On the vibration of a rotating shaft passing through the critical speed

By Saburo AIBA**

Several papers have been published concerning the vibration of a rotating shaft carrying a rotor at the time of passing through the critical speed. But most of them treat the rotor as a concentrated mass. In this paper, the equations of motion of the rotor when the shaft speed varies arbitrarily are introduced using a rotating coordinate system and taking into account the gyroscopic effect of the rotor. The equations are solved by Runge-Kutta-Gill's numerical method in various cases where the rotor and the bending rigidity of the shaft are symmetrical or asymmetrical and the shaft passes through the critical speed with a constant or a variable angular acceleration. The exact solutions expressed with Fresnel's integrals in the case when the rotor is a concentrated mass are compared with the results of calculation by the numerical method in order to check the accuracy of this method. In addition, the results of simulation by an analogue computer are shown.

1. Introduction

There have been published several papers concerning the vibration of a rotating shaft passing through the critical speed, but most of them treat the rotor as a concentrated mass neglecting the gyroscopic effect of the rotor. The bending rigidity of the shaft is also assumed to be symmetrical in those papers. E. G. Poloskow and A. P. Filippow have investigated this problem taking into account the gyroscopic effect of the rotor. But they have used the averaging method regarding the shaft system as a system having slowly varying parameters, the method, therefore, being an approximate one.

In this paper, the problem is treated taking into account the gyroscopic effect of the rotor and using Runge-Kutta-Gill's numerical method for a shaft of symmetrical or asymmetrical bending rigidity carrying a symmetrical rotor and for a shaft of symmetrical bending rigidity carrying an asymmetrical rotor. The relation between the amplitude of the rotor and the angular acceleration of the shaft passing through the critical speed is obtained, and the results are compared with those in the case where the rotor is a concentrated mass.

2. Equations of motion

We shall consider an overhung shaft carrying a rotor as shown in Fig. 1. Other systems such as a shaft supported at both ends can be treated similarly. The mass of the shaft is assumed to be negligible as compared with that of the rotor. We shall consider a sufficiently generalized case in which the rotor and the bending rigidity of the shaft are both symmetrical and the shaft speed varies arbitrarily. It is assumed, however, that the principal axes of the rotor perpendicular to the shaft are parallel with those of the cross section of the shaft when there is no dynamical unbalance. It is also assumed that the system receives no damping force.

We shall use the following notations.

$\xi$: Principal axis of the rotor through its center of gravity along the shaft.
$\eta, \zeta$: Principal axes of the rotor through its center of gravity perpendicular to the shaft.
$I_\phi$: Moment of inertia of the rotor about the $\phi$-axis.
$I_\phi, I_\zeta$: Moment of inertia about the $\zeta$- and $\eta$-axis respectively.
$M$: Mass of the rotor.
$\omega(t)$: Shaft speed.

In order to describe the motion of the rotor, we shall use a rectangular rotating coordinate system $xyz$ rotating at the same angular speed as that of the shaft. Let us take the center line of the bearing as the x-axis, and take y- and z-axes such that they coincide with $\eta$- and $\zeta$-axes respectively when there are no unbalances and the shaft is at rest. Let us denote by $\phi$, $\gamma$ the angles between $\phi$-axis and xz-, xy-plane respectively, and denote by $y, z$ the coordinates of the center of gravity of the rotor. Also we shall denote by $\psi, \alpha, \beta_\phi$ and $\beta_\zeta$ the components of the eccentricity and angular eccentricity of the rotor, that is, the values of $y, z, \beta$ and $\gamma$ at...
the state of rest. The values of \( \gamma, z, \beta, r, \psi, z, \beta \) and \( \tau \) are assumed to be small quantities of the first order.

Expressing by \( \omega_x, \omega_y \) and \( \omega_z \) the components of angular speed of the rotor about the axes \( \xi, \eta \) and \( \zeta \) respectively, and neglecting small quantities of order higher than the first, we get

\[
\begin{align*}
\omega_x &= 0 \\
\omega_y &= \gamma - \omega \beta \\
\omega_z &= -\omega \alpha - \omega \gamma 
\end{align*}
\]

(1)

We shall substitute these formulas in Euler's equations:

\[
\begin{align*}
I_{\omega x} + (I_{\eta} - I_{\zeta}) \omega_y \omega_x &= T_\xi \\
I_{\omega y} &= (I_{\zeta} - I_{\xi}) \omega_x \omega_y = T_\eta \\
I_{\omega z} + (I_{\xi} - I_{\zeta}) \omega_z \omega_z &= T_\zeta
\end{align*}
\]

(2)

where \( T_\xi, T_\eta \) and \( T_\zeta \) represent the moments of the force acting on the rotor about the \( \xi, \eta \), and \( \zeta \) axes respectively. Neglecting small quantities of order higher than the first, and further using

\[
\begin{align*}
T_\xi &= b_1'(\beta - \beta_0) + a_1'(y - y_0) - T_\eta \\
T_\eta &= b_2'(\gamma - \gamma_0) - a_1'(x - x_0) - T_\zeta
\end{align*}
\]

(3)

we obtain the following equations:

\[
\begin{align*}
I_{\theta x} + (I_{\eta} - I_{\zeta}) \omega_y \omega_x &= b_1'(\beta - \beta_0) + a_1'(y - y_0) \\
I_{\theta y} &= (I_{\zeta} - I_{\xi}) \omega_x \omega_y = b_2'(\gamma - \gamma_0) - a_1'(x - x_0) \\
I_{\theta z} + (I_{\xi} - I_{\zeta}) \omega_z \omega_z &= b_2'(\beta - \beta_0) - a_1'(x - x_0)
\end{align*}
\]

(4)

\[
I_{\omega x} = T
\]

(5)

in which \( T \) represents the torque applied to the shaft from outside of the bearing, and it is assumed that the vector of the torque is always directed towards the \( x \)-axis. The constants \( a_1' \) and \( b_1' \) \((i = 1, 2)\) are coefficients of the restoring moment exerted on the rotor by the shaft due to the displacement and angular displacement of the rotor. In the case of an overhung shaft,

\[
\begin{align*}
a_1' &= \frac{3Eh}{f} \\
b_1' &= -\frac{4Eh}{f}
\end{align*}
\]

(5)

where \( E \) is Young's modulus, \( h \) is the moment of inertia of area about the \( x \), \( y \)-axis, and \( f \) is the distance between the bearing and the center of gravity of the rotor.

The equations of motion of the center of gravity of the rotor are obtained by considering the centrifugal force, Coriolis' force and the virtual force due to the angular acceleration of the coordinate system, and can be written as follows:

\[
M\ddot{\gamma} + (M\alpha + a_1')\gamma - 2M\omega - b_1'\beta
\]

\[
- M\omega_x = -a_1\omega_0 - b_1\beta_0
\]

\[
M\ddot{\alpha} = (M\alpha + a_1')\alpha + 2M\omega_0 - b_1\gamma
\]

\[
M\alpha = e - a_1\alpha_0 - b_1\gamma_0
\]

The constants \( a_1 \) and \( b_1 \) \((i = 1, 2)\) are coefficients of the restoring forces exerted on the rotor by the shaft due to the displacement and angular displacement of the rotor. In the case of an overhung shaft, they are written as follows:

\[
\begin{align*}
a_1 &= -12EhI_0' \\
b_1 &= a_1' \\
a_2 &= -12EhI_0' \\
b_2 &= a_1'
\end{align*}
\]

(7)

3. The case in which the angular acceleration is constant

We shall consider the vibration of the rotor during a comparatively short period of passing through the critical speed, and assume that the torque \( T \) is constant during the period. Then we get, by the third equation of Eqs.(4), the following equations:

\[
\omega = \tau / I_0, \quad \omega = \omega_0 + \omega t
\]

(8)

where \( \tau \) is a positive or a negative constant. The positive value of \( \tau \) corresponds to a positive angular acceleration, and the negative value corresponds to a negative angular acceleration of the shaft, namely deceleration. In order to express Eqs.(4) and (6) in non-dimensional forms, let us put

\[
\begin{align*}
\eta &= \sqrt{3Eh} / I_0 \gamma, \quad \theta = \eta \psi, \quad \varphi = \omega_0 / \eta \\
\xi &= \omega_0 / \eta, \quad \alpha = z / \eta, \quad \beta = y / \eta \\
\gamma &= y / (4Eh / f), \quad \epsilon = x / \eta
\end{align*}
\]

(9)

\[
A = I_0 / I_0', \quad B = I_0 / I_0', \quad C = I_0 / I_0', \quad \kappa = I_0 / I_0
\]

where \( \kappa \) is the circular frequency of the eigen-vibration in the \( xy \)-plane when the rotor is a concentrated mass and \( \omega = 0 \). Then Eqs.(4) and (6) can be rewritten as follows:

\[
\begin{align*}
M\ddot{\gamma} + [(A-C)(\gamma + \alpha \theta) + (4Eh / f)] \beta
&+ (A-B-C)(\gamma + \alpha \theta) \gamma + (A-B) \alpha \gamma \\
&- 2\beta \gamma + (4Eh / f) \alpha \gamma
\end{align*}
\]

(10)

Table 1.
Relation between the step size \( h \) of integration used in Runge-Kutta-Gill's numerical method and the results of calculation.

| \( h \) | \( \psi_{\text{max}} \) | \( \varphi_{\text{max}} \) | \( \psi_{1.1 \times \text{max}} \) | \( \varphi_{1.1 \times \text{max}} \)
<table>
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<tr>
<td>0.02</td>
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<td>61.94</td>
<td>30.21</td>
<td>58.08</td>
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<td>41.60</td>
<td>61.95</td>
<td>30.21</td>
<td>58.10</td>
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<td>63.73</td>
<td>30.19</td>
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<tr>
<td>0.15</td>
<td>41.65</td>
<td>63.11</td>
<td>30.25</td>
<td>57.31</td>
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<tr>
<td>0.2</td>
<td>41.63</td>
<td>61.89</td>
<td>30.24</td>
<td>57.25</td>
</tr>
<tr>
<td>( A = 0.1, B = C = 0.05, \kappa = 1 ), ( \eta = 1.005 ), ( \omega_0 = 0.02 ), ( \omega_0 = 0.05 )</td>
<td></td>
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</table>

Table 1. Relation between the step size \( h \) of integration used in Runge-Kutta-Gill's numerical method and the results of calculation. The components of eccentricity and angular eccentricity are equated to zero except for \( \psi \), and the initial conditions are all assumed to be zero. (The same conditions are also valid for Table 2 and 5.)
where $\theta$ denotes $4d\vartheta$. The equations (10) are linear ones, but the coefficients are functions of the nondimensional time $\theta$. If we assume that the initial conditions are such that at $\vartheta = 0$, $\beta = \beta(0)$, $\gamma = \gamma(0)$ etc., the solution of Eq. (10) can be expressed as a superposition of the solutions each of which is obtained by preserving only one each of the twelve constants $\beta_n$, $\gamma_n$, $\varepsilon_n$, $\beta'(0)$, $\gamma'(0)$, $\beta''(0)$, $\gamma''(0)$, $\beta'''(0)$ and $\gamma'''(0)$ and equating the other eleven constants to zero.

At first we shall consider the case in which only $\beta$ is non-zero. The equation (10) is difficult to solve analytically, and we shall resort to Runge-Kutta-Gill’s numerical method. The step size $h$ of integration is taken equal to 0.05 for $a = 0.001-0.003$. In the time interval considered, the results remain almost the same even if $h$ is taken smaller, and by comparison of the numerical solution and the exact solution in the case of a shaft with a concentrated mass (cf. Chap. 4), it can be confirmed that the above mentioned value of $h$ is small enough.

Table 1 shows the relation between the step size $h$ and the calculated values of both the amplitude and angular amplitude of the rotor. The value of 0.001 for the parameter $a$ expressing the angular acceleration of the shaft corresponds to a case in which when $\omega = 2\pi \times 1000 / 60$, or 1000 rpm, it takes $\tau_{12} = 1/\omega = 9.55$ seconds to ac-

Fig. 2
Amplitude of an overhung shaft of symmetrical bending rigidity carrying a disc-type symmetrical rotor passing through the critical speed with a constant angular acceleration (Figs. 4, 5 and 9 also relate to the same shaft system).

Fig. 3
The case of a rotor-shaft system which is the same as that in Fig. 2 except that the rotor is of a cylinder-type.

Fig. 4
Angular amplitude of precessional motion of the rotor.

Fig. 5
Amplitude of the rotor passing through the critical speed with a constant angular deceleration.

critical speed band (CRB) : $\omega = 1.00 - 1.05$

Fig. 6
The case of a shaft of asymmetrical bending rigidity carrying a symmetrical rotor.

Critical speed band (CRB) : $\omega = 1.011 - 1.096$

Fig. 7
The case of a shaft of symmetrical bending rigidity carrying an asymmetrical rotor.
celerate the shaft from \( \omega = 0 \) to \( \omega = \omega_c \). From Eqs. (8) and (9), we get
\[
\sigma = \frac{p_0}{k} = \frac{T}{k} = T/3EIA, \tag{11}
\]
Therefore, \( \sigma \) is independent of the mass \( M \) of the rotor.

Figs. 2-7 are examples of the calculated results showing the relation between the shaft speed \( \omega \) and the amplitude \( \varepsilon = \sqrt{\omega^2 + \beta^2} \) or angular amplitude \( \phi = \sqrt{\beta^2 + \gamma^2} \) of the rotor. Here \( \omega_c = \omega_{cr} / p_0 \)

in which \( \omega_{cr} \) denotes the critical speed.

Fig. 3 is a diagram of the calculated results showing the relation between \( \beta \) and \( \gamma \) in the case where a shaft of symmetrical bending rigidity carrying a disc-type symmetrical rotor passes through the critical speed of the first order, and Fig. 3 is a similar diagram for a shaft carrying a cylinder-type rotor. Fig. 4 shows the relation between \( \phi \) and \( \beta \) in the same case as that of Fig. 2, and Fig. 5 is an \( \Omega - \gamma \) diagram for the same shaft system passing through the critical speed with a constant angular deceleration. Fig. 6 is a diagram showing the relation between \( \beta \) and \( \gamma \) in the case where a shaft of symmetrical bending rigidity carrying a symmetrical rotor passes through the principal critical speed band, and Fig. 7 is a similar diagram for a shaft of symmetrical bending rigidity carrying an asymmetrical rotor. The speed zone shaded in each of the figures is the principal critical speed band.

Strictly speaking, waves of higher frequency and minute amplitude are superposed upon these slowly varying curves, which are omitted in these figures. These minute waves are due to the eigen-vibrations of higher orders of the system, and they appear a little more apparently in the precessional vibrations than in the vibration of the center of gravity of the rotor. But the amplitude of the above-mentioned vibration of higher frequency is as small as 1-2% of the maximum overall amplitude in the case of precessional vibration, and is less than 1% in the case of the vibration of the center of gravity of the rotor. The amplitude of the above-mentioned vibration of higher frequency is small enough to appear prominently. But in such cases, however, the overall amplitude does not build up as the shaft passes through the critical speed (cf. Chap. 6).

In order to clarify the influence of the gyroscopic effect of a symmetrical rotor upon the maximum amplitude at the time of passing through the critical speed, solutions were obtained for various geometrically similar rotors in which \( B/A \) was constant and \( A \) took different values. The results of computation are shown in Table 2. Comparing the results with that obtained for the case in which the rotor is a concentrated mass \( (A = B = 0) \), we know that when the radius of the rotor is of the same order as the length of the shaft, the influence of the gyroscopic effect of the rotor upon the maximum amplitude becomes considerably large and can not be neglected. The value \( A = 0.5 \) corresponds to the case in which the length of the rotor is equal to the length of the shaft.

In the case where the eccentricity of the rotor is zero and the angular eccentricity is not zero, the maximum amplitude and maximum angular amplitude at the time of passing through the critical speed of the first order are far smaller than in the case where the angular eccentricity is zero and the eccentricity is not zero. For a cylindrical rotor, there exists a critical speed of the second order, and the effect of the angular eccentricity especially on the maximum angular amplitude of precessional motion of the rotor at the time of passing through the critical speed of the second order is comparatively large. Fig. 8 shows the amplitude and angular amplitude when the same shaft system as in Fig. 3 passes through the critical speed of the second order. Table 3 shows the maximum amplitude and maximum angular amplitude of a cylinder-type rotor at the time of passing through the critical speeds of the first and the second order in the case where only an eccentricity exists and the case where only an angular eccentricity exists. It was made clear by calculation

<table>
<thead>
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<th>( \Omega )</th>
<th>( \Omega_{cr} )</th>
<th>( \Omega_{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>1.056</td>
</tr>
<tr>
<td>0.2</td>
<td>1.759</td>
<td>1.312</td>
</tr>
<tr>
<td>0.4</td>
<td>1.259</td>
<td>1.144</td>
</tr>
<tr>
<td>0.6</td>
<td>1.142</td>
<td>1.082</td>
</tr>
</tbody>
</table>

Table 2. Influence of the gyroscopic effect of the rotor on the maximum amplitude in the case of an overhung shaft of symmetrical bending rigidity carrying a disc-type symmetrical rotor.

Here \( \omega \) is the value of \( \omega \) at which \( \varepsilon \) takes the maximum value \( \varepsilon_{max} \).

\[
\begin{align*}
N/A & = 0.55, \, \beta_{cr} = 0.1, \, \omega = 0.312. \\
\varepsilon_{max}/\varepsilon & = 1.27, \, 0.23, \, 0.23. \\
\varepsilon_{max}/\varepsilon & = 1.17, \, 1.17, \, 1.17.
\end{align*}
\]

Table 3. Maximum amplitude and maximum angular amplitude of the rotor in the case where an overhung shaft of symmetrical bending rigidity carrying a cylinder-type symmetrical rotor passes through the critical speeds of the first and the second order. The components of eccentricity and angular eccentricity are equated to zero except for \( \beta \) or \( \omega_c \), and the initial conditions are all assumed to be zero.

(1): The case in which the shaft passes through the critical speed of the first order.
(2): The case in which the shaft passes through the critical speed of the second order.
that in general the maximum angular amplitude is far greater when passing through the critical speed of the second order than when passing through the critical speed of the first order with the same angular acceleration. On the contrary, the maximum amplitude of the center of gravity of the rotor due to the eccentricity is smaller in the former case than in the latter, which is thought to be due to the difference between the first and second modes of eigen-vibrations.

As in the case where the gyroscopic effect of the rotor is not considered, the maximum amplitude becomes smaller and the difference between the critical speed and the shaft speed at which the amplitude becomes maximum becomes greater as the angular acceleration of the shaft passing through the critical speed becomes greater. Fig. 9 shows the relation between the angular acceleration $\dot{\alpha}$ and the maximum amplitude $r_{\text{max}}$ or the maximum angular amplitude $\varphi_{\text{max}}$ for the same shaft system as in Fig. 2. Further, it was made clear by the result of calculation that when the rotor or the bending rigidity of the shaft is asymmetrical, the maximum amplitude is far greater than in the case of a symmetrical shaft system. The situation is the same with regard to the angular amplitude of precessional motion of the rotor.

4. The case in which there is no gyroscopic effect

In order to compare the case in which there is the gyroscopic effect of the rotor with one in which there is no gyroscopic effect, as well as in order to compare the solution obtained by Runge-Kutta-Gill's numerical method with the exact solution expressed with Fresnel's integrals, we shall consider a case in which the rotor is a concentrated mass. The angular acceleration is assumed to be constant.

We shall substitute $\Delta B = \Delta C = 0$ in Eq. (10). From the first and second equations, we get

$$2(\beta - \beta_0) = 3(\eta - \eta_0)$$
$$2(\gamma - \gamma_0) = 3(\xi - \xi_0)$$

Substituting these relations in the third and fourth equations, we get

$$\eta'' - (2\beta - 1)\eta - 2\beta(2\eta' - \alpha \xi - \eta) = 0$$
$$\xi'' - (2\gamma - 1)\xi - 2\gamma(2\xi' - \alpha \eta + \gamma \xi) = 0$$

$$\eta(0) = \eta_0, \eta'(0) = \eta_1, \xi(0) = \xi_0, \xi'(0) = \xi_1$$

We shall also solve these equations by Runge-Kutta-Gill's numerical method. The step size $h$ of integration is set at 0.05 as in Chap. 3.

Fig. 10 is an example of the results of calculation showing the relation between $\eta$ and $\eta_0, \eta_1, \xi_0, \xi_1$ in which $\eta(0) = \eta_0, \eta'(0) = \eta_1, \xi(0) = \xi_0, \xi'(0) = \xi_1$. In this figure, $\eta_0$ and $\eta_1$ are written as $\eta$ and $\xi$, respectively.

5. Exact solution expressed with Fresnel's integrals

We shall consider a case in which the shaft has symmetrical bending rigidity and the rotor is a concentrated mass. Let us take a static rectangular coordinate system $xy$ in the plane of motion of the rotor so that its origin is located on the center line of the bearing. The angular acceleration of the shaft is assumed to be constant and positive. Then the equations of motion of the rotor can be written as follows:

Fig. 9
Relation between the angular acceleration and the maximum amplitude $r_{\text{max}}$ or the maximum angular amplitude $\varphi_{\text{max}}$ of the rotor.

Fig. 10
The case of a shaft of symmetrical bending rigidity carrying a concentrated mass.
where \( n \) is the eccentricity of the rotor, and \( k = \frac{3E/P}{2} \).

Putting \( z = x + iy \), Eq.(14) can be written as follows:

\[
Mz + k\left[x - r_0 \sin\left(\omega t + \frac{1}{2} \pi t^2\right)\right] = 0
\]

\[
My + k\left[y - r_0 \sin\left(\omega t + \frac{1}{2} \pi t^2\right)\right] = 0
\]

Using the same notations as in Chap.3 and putting \( z = \rho e^{i\theta} \), we can write Eq.(15) in the following form:

\[
z' + z = exp\left[i\theta \omega + \frac{1}{2} \pi \omega^2 \right]
\]

where ' denotes \( d/d\theta \).

The solution of this equation is

\[
z = z_0 + \cos \theta + B \sin \theta
\]

\[
z_0 = \int \left[ \sin(\theta - \psi) \exp\left[i(\theta - \psi) + \frac{1}{2} \pi \psi^2\right] d\psi
\]

where \( \psi \) represents the forced vibration due to the eccentricity. The second and third terms of the right side of Eq.(17) represent the free vibration which is determined by the initial conditions, \( A \) and \( B \) being arbitrary constants. By a simple calculation, we get

\[
z_0 = \frac{1}{2} \exp \left[i \theta \omega + \frac{1}{2} \pi \omega^2 \right] \left[ \int_0^\infty \exp \left[i \theta \omega + \frac{1}{2} \pi \omega^2 \right] d\omega
\]

where

\[
\psi = (\omega - 1)/\sqrt{\pi}, \quad \psi = (\omega + 1)/\sqrt{\pi}
\]

Expressing by \( z \) and \( y \), the real and imaginary parts of \( z \), namely the x- and y-components of vibration due to the eccentricity respectively, we get

\[
x = \frac{1}{2} \left[ A_1 \phi' + B_1 \phi'\right] + A_1 \phi + B_1 \phi
\]

\[
y = \frac{1}{2} \left[ A_2 \phi' + B_2 \phi'\right] + A_2 \phi + B_2 \phi
\]

where

\[
A_1 = C(\phi') - C(\phi'^*), \quad B_1 = S(\phi') - S(\phi'^*)
\]

\[
A_2 = C(\phi'^*) - C(\phi'), \quad B_2 = S(\phi'^*) - S(\phi')
\]

\[
f_1 = \sin \left[ \frac{1}{2} \omega (\omega - 1)^2 \right]
\]

\[
g_1 = \cos \left[ \frac{1}{2} \omega (\omega - 1)^2 \right]
\]

\[
f_2 = \sin \left[ \frac{1}{2} \omega (\omega + 1)^2 \right]
\]

\[
g_2 = \cos \left[ \frac{1}{2} \omega (\omega + 1)^2 \right]
\]

\[
C(x) = \frac{1}{2} \cos \frac{1}{2} \sqrt{\pi} \omega, \quad S(x) = \frac{1}{\sqrt{\pi}} \sin \frac{1}{2} \sqrt{\pi} \omega
\]

Here \( C(x) \) and \( S(x) \) are Fresnel's integrals, which can be evaluated with sufficient accuracy (order of \( 10^{-5} \)) by using approximate formulas in the series of polynomials \( n \).

Table 4 shows the values of \( r = \sqrt{x^2 + y^2} \) calculated by the formulas of Fresnel's integrals along with the results obtained by Runge-Kutta-Gill's numerical method in Chap.4. By this comparison, it is known that a value equal to or smaller than 0.05 is sufficient as the step size of integration of Runge-Kutta-Gill's numerical method.

6. Simulation by an analogue computer

Equations (10) and (13) can be easily simulated by an analogue computer. Fig.11 shows a circuit simulating Eq.(13) in the case where the rotor is a concentrated mass. The equations to be solved on the analogue computer are as follows:

\[
y'' = \frac{1}{C}\left(\Omega + x\omega\right)x'' - y
\]

\[
x'' = \frac{1}{C}\left(\Omega + x\omega\right)y'' + x - \frac{1}{C}\left(\Omega + x\omega\right)x
\]

\[
y'' = \frac{1}{C}\left(\Omega + x\omega\right)x'' - y
\]

where \( \tau \) is the machine time and \( \omega = \sigma \), \( \sigma \) being the scaling factor of time, and \( y = y_0 \), \( y_0 = y_0 \), \( \sigma \), \( \sigma \), etc., \( b \) being the scaling factor of dependent variables. The symbol ' denotes \( d/d\tau \).

Figs.12(a)-(c) are examples of the results of simulation. Figs.12(a)-(c) are the results of simulation of the cases

<table>
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<th>Fresnel</th>
<th>Runge-Kutta-Gill</th>
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<tr>
<td>( y )</td>
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Comparison of the exact solution expressed with Fresnel's integrals and the results obtained by Runge-Kutta-Gill's numerical method in the case where the rotor is a concentrated mass.

This table includes the values of \( \tau \) at \( \sigma = 1.1 \) when the initial conditions are all assumed to be zero.

![Analogue computer circuit](image)

\( M_\alpha = \text{multiplier}, F = \text{function generator} \)
in which only one among the six parameters \( \omega_s, \theta_0, \phi(0), \gamma(0), v(0), r(0) \) is non-zero respectively. In the Figs.12(b) and (c), especially in the latter, waves of higher frequency appear apparently. When the rotor is a concentrated mass, one of the two eigen-frequencies is zero and the other is \( 2\omega_s \), at the critical speed \( \omega_c \) observed in the rotating coordinate system. The minute vibration of higher frequency appearing in those figures is considered to be the latter eigen-vibration. In the cases of Figs. (b) and (c), buildup of the amplitude at the time of passing through the critical speed does not take place. This is easily understood because when we equate the eccentricity to zero in the equations of motion written in Chap.5, the forcing function of variable frequency disappears.

7. The case in which the angular acceleration is not constant

Up to the preceding chapter, we have assumed that the angular acceleration of the shaft is constant. Actually, however, the angular acceleration of the shaft is generally a function of the shaft speed and finally tends to zero, when the shaft speed reaches an ultimate constant value. As an example, let us consider the case in which the torque \( T \) decreases linearly with the shaft speed. We shall discuss the case in which the angular acceleration is positive, that is, the case of starting. We shall replace the third equation in Eq.(4) by

\[
\omega = \omega_\infty + (\omega_\infty - \omega_0)e^{-\alpha t} \]

where \( \alpha \) is a positive constant. Assuming \( \omega = \omega_0 \) at \( t=0 \), we get as the solution of Eq. (25):

\[
\omega = \omega_0 + (\omega_0 - \omega_\infty)e^{-\alpha t} \]

where \( \omega_0 = T_0/\omega_s \), which is the ultimate shaft speed. We shall denote by \( \omega \) the critical speed in the case of a symmetrical shaft system or the lower boundary of the critical speed band in the case of an asymmetrically shaft system, and denote by \( \omega_0 \) the angular acceleration of the shaft at \( \omega = \omega_0 \).

Using the same non-dimensional expressions as in the formula (9) and further putting

we easily obtain the following formulas from Eq.(26):

\[
\frac{\partial \omega}{\partial \phi} = \left[ \frac{\omega_0 - \omega_\infty}{\omega_0} \right] \frac{\partial \omega}{\partial \phi_\infty} \]

Using these formulas, we can write Eqs.(4) and (6) in the following non-dimensional form:

\[
B\phi'' + [(A-C)\phi(\theta) + (4/3)\beta \psi]^2 + (A-B-C)\phi(\theta)\psi - 2\psi = 2_\psi \]

\[
\phi'' + (A-B-C)\phi(\theta)\psi - 2\psi = 2_\psi \]

The solution of these equations can also be obtained by Runge-Kutta-Gill's numerical method. As an example, Table 5 shows the maximum values of \( \omega \) and \( \dot{\phi} \) for the same shaft system as that in Fig.2 and under such conditions as \( \omega_0 = 1.05\Omega, \omega_\infty = 3.5 \) and \( \alpha = 0.001-0.003 \). For the sake of comparison, the corresponding values obtained in the case of constant angular acceleration are also given in this table. The difference between the two cases are slight when \( \alpha = \alpha_0 \).

<table>
<thead>
<tr>
<th>( \alpha_0 )</th>
<th>( \Omega_{\max} )</th>
<th>( \phi_{\max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>0.001</td>
<td>47.76</td>
<td>60.84</td>
</tr>
<tr>
<td>0.002</td>
<td>48.49</td>
<td>61.72</td>
</tr>
<tr>
<td>0.003</td>
<td>48.78</td>
<td>61.42</td>
</tr>
</tbody>
</table>

\( A=3.4, B=1.3, C=0.001, \alpha = 1.0, \omega_0 = 0.05, \omega_\infty = 0.5, \omega_0 = 2.5 \).

Table 5. Comparison of the maximum amplitude and the maximum angular amplitude of the rotor between the cases where the shaft passes through the critical speed with a constant angular acceleration and with a variable angular acceleration.

(1) constant angular acceleration
(2) variable angular acceleration

Fig.12 Results of simulation by an analogue computer
8. Conclusion

In this paper, we introduced the equations of motion of the rotor when the shaft speed varies arbitrarily taking into account the gyroscopic effect of the rotor. Using these equations, the amplitude and angular amplitude of the rotor at the time of passing through the critical speed were obtained by Runge-Kutta-Gill's numerical method. In general, the maximum amplitude of the rotor at the time of passing through the critical speed becomes smaller as the angular acceleration of the shaft becomes larger. The amplitude is remarkably larger in the case of an asymmetrical shaft system in which the rotor or the bending rigidity of the shaft is asymmetrical as compared with the case of a symmetrical shaft system.

In this paper, we assumed that there are no damping effects. The author wishes to discuss the effect of the external and internal damping forces of the shaft system in another paper.

Appendix

The equation of precessional motion of a symmetrical rotor described in a static coordinate system is deduced by equating the rate of change of the angular momentum of the rotor about a coordinate axis perpendicular to the rotating shaft to the moment of the forces acting on the rotor about the axis. Applying a similar method to the rotating coordinate system described in Chap.2, we can deduce the same equations as Eqs.(4) without using Euler's equations. The method would be worthy of writing here in order to understand the relation between the equations based on the static coordinate system and those based on the rotating coordinate system.

We shall use the coordinate system xyz and \( \xi, \eta, \zeta \) described in Chap.2. Neglecting small quantities of order higher than the first, we get the following relations between the \( x, y, z \)- and \( \xi, \eta, \zeta \)-coordinates of an arbitrary point of the rotor:

\[
\begin{align*}
x &= \xi + \alpha \beta \eta - \alpha \zeta \\
y &= \gamma + \beta \zeta \\
z &= \gamma + \alpha \zeta
\end{align*}
\]  

where \( \gamma \) and \( \alpha \) denote the \( y \)- and \( z \)-coordinates of the center of gravity of the rotor. Denoting the moments about the \( z \)-axis of the centrifugal force, Coriolis's force and the virtual force due to the angular acceleration of the coordinate system by \( M_z, M_\phi \) and \( M_\theta \) respectively, we get:

\[
\begin{align*}
M_\phi &= -\omega_\phi \int \alpha \beta \eta \, d\alpha \beta \\
M_\theta &= 2\omega_\theta \int \alpha \beta \eta \, d\alpha \beta \\
M_z &= -\omega_\phi \int \alpha \beta \eta \, d\alpha \beta
\end{align*}
\]

which hold since the \( \xi, \eta, \zeta \)-axes are the principal axes of the rotor through its center of gravity.

\[
\begin{align*}
\int \xi \, dm &= \int \xi \, dm \\
\int \xi \, dm &= \int \xi \, dm
\end{align*}
\]

We get:

\[
\begin{align*}
M_x &= \omega_\phi \int (\xi \eta - \xi \zeta) \, d\alpha \beta \\
M_y &= 2\omega_\theta \int \xi \, dm \\
M_z &= \omega_\phi \int (\xi \eta - \xi \zeta) \, d\alpha \beta
\end{align*}
\]

Neglecting small quantities of order higher than the first, we get the following equation for the angular motion of the rotor about the \( z \)-axis:

\[
I \ddot \theta = M_x + M_\phi + M_z + M_\alpha (\eta - \eta_0)
\]

Substituting the formulas (32) in this equation, we get the first equation in Eqs. (4). The second equation in Eqs. (4) can be obtained similarly.

The moment about the \( x \)-axis of the virtual force due to the angular acceleration of the coordinate system is:

\[
M_x = -\omega_\phi \int (\xi \eta + \xi \zeta) \, d\alpha \beta
\]

Substituting Eq.(29) in this formula and considering the relations (31), we get:

\[
M_x = -\omega_\phi \int (\xi \eta + \xi \zeta) \, d\alpha \beta
\]

It is easily known that the moment of Coriolis's force about the \( x \)-axis is zero, and the moment of Coriolis's force about the \( x \)-axis is a small quantity of the second order. Therefore, for the angular acceleration \( \omega \) of the shaft, we get:

\[
T - \omega_\phi = 0
\]

which is the same as the third equation in Eqs. (4).

References