/7} c_3^{11/7} (1-c_3+c_3 k)^{3/7}\} \\ \varepsilon_1 &= c_3^2 (1-k) t_1 / \{c_1(2-c_3+c_3 k)\}, \dots \end{aligned} \right\} \quad (59)$$

where the double sign in front of t_0 can't be determined in advance because t_0 is an

asymptotic value in Eqs.(54) ~ (56). The dynamic pressure is calculated in the same way as the laminar flow.

3.3 Numerical examples

Calculated results of the velocity distributions of fluid between two disks for some sets of R_1, R_2, α_i which are equal to that in Figs.2 and 3 in the laminar case, are shown in Figs.8 and 9. The variations of the static and the dynamic pressures with radius are represented by the broken lines in Figs.4 and 5, respectively. In these calculations $k=0.5$ is used. The velocity distributions U and V depend slightly on the assumed value of k . However, the typical turbulent velocity distributions shown in Figs.8 and 9 are always obtained for reasonable values of k . In order to show the influence of k on the flow, the change of the flow in the α -direction for two cases,

Fig.8 Widthwise distribution of velocities of turbulent flow

Fig.9 Widthwise distribution of velocities of turbulent flow

$k=0.5$ and 1.0 , is indicated in Fig.10. The strength of the secondary flow is seen to be directly influenced by the value of k , but increase of P and P_d with radius differs little between two cases. It suggests that for the purpose of estimating the energy supplied to the fluid, the exact value of k may not be indispensable. The fine solid

Fig.10 Dependence of the flow on the coefficient k of turbulent flow and the perturbation solution in the turbulent case

Fig.11 Dependence of the flow on the Reynolds number R_1 in the turbulent case

Fig.12 Dependence of the flow on the Reynolds number R_2 in the turbulent case

lines in the figure are the results computed by means of the perturbation solution, Eqs. (58) and (59). It is found that this simplified approximate solution can be applied to a relatively small radial position, as in the laminar case.

The numerical examples showing the effect of the flow rate R_1 , the rotational speed of disks R_2 and the inner radius of disks α_i are given in Figs.11, 12 and 13, respectively. In the case of turbulent flow for large flow rate (Fig.8), the increase of the main stream velocity U in the inlet region is not remarkable, in spite of the boundary-layer growth on the disk walls, and the velocity within the boundary-layer is large compared with the laminar case, and hence, the pressure depression becomes little in this region (Fig.4). As shown in Fig.5, the dimensionless dynamic pressure P_d decreases, and the region of the total pressure decrease in the inlet region is wider than in the laminar case. As the radial position becomes large, V increases due to the action of the rotating disks and energy is given to the fluid. Since the defect of the circumferential velocity V from that of the disks is almost uniformly small between the flow passage, except in the immediate neighborhood of disk wall, which is typical of turbulent flow, the centrifugal force is more strongly distributed near the disk walls in the turbulent flow than in the laminar one. This causes a strong outward flow near the walls as shown in Figs.8 and 9. As the radial position becomes still larger, an inward flow occurs in the vicinity of the midplane between two disks. The asymptotic behavior of U for large α is $U \propto \alpha^{3/7}$, as is given in Eq. (58), whereas U is constant in the laminar case. In any case, the dimensional velocity u decreases in the large radial position, and in the region sufficiently far from the center axis, the fluid rotates with the rotating disks as if it were a rigid body. In the turbulent case of small flow rate (Fig.9), energy

Fig.13 Dependence of the flow on the dimensionless inner radius of disks α_i in the turbulent case

is given to the fluid from the smaller radial position and the pressure rises more rapidly than in the case of a large flow rate, which is the same tendency as with the laminar case. The velocity V in the vicinity of the midplane between two disks is smaller, and correspondingly, the pressure rise is smaller than in the laminar case at the same radial position.

As in the laminar case, the fluid experiences energy dissipation near the entrance periphery and an energy supply to the fluid begins to take place a little downstream. When the flow rate is small and both the inner radius and the speed of rotation are large, the radial span in which the energy loss occurs becomes short; this means that the effect of the inner periphery manifests itself only in the vicinity of the inner periphery. In further downstream region, the velocity of fluid relative to the disks decreases and the pressure variation approaches the asymptotic one; i.e., it rises in proportion to the square of radius with small negative shift. As the static pressure doesn't change in the z -direction, in the framework of the present calculation, the effect of Coriolis' force produces a secondary flow in the left-hand side relative to the positive flow direction in the midplane.

In case that the flow rate is large and both the rotational speed and the inner radius of disks are small, the flow becomes similar to one within a stationary ring diffuser; near the entrance periphery, the frictional effect of the disks upon the fluid is weak and the pressure recovery by decrease of the flow velocity due to enlargement of the flow passage plays the principal role in the determination of the static pressure distribution. In this region the total pressure decreases, and the rotating disks work only to dissipate energy. A secondary flow occurs in the opposite side to the former case because the effect of the centrifugal force due to the curve of the streamline exceeds that of Coriolis' force. The rotational effects of disks begin to manifest themselves as the radial position is larger, and the flow attains the regime which appears in the fully developed region when the flow rate is smaller, the rotational speed of disks is larger and the inner radius is larger. Namely, the effect of the centrifugal force due to the curve of the streamline decreases, that of Coriolis' force increases relatively, and the direction of the secondary flow is inversed and the total pressure rises.

When R_1 , R_2 , α_i are fixed as shown in Fig.13, the inner peripheral edge effects manifest themselves weakly at a certain radial position independently of the inner radius α_i , and the flow at larger radial position is determined independently of α_i and can be represented in the power series of $[R_1^3/(R_2^4\alpha^3)]^{1/4}$. The relative velocity $-u_c$, which is determined by the first equation of Eq.(47), in the midplane between two disks takes a minimum value when $\alpha_i = \sqrt{R_1/R_2}$, as shown in Fig.14. In the example of Fig.13,

this occurs when $\sqrt{R_1/R_2}=10$. When $\alpha_i < \sqrt{R_1/R_2}$, the velocity $-u_c$ decreases until it arrives at a value approximately equal to that at $\alpha_i = \sqrt{R_1/R_2}$ and the total pressure doesn't increase. In this case the flow becomes similar to one within a stationary ring diffuser and the rotating disks in this region play a role of only dissipating energy. As the radial position α is larger, the energy is gradually given to the fluid. In the alternative case, when $\alpha_i > \sqrt{R_1/R_2}$, the total pressure increases just from the inner periphery, though the velocity $-u_c$ increases and the static pressure slightly decreases, because the effect of boundary-layer growth is more responsible than that of enlargement of the flow passage. When α_i is too large as in the case of $\alpha_i = 30$ (Fig.13), the static pressure rise becomes less than that for even smaller α_i . After all, if R_1 and R_2 are fixed, the optimum value of α_i which gives maximum value to the total pressure rise exists and the value seems to be $\alpha_i = (1 \sim 2)\sqrt{R_1/R_2}$.

4. Conclusions

In order to establish an accurate theory of performance and to obtain optimum values in designing parameters of a disk friction pump, an analysis was made of a flow between two infinitely large corotating disks with throughflow supplied at the inner periphery of finite radius.

The flow field is assumed to be either everywhere laminar or everywhere turbulent. The laminar case is treated by dividing the field into three regions; the region in the vicinity of the entrance periphery to which Görtler series is applied, the region far from the center axis for which analytical perturbation solution is used and the mid-region connecting the above two extreme regions for which finite difference computation is carried out. The combination of these three solutions gives a complete scheme of the flow field with satisfactory indications for the engineering purposes. Meanwhile, the turbulent case is treated by means of momentum equations with elaborated curvilinear coordinate axes. This solution shows a physically acceptable flow pattern and it was found that the non-dimensional

Fig.14 Relation between the flow velocity, $-u_{ci}$, on the midplane between two disks at the inlet periphery and the dimensionless inner radius α_i

inner radius α_i , giving maximum total head falls in the region $\alpha_i = (1 \sim 2)\sqrt{R_1/R_2}$. Furthermore, it was suggested that the increase of the static pressure is larger in the laminar flow than in the turbulent one, and that in both the laminar and the turbulent flow, the fluid rotates with the rotating disks as a rigid body in the region far from the center axis. The radial span of the inlet region was found to become shorter as the flow rate is smaller.

References

- (1) *Scientific American*, (1911-9-30), p.296.
- (2) Nendl, D., Diss. Dr.-Ing. v. T. H. Aachen, (1966).
- (3) Нуликов, Г.С., Гидравлические Машины, No.1 (1967), p.109.
- (4) Hasinger, S.H. and Kehrt, L.G., *Trans. ASME. Ser.A*, Vol.85, No.3 (1963-7), p.201.
- (5) Мисюра, В.И. и Дронов, Ю.В., Изв. вуз. Авиаци. техн., Vol.15, No.1 (1972), p.146.
- (6) Barske, U.M., *Proc. Inst. Mech. Engrs.*, Vol.174, No.11 (1960), p.437.
- (7) Vannérus, T., *Allg. Wärmetechn.*, Bd.6. Nr.12 (1955), S.257.
- (8) Rice, W., *Trans. ASME, Ser.A*, Vol.85, No.3 (1963-7), p.191.
- (9) Васильцов, З. А. и Невелич, В. В., Герметические электронасосы, (1968), p.176, Машиностроение.
- (10) Murata, S., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.18, No.73 (1952), p.25.
- (11) Breiter, M.C. and Pohlhausen, K., ARL, USAF, Dayton, Ohio, Rep.No.ARL62-318 (1962-3).
- (12) Peube, J. L. et Kreith, F., *Jour. Mécanique*, Vol.5, No.2 (1966-6), p.261.
- (13) Matsch, L. and Rice, W., *Trans. ASME. Ser.E*, Vol.35, No.1 (1968-3), p.155.
- (14) Мисюра, В.И., Изв. АН СССР, Механика жидкости и газа, No.5 (1972), p.178.
- (15) Boyd, K.E. and Rice, W., *Trans. ASME, Ser.E*, Vol.35, No.2 (1968-6), p.229.
- (16) Byal'yi, B.N., et al., *Russian Engr. Jour.*, Vol.51, No.9 (1971), p.56.
- (17) Boyack, B.E. and Rice, W., *Trans. ASME, Ser.D*, Vol.93, No.3 (1971-9), p.350.
- (18) Görtler, H., *J. Math. u. Mech.*, Bd.6, Nr.1 (1957), S.1.
- (19) Ishizawa, S., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.30, No.219 (1964-11), p.1301.
- (20) Köhler, M., Diss. Dr.-Ing. v. T. H. Karlsruhe, (1969-7).
- (21) Bakke, E., et al., *J. Fluid Mech.*, Vol.58 (1973), p.209.