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The Behaviour of an Infinite Circular Bar subjected to Impulsive Bending Load*

By Kichinosuke TANAKA** and Chikashi MOTOYAMA***

Stresses, shearing force and displacement of the center line of an infinite circular bar are analyzed in the case that an impulsive load is applied normally to its surface. The analysis is based on the fundamental equations of three-dimensional elasticity. Laplace transform and Fourier transform techniques are used in the analysis. The inverse transforms are carried out with the help of stationary phase method. The results obtained by the analysis are compared with approximate theories (elementary theory, Rayleigh's theory and Timoshenko's theory). The applicabilities of the approximate theories are discussed.

1. Introduction

There are many studies of an infinite circular bar to whose side surface the static loads are applied based on the fundamental equations of three-dimensional elasticity⁽¹⁾⁽²⁾. Modes of these static loads are roughly grouped into three: one is normal loading mode by radial stress and the others are shearing loading modes by tangential or axial shear stress.

This paper deals with the problems of an infinite circular bar to whose side surface the impulsive loads are applied by radial stress, and analyzes its stresses, shearing forces and displacement of the center line based on the fundamental equations of three-dimensional elasticity. The impulsive load generates both dilatational waves and shear waves in the bar, which are transmitted as spherical waves, being reflected on the side surface. This is equivalent to the phenomenon that innumerable sinusoidal plane waves whose frequencies and wave lengths differ from one another are transmitted through the bar.

Laplace transform⁽³⁾ and Fourier transform⁽³⁾ techniques are utilized for the analysis. Utilization of these techniques means that the phenomena are regarded as superposition of above-mentioned sinusoidal plane waves. The inverse transforms are expressed by double Fourier integrals, whose integrands have so complex expressions containing transcendental functions, e.g. Bessel functions in circular cylindrical coordinates that complete calculations of the integrals are difficult in general. In this paper, asymptotic solu-

tions becoming more and more reasonable with the lapse of time are obtained by use of group velocity approximation (or stationary phase method)⁽⁴⁾⁽⁵⁾. A physical meaning of the group velocity approximation is that the phenomena expressed by superposition of sinusoidal waves are governed by the density of waves in the same phase.

2. Analysis

2.1 Fundamental equations

The cylindrical coordinates r , θ and z are used in the following. The z axis coincides with the axis of the circular bar of radius a . Time is denoted by t . The boundary conditions which hold at the surface are

$$\left. \begin{aligned} \sigma_{rr}(a, \theta, z, t) &= \sum_{n=0}^{\infty} F_n \cos(n\theta) G(z, t) \\ \sigma_{r\theta}(a, \theta, z, t) &= 0, \quad \sigma_{rz}(a, \theta, z, t) = 0 \end{aligned} \right\} \dots (1)$$

$G(z, t)$ is an arbitrary function and satisfies the condition $\lim_{z \rightarrow \infty} G(z, t) = 0$. The distribution of σ_{rr} at the surface is shown in Fig. 1 for $n = 0, 1$ and 2 . The surface forces are self-balanced except in the case of $n = 1$. It is supposed that the component of $n = 1$ plays a predominant role in the stress distributions on the distant cross-sections from the loaded point. The equation of motion of elastic body is expressed as

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \rho \ddot{\mathbf{u}} \dots \dots \dots (2)$$

Fig. 1 Distribution of $\sigma_{rr}(r=a)$

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** Professor, Faculty of Engineering, Kyoto University, Sakyo-ku, Kyoto.

*** Engineer, Ship Initial Designing Dept., Kawasaki Heavy Industries.

where u is the displacement vector, λ and μ , Lamé's constants, ρ , the density, and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

The solution of Eq. (2) is given as⁽⁶⁾

$$u = \nabla \varphi + \nabla \times \{ \nabla \times (\psi e_z) + \nabla \times \nabla \times (\chi e_z) \} \quad (3)$$

where e_z is the unit vector directed to the z axis. The functions φ , ψ and χ satisfy the equations

$$c_d^2 \nabla^2 \varphi = \ddot{\varphi}, \quad c_s^2 \nabla^2 \psi = \ddot{\psi}, \quad c_s^2 \nabla^2 \chi = \ddot{\chi} \quad (4)$$

where

$$\left. \begin{aligned} c_d &= \sqrt{(\lambda + 2\mu)/\rho} \text{ (dilatational wave velocity)} \\ c_s &= \sqrt{\mu/\rho} \text{ (shear wave velocity)} \end{aligned} \right\} \quad (5)$$

The displacements and the stresses can be expressed by φ , ψ and χ [Appendix (1)]. By considering the forms of the surface boundary conditions of Eq. (1), φ , ψ and χ are expressed as

$$\left. \begin{aligned} \varphi(r, \theta, z, t) &= \sum_{n=0}^{\infty} \varphi_n(r, z, t) \cos(n\theta) \\ \psi(r, \theta, z, t) &= \sum_{n=0}^{\infty} \psi_n(r, z, t) \cos(n\theta) \\ \chi(r, \theta, z, t) &= \sum_{n=1}^{\infty} \chi_n(r, z, t) \sin(n\theta) \end{aligned} \right\} \quad (6)$$

Applying the Fourier transformation and the Laplace transformation

$$\tilde{f}(\gamma) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) \exp(-i\gamma z) dz \quad (7)$$

$$\tilde{f}(p) \equiv \int_0^{\infty} f(t) \exp(-pt) dt \quad (8)$$

to Eq. (4) and using the relations of Eq. (6), the following equations are obtained,

$$\left. \begin{aligned} \tilde{\varphi}_n(r, \gamma, p) &= A_n(p, \gamma) J_n(q_1, r) \quad (n \geq 0) \\ \tilde{\psi}_n(r, \gamma, p) &= B_n(p, \gamma) J_n(q_2, r) \quad (n \geq 0) \\ \tilde{\chi}_n(r, \gamma, p) &= C_n(p, \gamma) J_n(q_2, r) \quad (n \geq 1) \end{aligned} \right\} \quad (9)$$

where J_n is a Bessel function of order n and q_1 and q_2 are denoted as

$$q_1^2 \equiv -(p^2/c_d^2 + \gamma^2), \quad q_2^2 \equiv -(p^2/c_s^2 + \gamma^2) \quad (10)$$

A_n , B_n and C_n are functions of parameters p and γ , and are determined by the boundary conditions. By use of Eqs. (1), (6) and (9), we have

$$\left. \begin{aligned} A_n(p, \gamma) &= -\tilde{G}(p, \gamma) (F_n a^2 / \mu) \\ &\times \tilde{J}(A_n) \tilde{J}_n(Q, \zeta) \quad (n \geq 0) \\ B_n(p, \gamma) &= -2i\tilde{G}(p, \gamma) (F_n a^3 / \mu) \\ &\times \tilde{J}(B_n) \tilde{J}_n(Q, \zeta) \quad (n \geq 0) \\ C_n(p, \gamma) &= -2\tilde{G}(p, \gamma) (F_n a^4 / \mu) \\ &\times \tilde{J}(C_n) \tilde{J}_n(Q, \zeta) \quad (n \geq 1) \end{aligned} \right\} \quad (11)$$

Introducing the non-dimensional quantities,

$$\alpha \equiv q_1 a, \quad \beta \equiv q_2 a, \quad Q \equiv -ipa/c_s, \quad \zeta \equiv \gamma a \quad (12)$$

$\tilde{A}(A_n)$, $\tilde{A}(B_n)$, $\tilde{A}(C_n)$ and $\tilde{A}$ are expressed by these quantities [Appendix (2)] and are

determined irrespective of the surface boundary conditions.

Introducing the non-dimensional quantities,

$$R \equiv r/a, \quad Z \equiv z/a, \quad T \equiv c_s t/a \quad (13)$$

the transforms of the stress components are given as

$$\left. \begin{aligned} &\{\bar{\sigma}_{rr}(R, \theta, \zeta, Q), \bar{\sigma}_{\theta\theta}, \bar{\sigma}_{zz}, \bar{\sigma}_{rz}\} \\ &= \sum_{n=0}^{\infty} \{\bar{\sigma}_{rr}^{(n)}(R, \zeta, Q), \bar{\sigma}_{\theta\theta}^{(n)}, \\ &\quad \bar{\sigma}_{zz}^{(n)}, \bar{\sigma}_{rz}^{(n)}\} \cos(n\theta) \\ &\{\bar{\sigma}_{\theta z}(R, \theta, \zeta, Q), \bar{\sigma}_{r\theta}\} \\ &= \sum_{n=1}^{\infty} \{\bar{\sigma}_{\theta z}^{(n)}(R, \zeta, Q), \bar{\sigma}_{r\theta}^{(n)}\} \sin(n\theta) \end{aligned} \right\} \quad (14)$$

The shear force, bending moment and displacement of the center line are defined as

$$\left. \begin{aligned} S(z, t) &\equiv \int_0^a \int_0^{2\pi} (\sigma_{rz} \cos \theta - \sigma_{\theta z} \sin \theta) r d\theta dr \\ M(z, t) &\equiv \int_0^a \int_0^{2\pi} \sigma_{rz} r^2 \cos \theta d\theta dr \\ u(z, t) &\equiv \lim_{r \rightarrow 0, \theta \rightarrow 0} u_r(r, \theta, z, t) \end{aligned} \right\} \quad (15)$$

In the following, the arbitrary function $G(z, t)$ in Eq. (1) is considered to be defined as

$$G(z, t) = \hat{o}(z) H(t) \quad (16)$$

or

$$G(z, t) = \hat{o}(z) \hat{o}(t) \quad (17)$$

where $\hat{o}$ is a Dirac delta function and H a Heaviside step function.

Under the condition of Eq. (16) or (17), the expressions for the transforms of stresses, shear force and displacement are obtained as Appendix (3). This shows that only the component of $n=1$ contributes to S and u .

The inverse transformation is defined as

$$\begin{aligned} f(Z, T) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(i\zeta Z) d\zeta \\ &\times \int_{-\infty}^{\infty} \tilde{f}(\zeta, Q) \exp(iQT) dQ \end{aligned} \quad (18)$$

The integral is calculated at first in the complex Q plane, and then, in the complex ζ plane. The residue theorem and Jordan's theorem are applied for the calculation in the Q plane.

The integrand has the poles which satisfy the equation $\tilde{A}_n(Q, \zeta) = 0$.

For the case of Eq. (16), $Q=0$ is also the pole.

2.2 Contribution from $Q=0$

To show the contribution from $Q=0$, suffix s is added to the left lower side of σ , u and S . The function $H(t)$ is omitted in the following.

(i) $n=0$

The stresses are expressed as

$$\begin{aligned}
 s\sigma_{rr}^{(0)}(R, Z, T) &= \frac{F_0}{2\pi a} \int_{-\infty}^{\infty} [AAA_1^{(0)}(\zeta) I_0(R\zeta) \\
 &+ \{AAA_2^{(0)}(\zeta)R + AAA_3^{(0)}(\zeta)/R\} I_1(R\zeta)] \\
 &\times \{\exp(i\zeta Z)/E_0(\zeta)\} d\zeta \\
 &= (iF_0/a) \sum [AAA_1^{(0)}(\zeta) I_0 + \{AAA_2^{(0)}(\zeta)R \\
 &+ AAA_3^{(0)}(\zeta)/R\} I_1] \{\exp(i\zeta Z)/E_0'(\zeta)\} \\
 &\quad (R < 1) \\
 s\sigma_{\theta\theta}^{(0)} &= \frac{iF_0}{a} \sum \{BBB_1^{(0)}(\zeta) I_0 \\
 &+ BBB_2^{(0)}(\zeta) I_1/R\} \{\exp(i\zeta Z)/E_0'(\zeta)\} \\
 s\sigma_{zz}^{(0)} &= \frac{iF_0}{a} \sum \{CCC_1^{(0)}(\zeta) I_0 \\
 &+ CCC_2^{(0)}(\zeta) I_1/R\} \{\exp(i\zeta Z)/E_0'(\zeta)\} \\
 s\sigma_{r\theta}^{(0)} &= 0, \quad s\sigma_{\theta z}^{(0)} = 0 \\
 s\sigma_{rz}^{(0)} &= -\frac{F_0}{a} \operatorname{sgn}(Z) \sum \{CAA_1^{(0)}(\zeta) R I_0 \\
 &+ CAA_2^{(0)}(\zeta) I_1\} \{\exp(i\zeta Z)/E_0'(\zeta)\} \\
 &\quad \dots\dots\dots (19)
 \end{aligned}$$

At $R=1$,

$$\begin{aligned}
 s\sigma_{rr}^{(0)}(1, Z, T) &= \frac{F_0}{2\pi a} \int_{-\infty}^{\infty} \exp(i\zeta Z) d\zeta \\
 &= \frac{F_0}{a} \delta(Z) \\
 s\sigma_{rz}^{(0)}(1, Z, T) &= 0 \\
 &\quad \dots\dots\dots (20)
 \end{aligned}$$

where $\operatorname{sgn}(Z) = +1 (Z > 0)$, $-1 (Z < 0)$. I_0 and I_1 are modified Bessel functions of zero and the first order, respectively. Introducing the parameter

$$k = c_s/c_d \quad \dots\dots\dots (21)$$

$AAA_i^{(0)}(\zeta)$ ($i=1\sim 3$), $BBB_i^{(0)}(\zeta)$ ($i=1, 2$), $CCC_i^{(0)}(\zeta)$ ($i=1, 2$), $CAA_i^{(0)}(\zeta)$ ($i=1, 2$) and $E_0(\zeta)$ are expressed as functions of ζ and k [Appendix (4)]. $\sum$ defines the total summation concerning all zero points [Appendix (5)] in the upper half plane of $E_0(\zeta)$:

$\zeta=0$ which is one of the zero points of $E_0(\zeta)$ is not a singular point of the integrands, and $E_0(\zeta)$ has no zero point on the real axis. Therefore, the stresses represented by Eqs. (19) and (20) are important only in the region near the loaded point. The stress distributions in the Z -direction at $R=0, 0.8$ and 1.0 are shown in Fig. 2.

(ii) $n=1$

The stresses, shear force and displacement are given by the following equations:

$$\begin{aligned}
 s\sigma_{rr}^{(1)}(R, Z, T) &= \frac{F_1}{2\pi a} \int_{-\infty}^{\infty} [\{AAA_1^{(1)}(\zeta)R + AAA_2^{(1)}(\zeta)/R\} I_0(R\zeta) \\
 &+ \{AAA_3^{(1)}(\zeta) + AAA_4^{(1)}(\zeta)/R^2\} I_1(R\zeta)] \{\exp(i\zeta Z)/E_1(\zeta)\} d\zeta \\
 &= \frac{iF_1}{a} \sum [\{AAA_1^{(1)}(\zeta)R + AAA_2^{(1)}(\zeta)/R\} I_0 + \{AAA_3^{(1)}(\zeta) \\
 &+ AAA_4^{(1)}(\zeta)/R^2\} I_1] \{\exp(i\zeta Z)/E_1'(\zeta)\} \quad (0 < R < 1) \\
 s\sigma_{\theta\theta}^{(1)} &= \frac{iF_1}{a} \sum \{BBB_1^{(1)}(\zeta) I_1 + BBB_2^{(1)}(\zeta) I_2/R\} \{\exp(i\zeta Z)/E_1'(\zeta)\} \\
 s\sigma_{zz}^{(1)} &= \frac{F_1}{2\pi a} \int_{-\infty}^{\infty} [\{CCC_1^{(1)}(\zeta) R I_0 + CCC_2^{(1)}(\zeta) I_1\} / E_1(\zeta)] \exp(i\zeta Z) d\zeta \\
 s\sigma_{\theta z}^{(1)} &= \operatorname{sgn}(Z) \frac{F_1}{a} \left[\frac{3+2\nu}{4(1+\nu)} \left(1 - \frac{1-2\nu}{3+2\nu} R^2 \right) - \sum \{BCC_1^{(1)}(\zeta) I_0 \right. \\
 &\quad \left. + BCC_2^{(1)}(\zeta) I_1/R\} \{\exp(i\zeta Z)/E_1'(\zeta)\} \right]
 \end{aligned}$$

Fig. 2 Distributions of $s\sigma_{rr}^{(0)}$, $s\sigma_{zz}^{(0)}$, $s\sigma_{\theta\theta}^{(0)}$ and $s\sigma_{rz}^{(0)}$ in axial direction

$$\begin{aligned}
{}_s\sigma_{rz}^{(1)} &= \text{sgn}(Z) \frac{F_1}{a} \left[\frac{3+2\nu}{4(1+\nu)} (R^2-1) - \sum [CAA_1^{(1)}(\zeta)R + CAA_2^{(1)}(\zeta)/R] I_1 \right. \\
&\quad \left. + CAA_3^{(1)}(\zeta)I_2 \right] \{\exp(i\zeta|Z|)/E_1'(\zeta)\} \\
{}_su(Z, T) &= \frac{F_1}{2\pi\mu} \int_{-\infty}^{\infty} \{uuu(\zeta)/E_1(\zeta)\} \exp(i\zeta Z) d\zeta \\
{}_sS(Z, T) &= -\text{sgn}(Z) \pi a F_1/2
\end{aligned}
\tag{22}$$

At $R=0$ and 1,

$${}_s\sigma_{rr}^{(1)}(0, Z, T) = 0$$

$$\begin{aligned}
{}_s\sigma_{rr}^{(1)}(1, Z, T) &= \frac{F_1}{2\pi a} \int_{-\infty}^{\infty} \exp(i\zeta Z) d\zeta \\
&= \frac{F_1}{a} \delta(Z) \\
{}_s\sigma_{rz}^{(1)}(1, Z, T) &= 0
\end{aligned}
\tag{23}$$

$AAA_i^{(1)}$ ($i=1\sim 4$), $BBB_i^{(1)}$ ($i=1, 2$), $CCC_i^{(1)}$ ($i=1, 2$), $BCC_i^{(1)}$ ($i=1, 2$), $CAA_i^{(1)}$ ($i=1\sim 3$), uuu and E_1 are functions of k and ζ [Appendix (6)]. Σ defines the total summation concerning all zero points except $\zeta=0$ in the upper half ζ plane of $E_1(\zeta)$ [Appendix (7)].

${}_s\sigma_{\theta z}^{(1)}$, ${}_s\sigma_{rz}^{(1)}$ and ${}_sS$ have the contributions from $\zeta=0$ and the contributions are given by the terms except Σ terms in Eq. (22).

Because $\zeta=0$ is the pole of the second and the fourth orders for ${}_s\sigma_{zz}^{(1)}$ and ${}_su$, respectively, ${}_s\sigma_{zz}^{(1)}$ and ${}_su$ are replaced by ${}_s\sigma_{zz}^{(1)'}$ and ${}_su'$ respectively. They are defined as

$$\begin{aligned}
{}_s\sigma_{zz}^{(1)'} &= \frac{F_1}{2\pi a} \int_{-\infty}^{\infty} [(CCC_1^{(1)}(\zeta)RI_0 + CCC_2^{(1)}(\zeta)I_1)/E_1(\zeta) - 4R/\zeta^2] \exp(i\zeta Z) d\zeta \\
{}_su' &= \frac{F_1}{2\pi\mu} \int_{-\infty}^{\infty} \left\{ uuu(\zeta)/E_1(\zeta) - \frac{2}{1+\nu} \frac{1}{\zeta^4} \right\} \exp(i\zeta Z) d\zeta
\end{aligned}
\tag{22}'$$

The differences between ${}_s\sigma_{zz}^{(1)}$ and ${}_s\sigma_{zz}^{(1)'}$ and between ${}_su$ and ${}_su'$ are added as the first term in Eq. (32) and Eq. (30), respectively. Addition of these terms makes possible the existence of the integral values of ${}_s\sigma_{zz}^{(1)}$ and ${}_su$. Σ terms in Eq. (22) are important only in the region near the loaded point. The stress distributions in the R -direction at $Z=0.4$ and 1.0 are shown in Fig. 3.

(iii) $n=2$

The stresses are expresses as

$$\begin{aligned}
{}_s\sigma_{rr}^{(2)}(R, Z, T) &= \frac{F_2}{2\pi a} \int_{-\infty}^{\infty} [(AAA_1^{(2)}(\zeta)R + AAA_2^{(2)}(\zeta)/R) I_1(R\zeta) \\
&\quad + \{AAA_3^{(2)}(\zeta) + AAA_4^{(2)}/R^2\} I_2(R\zeta)] \{\exp(i\zeta Z)/E_2(\zeta)\} d\zeta \\
&= \frac{iF_2}{a} \sum [(AAA_1^{(2)}(\zeta)R + AAA_2^{(2)}(\zeta)/R) I_1 + \{AAA_3^{(2)}(\zeta) \\
&\quad + AAA_4^{(2)}/R^2\} I_2] \{\exp(i\zeta|Z|)/E_2'(\zeta)\} \\
{}_s\sigma_{\theta\theta}^{(2)} &= \frac{iF_2}{a} \sum [BBB_1^{(2)}(\zeta)I_1/R + \{BBB_2^{(2)}(\zeta) + BBB_3^{(2)}(\zeta)/R^2\} I_2] \\
&\quad \times \{\exp(i\zeta|Z|)/E_2'(\zeta)\} \\
{}_s\sigma_{zz}^{(2)} &= \frac{iF_2}{a} \sum \{CCC_1^{(2)}(\zeta)RI_1 + CCC_2^{(2)}(\zeta)I_2\} \{\exp(i\zeta|Z|)/E_2'(\zeta)\} \\
{}_s\sigma_{\theta z}^{(2)} &= -\frac{F_2}{a} \text{sgn}(Z) \sum \{BCC_1^{(2)}(\zeta)I_1 + BCC_2^{(2)}(\zeta)I_2/R\} \{\exp(i\zeta|Z|)/E_2'(\zeta)\} \\
{}_s\sigma_{rz}^{(2)} &= -\frac{F_2}{a} \text{sgn}(Z) \sum [CAA_1^{(2)}(\zeta)I_1 + \{CAA_2^{(2)}(\zeta)/R + CAA_3^{(2)}(\zeta)R\} I_2] \\
&\quad \times \{\exp(i\zeta|Z|)/E_2'(\zeta)\}
\end{aligned}
\tag{24}$$

Fig. 3 Distributions of ${}_s\sigma_{rr}^{(1)}$, ${}_s\sigma_{\theta\theta}^{(1)}$, ${}_s\sigma_{zz}^{(1)}$, ${}_s\sigma_{\theta z}^{(1)}$ and ${}_s\sigma_{rz}^{(1)}$ in radial direction

At $R=1$,

$$\left. \begin{aligned} {}_s\sigma_{rr}^{(2)}(1, Z, T) &= \frac{F_2}{2\pi a} \int_{-\infty}^{\infty} \exp(i\zeta Z) d\zeta \\ &= \frac{F_2}{a} \delta(Z), \\ {}_s\sigma_{rz}(1, Z, T) &= 0 \end{aligned} \right\} \dots (25)$$

$AAA_i^{(2)} (i=1\sim 4)$, $BBB_i^{(2)} (i=1\sim 3)$, $CCC_i^{(2)} (i=1, 2)$, $BCC_i^{(2)} (i=1, 2)$, $CAA_i^{(2)} (i=1\sim 3)$ and E_2 are functions of ζ and k [Appendix (8)]. Σ defines the total summation concerning all zero points in the upper half ζ plane of $E_2(\zeta)$ [Appendix (9)]. The stresses expressed by Eq. (24) are important only in the region near the loaded point.

(iv) $n \geq 3$

$E_n(\zeta)$ has one real root of $\zeta=0$ and infinite numbers of imaginary, and complex roots. The integrands have no singularities at $\zeta=0$. Therefore, the stresses for $n \geq 3$ are important only in the region near the loaded point.

2.3 Contribution from $\mathcal{Q}$ satisfying $\bar{J}_n(\mathcal{Q}, \zeta) = 0$

$\bar{J}_n(\mathcal{Q}, \zeta)$ is a function of $\mathcal{Q}$, ζ and k (or Poisson's ratio ν). $\bar{J}_n(\mathcal{Q}, \zeta) = 0$ has always the roots of $\mathcal{Q}=0$, $\mathcal{Q}=\pm\zeta$ (whose phase velocity corresponds to the shear wave velocity) and $\mathcal{Q}=\pm\zeta/k$ (whose phase velocity corresponds to the dilatational wave velocity). These self-evident roots

are excluded from the following discussion of the roots of $\bar{J}_n(\mathcal{Q}, \zeta) = 0$. Because the integral path is on the real axis in the complex ζ plane, ζ can be assumed as the real number. And only the real roots of $\mathcal{Q}$ are considered in view of the character of this problem.

$\bar{J}_n(\mathcal{Q}, \zeta)$ is an even function of $\mathcal{Q}$ and ζ , and the relations of $\mathcal{Q}$ and ζ drawn on $(\zeta, \mathcal{Q})$ plane are symmetric about both ζ -axis and $\mathcal{Q}$ -axis. Thus, it is sufficient to investigate the characters of the roots of $\bar{J}_n(\mathcal{Q}, \zeta) = 0$ only in the region of $\zeta, \mathcal{Q} \geq 0$. When $\mathcal{Q} = \mathcal{Q}_n^m(\zeta)$ ($\mathcal{Q} \geq 0, -\infty \leq \zeta \leq \infty$) is the m -th root of $\bar{J}_n(\mathcal{Q}, \zeta) = 0$ [$0 \leq \mathcal{Q}_n^1(0) < \mathcal{Q}_n^2(0) < \dots < \mathcal{Q}_n^{m-1}(0) < \mathcal{Q}_n^m(0) < \mathcal{Q}_n^{m+1}(0) < \dots$], $\mathcal{Q} = -\mathcal{Q}_n^m(\zeta)$ is also roots and these are generally simple roots. ζ - $\mathcal{Q}$ relations are known by the name of the first mode, second mode, corresponding to $m=1, m=2, \dots$ respectively and show each characteristic property [Appendix (10)].

The roots of $\bar{J}_n(\mathcal{Q}, \zeta) = 0$ ($n=0, 1, 2$) calculated by the computer are shown in Fig. 4. The ζ - $\mathcal{Q}$ relations for $n \geq 3$ are similar to those for $n=2$. Linear relations shown by the lines of OA, OB and OC in the figures show the phase velocities equal to Rayleigh's surface wave velocity, shear wave velocity and dilatational wave velocity, respectively. The contributions from $\mathcal{Q}$ satisfying $\bar{J}_n(\mathcal{Q}, \zeta) = 0$ are denoted by attaching the suffix d to the left lower side of σ , u and S . In the case of $G = \delta(z)\delta(t)$, ${}_d\sigma$, ${}_du$ and ${}_dS$ are expressed as

$$\left. \begin{aligned} {}_d\sigma_{ij}^{(n)}(R, Z, T) &= \frac{F_n}{\pi a} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \left[R_i R_j^{(n)}(R, \zeta, \mathcal{Q}) / \mathcal{Q} \frac{\partial}{\partial \mathcal{Q}} \bar{J}_n(\mathcal{Q}, \zeta) \right] \{ \exp i(\mathcal{Q}T + \zeta Z) \} \Big|_{\mathcal{Q}=\pm \mathcal{Q}_n^m(\zeta)} d\zeta \\ {}_du(Z, T) &= \frac{F_1}{2\pi \mu} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \left[uu(\mathcal{Q}, \zeta) / \mathcal{Q} \frac{\partial}{\partial \mathcal{Q}} \bar{J}_1(\mathcal{Q}, \zeta) \right] \{ \exp i(\mathcal{Q}T + \zeta Z) \} \Big|_{\mathcal{Q}=\pm \mathcal{Q}_1^m(\zeta)} d\zeta \\ {}_dS(Z, T) &= \frac{F_1 a}{i} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \left[SS(\mathcal{Q}, \zeta) / \mathcal{Q} \frac{\partial}{\partial \mathcal{Q}} \bar{J}_1(\mathcal{Q}, \zeta) \right] \{ \exp i(\mathcal{Q}T + \zeta Z) \} \Big|_{\mathcal{Q}=\pm \mathcal{Q}_1^m(\zeta)} d\zeta \end{aligned} \right\} \dots (26)$$

in which the function of $H(T)$ is omitted. In the case of $G = \delta(z)\delta(t)$, the integrand should be multiplied by $i\epsilon\mathcal{Q}/a$.

In relation to 2.2 (ii), $(2R/\zeta^2)\exp(i\zeta Z)$ and $2/(1+\nu)\zeta^4\exp(i\zeta Z)$ are added to the integrands of ${}_d\sigma_{zz}^{(1)}$ and ${}_du$ respectively, in the case of $m=1$. In the case of $G = \delta(z)\delta(t)$ no modification of the integrands is needed. The respective suffixes i and

Fig. 4 Relations between ζ and $\mathcal{Q}$

i are arbitrary ones of r, θ and z . R_i indicates A, B or C corresponding to $i = r, \theta$ or z , respectively. $R_i R_j^{(n)}$ represents $AA^{(n)}, BB^{(n)}$ and others defined in Appendix (3). The integrations in Eq. (26) have a form of

$$I(T) = \int_A^B \chi(\zeta) \exp\{iT\psi(\zeta)\} d\zeta \quad \dots\dots\dots (27)$$

In general, for the complexity of $\chi(\zeta)$ it is difficult to carry out the integration of Eq. (27). In such cases, the asymptotic values are obtained by use of the group velocity approximation [Appendix (11)].

The function ψ in Eq. (27) corresponds to $\pm Q_n^{(m)}(\zeta) + \zeta Z/T$ in Eq. (26). $\psi' = 0$ corresponds to $\pm (dQ_n^{(m)}/d\zeta)_{\zeta=\zeta_0} = Z/T$, and $\psi^{(i)} = 0$ ($i \geq 2$) to $(d^i Q_n^{(m)}/d\zeta^i)_{\zeta=\zeta_0} = 0$. $(dQ_n^{(m)}/d\zeta)$ is called the group velocity and is a function of ζ - $(dQ_n^{(m)}/d\zeta)$ are shown in Fig. 5 for $n=0$ and $m=1 \sim 5$, in Fig. 6 for $n=1$ and $m=1, 2$, in Fig. 7 for $n=1$ and $m=3 \sim 8$ and in Fig. 8 for $n=2$ and $m=1 \sim 5$.

2.4 Calculation of $sS(Z, T)$

sS in Eq. (26) is given by the following expression (28), in which the results for the cases (b)~(e) are obtained by the group velocity approximation.

Fig. 5 Relations between $(dQ_0^m/d\zeta)$ and ζ

Fig. 6 Relations between $(dQ_1^m/d\zeta)$ and ζ

(i) Contribution from the first mode $m=1$

(a) $|Z|/T < \delta$

$$\pi a F_1 \operatorname{sgn}(Z) \{C(B) + S(B)\} / 2 \quad (27)$$

(b) $\delta \leq |Z|/T < 0.932 (= c_R/c_s |_{\nu=1/3})$

$$\pi a F_1 \operatorname{sgn}(Z) \left\{ \frac{1}{2} - \sqrt{\frac{8}{\pi}} \frac{SS^{(1)}(\zeta_1)}{\sqrt{\frac{d^3 Q_1^{(1)}}{d\zeta^3}(\zeta_1)}} \frac{1}{\sqrt{T}} \right. \\ \left. \times \sin \left[T \left\{ Q_1^{(1)}(\zeta_1) - \frac{|Z|}{T} \zeta_1 \right\} + \frac{\pi}{4} \right] + O\left(\frac{1}{T}\right) \right\}$$

(c) $0.932 \leq |Z|/T < \text{max. group velocity}$

$$\pi a F_1 \operatorname{sgn}(Z) \left\{ \frac{1}{2} - \sqrt{\frac{8}{\pi}} \sum_{r=1,2} \frac{SS^{(1)}(\zeta_{1r})}{\sqrt{\frac{d^3 Q_1^{(1)}}{d\zeta^3}(\zeta_{1r})}} \right. \\ \left. \times \frac{1}{\sqrt{T}} \sin \left[T \left\{ Q_1^{(1)}(\zeta_{1r}) - \frac{|Z|}{T} \zeta_{1r} \right\} - (-1)^r \frac{\pi}{4} \right] + O\left(\frac{1}{T}\right) \right\}$$

(d) $|Z|/T \cong \text{max. group velocity}$

$$\pi a F_1 \operatorname{sgn}(Z) \left\{ \frac{1}{2} - \frac{4SS^{(1)}(\zeta_2)}{\sqrt{\frac{1}{2} \frac{d^3 Q_1^{(1)}}{d\zeta^3}(\zeta_2)}} \right. \\ \left. \times \frac{1}{\sqrt{T}} \operatorname{Ai} \left[\left\{ \frac{dQ_1^{(1)}}{d\zeta}(\zeta_2) - \frac{|Z|}{T} \right\} \right. \right. \\ \left. \left. \times \frac{\sqrt[3]{T^2}}{\sqrt{\frac{1}{2} \frac{d^3 Q_1^{(1)}}{d\zeta^3}(\zeta_2)}} \right] \sin \left[T \left\{ Q_1^{(1)}(\zeta_2) - \zeta_2 |Z|/T \right\} \right] + O\left(\frac{1}{T}\right) \right\}$$

(e) $|Z|/T > \text{max. group velocity}$

$$\pi a F_1 \operatorname{sgn}(Z) \frac{1}{2} + O(1/T) \quad \dots\dots\dots (28)$$

$$\text{where } B \equiv \sqrt{\frac{c_s}{c_E}} \frac{|Z|}{\sqrt{T}} (= \sqrt{\frac{|z|}{c_E a t}})$$

$$\left. \begin{aligned} C(B) &\equiv \sqrt{\frac{2}{\pi}} \int_0^B \cos x^2 dx \\ S(B) &\equiv \sqrt{\frac{2}{\pi}} \int_0^B \sin x^2 dx \\ \operatorname{Ai}(x) &= \frac{1}{\pi} \int_0^\infty \cos \left(sx + \frac{s^3}{3} \right) ds \end{aligned} \right\} \dots\dots\dots (29)$$

Fig. 7 Relations between $(dQ_1^m/d\zeta)$ and ζ

c_E is the one-dimensional longitudinal wave velocity and is equal to $\sqrt{E/\rho}$ (E is Young's modulus). ζ_1 and ζ_2 are denoted as follows. In Fig. 6, let us draw a parallel line to the abscissa crossing the ordinate at $|Z|/T$ and read the abscissa ζ_0 of the crossing point of the above-mentioned line and the curve. When $(d^2 Q_1/d\zeta^2)_{\zeta=\zeta_0} \neq 0$ is satisfied, let ζ_0 be denoted by ζ_1 (when two values of ζ_0 are obtained, let two ζ_1 be denoted by ζ_{11} and ζ_{12}). When $(d^2 Q_1/d\zeta^2)_{\zeta=\zeta_0} = 0$ and $(d^3 Q_1/d\zeta^3)_{\zeta=\zeta_0} \neq 0$ are satisfied, let ζ_0 be denoted by ζ_2 . When ζ is small, $Q_1(\zeta)$ can be expressed by approximate form written in Appendix (10).

The shear force distribution at $T=3$ obtained by adding Eq. (22) to Eq. (28) is shown in Fig. 9. The broken lines in the figure correspond to the cases (a)~(d) in Eq. (28), and the full line is obtained by connecting the broken lines smoothly.

(ii) Contribution from the higher modes ($m \geq 2$)

The contribution from the second mode ($m=2$) is illustrated in Fig. 9. It is small and can be neglected in the region of $|Z|/T$ smaller than about 0.7, but cannot be neglected in the region where the value of $|Z|/T$ is nearly equal to the maximum group velocity.

In the case of $m \geq 3$, there are many extremities as shown in Fig. 7. In the region where the value of $|Z|/T$ is nearly equal to the maximum velocity, there are contributions of $O(T^{-1/3})$ at least. But as the extremities exist in the region of rather high frequencies, the contributions are $O(1/Q)$ at least, and should be smaller than those of the lower modes. Therefore, Fig. 9 can illustrate the basically predominant features of the shear force.

2.5 Shear force distribution in the region $|Z|/T < \delta$

In the region $|Z|/T < \delta$, the final solution of shear force can be obtained by adding Eq. (22) to region (a) of Eq. (28). Fig. 10 shows the shear force distribution taking the parameter B as the abscissa. In the case of $G = \delta(z)\delta(t)$, the Fourier transform of S is finite in all region of ζ . Therefore, the region (a) in Eq. (28) does not exist and the solution represents only the vibration whose amplitude diminishes with the time T in all region of $|Z|/T$.

2.6 Displacement of the center line

The displacement of the center line depends only on the component of $n=1$. In the region $|Z|/T > \delta$, the solution gives the vibration whose amplitude diminishes with the time T . In the region $|Z|/T < \delta$, the contribution from the first mode is given in the form in the case of $G = \delta(z)H(t)$

$$\begin{aligned} u(Z, T) &= \frac{4F_1}{\pi E} \int_0^\infty \left[[1 - \cos\{\sqrt{(1+\nu)/2} T \zeta^2\}] / \zeta^4 \right] \cos(\zeta Z) d\zeta + O(\sqrt{T}) \\ &= \frac{4F_1}{\pi E} \left\{ \sqrt{(1+\nu)/2} T \right\}^{3/2} \int_0^\infty \frac{1 - \cos x^2}{x^4} \cos(\sqrt{2} B x) dx + O(\sqrt{T}) \dots \dots \dots (30) \end{aligned}$$

and also in the case of $G = \delta(z)\delta(t)$

$$\begin{aligned} u(Z, T) &= \frac{2F_1 c_E}{\pi a E} \int_0^\infty \left[[\sin\{\sqrt{(1+\nu)/2} T \zeta^2\}] / \zeta^2 \right] \cos(\zeta Z) d\zeta + O\left(\frac{1}{\sqrt{T}}\right) \\ &= \frac{F_1 c_E}{a E} |Z| \left\{ S\left(\frac{B}{\sqrt{2}}\right) - C\left(\frac{B}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} \frac{1}{B} \sin\left(\frac{B^2}{2} + \frac{\pi}{4}\right) \right\} + O\left(\frac{1}{\sqrt{T}}\right) \dots \dots \dots (31) \end{aligned}$$

Fig. 8 Relations between $(dQ_2^m/d\zeta)$ and ζ

Fig. 9 Distributions of shearing forces

Fig. 10 Distribution of shearing forces

Eq. (30) and Eq. (31) show that the displacements at $Z=0$ are $O(T\sqrt{T})$ and $O(\sqrt{T})$, and the velocities at $Z=0$ are $O(\sqrt{T})$ and $O(1/\sqrt{T})$ respectively.

2.7 Stresses

The features of $\sigma_{\theta z}$ and σ_{rz} are similar to that of the shear force S . In the case of $G=\delta(z)H(t)$, the contributions from the first mode of the component of $n=1$ are predominant in the region $|Z|/T < \delta$ and are given by

$$\left. \begin{aligned} d\sigma_{\theta z}^{(1)}(R, Z, T) &\sim -\frac{F_1}{a} \operatorname{sgn}(Z) \frac{3+2\nu}{4(1+\nu)} \left(1 - \frac{1-2\nu}{3+2\nu} R^2\right) \{C(B) + S(B)\} \\ d\sigma_{rz}^{(1)}(R, Z, T) &\sim -\frac{F_1}{a} \operatorname{sgn}(Z) \frac{3+2\nu}{4(1+\nu)} (R^2 - 1) \{C(B) + S(B)\} \\ d\sigma_{zz}^{(1)}(R, Z, T) &\sim \frac{4F_1}{\pi a} \int_0^\infty \left[[1 - \cos\{\sqrt{(1+\nu)/2} T \zeta^2\}] / \zeta^2 \right] \cos(Z\zeta) d\zeta R \\ &\sim \frac{4F_1}{\pi a} \{\sqrt{(1+\nu)/2} T\}^{1/2} \int_0^\infty \frac{1 - \cos x^2}{x^2} \cos(\sqrt{2} Bx) dx R \end{aligned} \right\} \dots (32)$$

The distributions of $\sigma_{\theta z}$, σ_{rz} and σ_{zz} obtained by adding Eq. (22), from which Σ term is excluded, to Eq. (32) are shown in Figs. 11, 12 and 13, respectively. In the case of $G=\delta(z)H(t)$, all the stresses except $\sigma_{\theta z}^{(1)}$, $\sigma_{rz}^{(1)}$, $\sigma_{zz}^{(1)}$ for $m=1$ vibrate with the amplitudes which diminish with time T .

In the case of $G=\delta(z)\delta(t)$, the stresses vibrate diminishingly in all region of $|Z|/T$.

3. Discussion

3.1 Discussion on predominant component

In the preceding section, it has been clarified that the most predominant component of Fourier expansions of external load is the component of $n=1$ except in

the neighborhood of the loaded point. Some of the asymptotic solutions obtained in the preceding section are important because they do not diminish with the lapse of time in the region $|Z|/T < \delta$, where the first mode ($m=1$) of Fourier component $n=1$ is the most predominant.

3.2 Discussion on δ

On calculating roots of $\bar{J}_n(Q, \zeta) = 0$ analytically, approximation such as shown in Appendix (10), that is,

$$Q_1'(\zeta) \sim \left(\frac{1+\nu}{2}\right)^{1/2} \zeta^2$$

is postulated in the region of small $|\zeta|$. In order to allow large values of δ , $Q_1'(\zeta)$ should be approximated to a higher order of ζ . Even if the value of δ is small, an applicable region of $|Z|$ becomes wider with the lapse of time T .

3.3 Comparison of approximate theories

Comparison has been made between the exact theory and several approximate theories from a viewpoint of dispersion relations⁽⁸⁾. In Fig. 6, group velocities obtained by several approximate theories are shown together with the exact theory.

Elementary theory and Rayleigh's theory which takes rotatory inertia into consideration have only one mode conforming to the first mode of the present exact theory in the region of small ζ . Therefore, elementary theory is sufficiently applicable in the region $|Z|/T < \delta$ if the

Fig. 11 Distribution of $\sigma_{\theta z}$

Fig. 12 Distribution of σ_{rz}

Fig. 13 Distribution of σ_{zz}

first mode is predominant.

Timoshenko's theory taking both the rotatory inertia and the shear deformation into consideration has two modes. The first mode well conforms to that of the exact theory in the region of all ζ , but second mode does not except in the region of small ζ . Timoshenko's theory is sufficiently applicable in the region of all $|Z|/T$ if the first mode is predominant, but not applicable except in the region of small $|Z|/T$ if the second mode is predominant.

4. Conclusions

(1) Stresses, shearing forces and deflec-

tion of the center line of the infinite circular bar to whose surface the impulsive loads are normally applied, are clarified. Asymptotic solutions are obtained in two different load cases.

(2) The most predominant component of Fourier series of load is the first mode ($m=1$) of component $n=1$ except in the neighborhood of the loaded point in the normal loading.

(3) Stresses are maximum in the neighborhood of the loaded point in the normal loading.

(4) Elementary theory gives good approximations under appropriate loading conditions.

Appendix

$$(1) \quad \begin{aligned} u_r &= \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} - \frac{1}{c_s^2} \frac{1}{r} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \chi}{\partial \theta} \right), \quad u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \chi}{\partial r} \right), \quad u_z = \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2} \\ \sigma_{rr} &= \frac{\lambda}{c_d^2} \ddot{\varphi} + 2\mu \left\{ \frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^3 \psi}{\partial r^2 \partial z} + \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{r^2} \frac{\partial \chi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \chi}{\partial r \partial \theta} \right) \right\} \\ \sigma_{\theta\theta} &= \frac{\lambda}{c_d^2} \ddot{\varphi} + 2\mu \left\{ \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^3 \psi}{\partial \theta^2 \partial z} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{r} \frac{\partial^2 \chi}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} \right) \right\} \\ \sigma_{zz} &= \frac{\lambda}{c_d^2} \ddot{\varphi} + 2\mu \left\{ \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^3 \psi}{\partial z^3} - \frac{1}{c_s^2} \frac{\partial^3 \psi}{\partial t^2 \partial z} \right\} \\ \sigma_{r\theta} &= \mu \left\{ \frac{2}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial \varphi}{\partial \theta} + \frac{2}{r} \frac{\partial^3 \psi}{\partial r \partial \theta \partial z} - \frac{2}{r^2} \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \right) \right\} \\ \sigma_{rz} &= \mu \left\{ 2 \frac{\partial^2 \varphi}{\partial r \partial z} + 2 \frac{\partial^3 \psi}{\partial r \partial z^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{r} \frac{\partial^2 \chi}{\partial \theta \partial z} + \frac{\partial \psi}{\partial r} \right) \right\} \\ \sigma_{\theta z} &= \mu \left\{ \frac{2}{r} \frac{\partial^2 \varphi}{\partial \theta \partial z} + \frac{2}{r} \frac{\partial^3 \psi}{\partial \theta \partial z^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial^2 \chi}{\partial r \partial z} \right) \right\} \end{aligned}$$

(2) In the case of $n=0$

$$\bar{A}(A_0) \equiv (\beta^2 - \zeta^2) \beta J_1(\beta), \quad \bar{A}(B_0) \equiv -\zeta \alpha J_1(\alpha)$$

In the case of $n \geq 1$

$$\begin{aligned} \bar{A}(A_n) &\equiv 2n^2 \zeta^2 J_n(\beta) - \{2n^2 - (\beta^2 - \zeta^2)\} \beta^3 J_n(\beta) J_n'(\beta) + 2(\beta^2 - \zeta^2) \beta^2 \{J_n'(\beta)\}^2 \\ \bar{A}(B_n) &\equiv n^2 \zeta J_n(\alpha) J_n(\beta) + (n^2 - \beta^2) \zeta \alpha J_n'(\alpha) J_n(\beta) - 2\zeta \alpha \beta J_n'(\alpha) J_n'(\beta) \\ \Omega^2 \bar{A}(C_n) &\equiv n \alpha \beta \Omega^2 J_n'(\alpha) J_n'(\beta) - n(\beta^2 - \zeta^2) \beta J_n(\alpha) J_n'(\beta) - 2n \zeta^2 \alpha J_n'(\alpha) J_n(\beta) \end{aligned}$$

In the case of $n=0$

$$\bar{A}_0(\Omega, \zeta) \equiv \{(\beta^2 - \zeta^2) J_0(\alpha) - 2\alpha J_1(\alpha)\} (\beta^2 - \zeta^2) \beta J_1(\beta) + 4\zeta^2 \alpha J_1(\alpha) \{\beta^2 J_0(\beta) - \beta J_1(\beta)\}$$

In the case of $n \geq 1$

$$\begin{aligned} \bar{A}_n(\Omega, \zeta) &\equiv f_1^n \beta^2 J_n(\alpha) J_{n-1}^2(\beta) + f_2^n \alpha \beta J_{n-1}(\alpha) J_{n-1}(\beta) J_n(\beta) + f_3^n \beta J_n(\alpha) J_{n-1}(\beta) J_n(\beta) \\ &\quad + f_4^n \alpha J_{n-1}(\alpha) J_n^2(\beta) + f_5^n J_n(\alpha) J_n^2(\beta) + f_6^n \alpha J_{n-1}(\alpha) J_{n-1}^2(\beta) \end{aligned}$$

where

$$\begin{aligned} f_1^n &\equiv 2(\beta^2 - \zeta^2)^2 + 4(\beta^2 + \zeta^2)(n-1)n(n+1), \quad f_2^n \equiv 2\beta^2(\beta^2 + 5\zeta^2) + 8(\beta^2 + \zeta^2)(n-1)n(n+1) \\ f_3^n &\equiv \beta^6 - \{2\zeta^2 + 2n(2n+3)\} \beta^4 + \{\zeta^4 + (2n-1)2n\zeta^2 - 4(n-1)n^2(n+1)\} \beta^2 - 4n\zeta^4 - 8(n-1)n^2(n+1)\zeta^2 \\ f_4^n &\equiv -2\beta^2\{(n-2\zeta^2)\beta^2 + n(4n+5)\zeta^2 + 2(n-1)n^2(n+1)\} \\ f_5^n &\equiv -n\beta^2[\beta^4 + \{2\zeta^2 - 4n(n+1)\}\beta^2 + \zeta^4 - 4n(n+1)\zeta^2], \quad f_6^n \equiv -4(n-1)(n+1)\beta^2(\beta^2 + \zeta^2) \end{aligned}$$

(3) When Eq. (16) is satisfied, transformed stresses, shearing force and displacement of the center line are given as follows. When Eq. (17) is satisfied, $i c_s \Omega / a$ should be multiplied to the right-hand side terms of the following expressions.

$$\begin{aligned}\bar{\sigma}_{rr}^{(n)}(R, \zeta, \Omega) &\equiv (F_n/\pi a)[AA^{(n)}(R, \zeta, \Omega)/\Omega\bar{J}_n(\Omega, \zeta)], & \bar{\sigma}_{\theta\theta}^{(n)} &\equiv (F_n/\pi a)[BB^{(n)}/\Omega\bar{J}_n] \\ \bar{\sigma}_{zz}^{(n)} &\equiv (F_n/\pi a)[CC^{(n)}/\Omega\bar{J}_n], & \bar{\sigma}_{r\theta}^{(n)} &\equiv (F_n/\pi a)[AB^{(n)}/\Omega\bar{J}_n] \\ \bar{\sigma}_{\theta z}^{(n)} &\equiv (F_n/\pi a)[BC^{(n)}/\Omega\bar{J}_n], & \bar{\sigma}_{rz}^{(n)} &\equiv (F_n/\pi a)[CA^{(n)}/\Omega\bar{J}_n] \\ \tilde{S}(\zeta, \Omega) &\equiv (aF_1/i)[SS(\zeta, \Omega)/\Omega\bar{J}_1], & \tilde{u}(\zeta, \Omega) &\equiv (F_1/2\pi\mu)[uu(\zeta, \Omega)/\Omega\bar{J}_1]\end{aligned}$$

where

$$\begin{aligned}AA^{(n)}(R, \zeta, \Omega) &\equiv \left\{ \frac{1-2k^2}{2} \Omega^2 + \left(\alpha^2 - \frac{(n-1)n}{R^2} \right) \right\} J_n(R\alpha) - \frac{\alpha}{R} J_{n+1}(R\alpha) \bar{J}(A_n) \\ &\quad + 2\zeta \left\{ \left(\frac{(n-1)n}{R^2} - \beta^2 \right) J_n(R\beta) + \frac{\beta}{R} J_{n+1}(R\beta) \right\} \bar{J}(B_n) + 2n \left\{ \frac{1-n}{R^2} J_n(R\beta) + \frac{\beta}{R} J_{n+1}(R\beta) \right\} \Omega^2 \bar{J}(C_n) \\ BB^{(n)} &\equiv \left\{ \left(\frac{(n-1)n}{R^2} + \frac{1-2k^2}{2} \Omega^2 \right) J_n(R\alpha) + \frac{\alpha}{R} J_{n+1}(R\alpha) \right\} \bar{J}(A_n) - \frac{2\zeta}{R} \left\{ \frac{(n-1)n}{R} J_n(R\beta) \right. \\ &\quad \left. + \beta J_{n+1}(R\beta) \right\} \bar{J}(B_n) + \frac{2n}{R} \left\{ \frac{n-1}{R} J_n(R\beta) - \beta J_{n+1}(R\beta) \right\} \Omega^2 \bar{J}(C_n) \\ CC^{(n)} &\equiv \left\{ \frac{1-2k^2}{2} \Omega^2 + \zeta^2 \right\} J_n(R\alpha) \bar{J}(A_n) + 2\zeta (\Omega^2 - \zeta^2) J_n(R\beta) \bar{J}(B_n) \\ AB^{(n)} &\equiv - \left\{ \frac{n(1-n)}{R^2} J_n(R\alpha) + \frac{n}{R} \alpha J_{n+1}(R\alpha) \right\} \bar{J}(A_n) + 2\zeta \left\{ \frac{n(1-n)}{R^2} J_n(R\beta) \right. \\ &\quad \left. + \frac{n}{R} \beta J_{n+1}(R\beta) \right\} \bar{J}(B_n) + \left\{ \left(\frac{2(n-1)n}{R^2} - \beta^2 \right) J_n(R\beta) + \frac{2}{R} \beta J_{n+1}(R\beta) \right\} \Omega^2 \bar{J}(C_n) \\ BC^{(n)} &\equiv - \frac{n}{R} \zeta J_n(R\alpha) \bar{J}(A_n) - \frac{n}{R} (\Omega^2 - 2\zeta^2) J_n(R\beta) \bar{J}(B_n) - \beta \zeta J_n'(R\beta) \Omega^2 \bar{J}(C_n) \\ CA^{(n)} &\equiv \left[\zeta \left\{ \alpha J_{n+1}(R\alpha) - \frac{n}{R} J_n(R\alpha) \right\} \bar{J}(A_n) + (\Omega^2 - 2\zeta^2) \left\{ \beta J_{n+1}(R\beta) - \frac{n}{R} J_n(R\beta) \right\} \bar{J}(B_n) \right. \\ &\quad \left. - \frac{n}{R} \zeta J_n(R\beta) \Omega^2 \bar{J}(C_n) \right] i \\ SS(\zeta, \Omega) &\equiv \zeta \alpha J_1(\alpha) \bar{J}(A_1) + \alpha (\beta^2 - \zeta^2) J_1(\beta) \bar{J}(B_1) + \alpha \zeta \Omega^2 J_1(\beta) \bar{J}(C_1) \\ uu(\zeta, \Omega) &\equiv - \frac{\alpha}{2} \bar{J}(A_1) + \beta \zeta \bar{J}(B_1) - \beta \Omega^2 \bar{J}(C_1)\end{aligned}$$

$$\begin{aligned}(4) \quad AAA_1^{(0)}(\zeta) &\equiv (1-k^2)\zeta^2 I_0(\zeta) + \zeta(1-k^2)I_1(\zeta), & AAA_2^{(0)}(\zeta) &\equiv (k^2-1)\zeta^2 I_1 \\ AAA_3^{(0)}(\zeta) &\equiv (k^2-1)\zeta I_0 - I_1, & BBB_1^{(0)} &\equiv -k^2\zeta I_1, & BBB_2^{(0)} &\equiv (1-k^2)\zeta I_0 + I_1 \\ CCC_1^{(0)} &\equiv -(1-k^2)\zeta^2 I_0 + 2(1-k^2)\zeta I_1, & CCC_2^{(0)} &\equiv (1-k^2)\zeta^2 I_1, & CAA_1^{(0)} &\equiv -(1-k^2)\zeta^2 I_1 \\ CAA_2^{(0)} &\equiv (1-k^2)\zeta^2 I_0, & E_0 &\equiv (1-k^2)\zeta^2(I_0^2 - I_1^2)\end{aligned}$$

(5) $E_0(\zeta)=0$ has neither real root except $\zeta=0$ nor imaginary root, but has complex roots in general. These complex roots exist innumerable and are in sets of four forms of $\pm(a+ib)^{(1)}$. The roots of $E_0(\zeta)=0$ which are $Re\{\zeta\}>0$ and $Im\{\zeta\}>0$ are calculated by Ichimatsu's method⁽⁹⁾. The results are shown in Table 1. Poisson's ratio is assumed to be $1/3$.

$$\begin{aligned}(6) \quad AAA_1^{(1)}(\zeta) &\equiv (2-2k^2)\zeta^3 I_0^2(\zeta) + \{(-5+5k^2)\zeta^2 - (1-k^2)\zeta^4\} I_0(\zeta) I_1(\zeta) \\ &\quad + \{(2-2k^2)\zeta + (1-k^2)\zeta^3\} I_1^2(\zeta) \\ AAA_2^{(1)} &\equiv -k^2\zeta^2 I_0 I_1 + \{k^2\zeta + (-1+k^2)\zeta^3\} I_1^2 \\ AAA_3^{(1)} &\equiv (-4+4k^2)\zeta^2 I_0^2 + (10-9k^2)\zeta I_0 I_1 + \{(-4+2k^2) + (2-2k^2)\zeta^2 + (1-k^2)\zeta^4\} I_1^2 \\ AAA_4^{(1)} &\equiv 2k^2\zeta I_0 I_1 + \{-2k^2 + (2-2k^2)\zeta^2\} I_1^2 \\ BBB_1^{(1)} &\equiv 2k^2\zeta^2 I_0^2 - k^2(5\zeta + \zeta^3) I_0 I_1 + k^2(2 + \zeta^2) I_1^2, & BBB_2^{(1)} &\equiv k^2\zeta^2 I_0 I_1 + \{-k^2\zeta + (1-k^2)\zeta^3\} I_1^2 \\ CCC_1^{(1)} &\equiv (k^2-1)\{2\zeta^3 I_0^2 - (5\zeta^2 + \zeta^4) I_0 I_1 + (2\zeta + \zeta^3) I_1^2\} \\ CCC_2^{(1)} &\equiv (-2+2k^2)\zeta^2 I_0^2 + \{(5-6k^2)\zeta + (3-3k^2)\zeta^3\} I_0 I_1 + \{(-2+4k^2) \\ &\quad + (-5+5k^2)\zeta^2 + (-1+k^2)\zeta^4\} I_1^2 \\ BCC_1^{(1)} &\equiv k^2\zeta^2 I_0^2 + \{-3k^2\zeta + (1-k^2)\zeta^3\} I_0 I_1 + \{2k^2 + (-2+2k^2)\zeta^2\} I_1^2 \\ BCC_2^{(1)} &\equiv -k^2\zeta I_0^2 + \{2k^2 + (1-k^2)\zeta^2\} I_0 I_1 + \{(-2+2k^2)\zeta - (1-k^2)\zeta^3\} I_1^2 \\ CAA_1^{(1)} &\equiv (1-k^2)\zeta\{2\zeta^2 I_0^2 - (5\zeta + \zeta^3) I_0 I_1 + (2 + \zeta^2) I_1^2\}\end{aligned}$$

$$CAA_2^{(1)} \equiv [-k^2\zeta^2 I_0^2 + (4k^2\zeta - (3-3k^2)\zeta^3)I_0 I_1 + (-4k^2 + (6-6k^2)\zeta^2 + (1-k^2)\zeta^4)I_1^2]/\zeta$$

$$CAA_3^{(1)} \equiv (k^2\zeta + (-2+2k^2)\zeta^3)I_0 I_1 + (-2k^2 + (4-4k^2)\zeta^2 + (1-k^2)\zeta^4)I_1^2$$

$$uuu \equiv -\zeta I_0^2 + ((5/2+k^2/2) + (-1/2+3k^2/4)\zeta^2)I_0 I_1 - ((1+k^2)/\zeta + (-3/2+7k^2/4)\zeta + (-1/4+k^2/4)\zeta^3)I_1^2$$

$$E_1 \equiv (2-2k^2)\zeta^3 I_0^3 - ((9-8k^2)\zeta^2 + (1-k^2)\zeta^4)I_0^2 I_1 + (12-8k^2)\zeta I_0 I_1^2 + (-4 + (4-4k^2)\zeta^2 + (1-k^2)\zeta^4)I_1^3$$

(7) $E_1(\zeta)=0$ has only one real root $\zeta=0$, innumerable imaginary roots $\pm ic$ and innumerable complex roots $\pm(a \pm ib)$. The roots of $E_1(\zeta)=0$ which are $Re(\zeta) \geq 0$ and $Im(\zeta) > 0$ are shown in Table 2. These roots are simple roots except $\zeta=0$. Poisson's ratio is assumed to be $1/3$.

$$(8) \quad AAA_1^{(2)}(\zeta) \equiv (-2+2k^2)\zeta^4 I_1^2(\zeta) + ((12-12k^2)\zeta^3 + (1-k^2)\zeta^5)I_1(\zeta)I_2(\zeta) + ((-12+12k^2)\zeta^2 + (-2+2k^2)\zeta^4)I_2^2(\zeta)$$

$$AAA_2^{(2)} \equiv 6\zeta^2 I_1^2 + ((-12-12k^2)\zeta + (3-2k^2)\zeta^3)I_1 I_2 + (24k^2 + (-6+4k^2)\zeta^2 + (1-k^2)\zeta^4)I_2^2$$

$$AAA_3^{(2)} \equiv (6-6k^2)\zeta^3 I_1^2 + ((-36+32k^2)\zeta^2 + (-1+k^2)\zeta^4)I_1 I_2 + ((24-12k^2)\zeta + (-6+6k^2)\zeta^3 + (-1+k^2)\zeta^5)I_2^2$$

$$AAA_4^{(2)} \equiv -12\zeta I_1^2 + (24k^2 - 6\zeta^2)I_1 I_2 + ((-12+24k^2)\zeta + (-6+6k^2)\zeta^3)I_2^2$$

$$BBB_1^{(2)} \equiv -6\zeta^2 I_1^2 + ((12+12k^2) + (-3+2k^2)\zeta^3)I_1 I_2 + (-24k^2 + (6-4k^2)\zeta^2 + (-1+k^2)\zeta^4)I_2^2$$

$$BBB_2^{(2)} \equiv -2k^2\zeta^3 I_1^2 + k^2(12\zeta^2 + \zeta^4)I_1 I_2, \quad BBB_3^{(2)} \equiv 12\zeta I_1^2 + (-24k^2 + 6\zeta^2)I_1 I_2$$

$$CCC_1^{(2)} \equiv (2-2k^2)\zeta^4 I_1^2 + ((-12+12k^2)\zeta^3 + (-1+k^2)\zeta^5)I_1 I_2 + ((12-12k^2)\zeta^2 + (2-2k^2)\zeta^4)I_2^2$$

$$CCC_2^{(2)} \equiv (4k^2\zeta^2 + (-2+2k^2)\zeta^4)I_1 I_2 + ((12-24k^2)\zeta + (12-12k^2)\zeta^3 + (1-k^2)\zeta^5)I_2^2$$

$$BCC_1^{(2)} \equiv -2k^2\zeta^3 I_1^2 + ((-6\zeta^2 - 2\zeta^4) + k^2(16\zeta^2 + 2\zeta^4))I_1 I_2$$

$$+ ((12\zeta + 6\zeta^3) + k^2(-24\zeta - 6\zeta^3))I_2^2$$

$$BCC_2^{(2)} \equiv 4k^2\zeta^2 I_1^2 + (12-24k^2)\zeta I_1 I_2$$

$$+ ((12\zeta^2 + 2\zeta^4) - k^2(12\zeta^2 + 2\zeta^4))I_2^2$$

Table 1 Roots of $E_0(\zeta)=0$

real part	imaginary part	real part	imaginary part
1.358 65	2.737 95	2.894 97	81.657 06
1.637 21	6.065 95	2.913 81	84.799 45
1.828 04	9.270 53	2.931 97	87.941 78
1.967 10	12.445 25	2.949 50	91.084 06
2.076 18	15.607 60	2.966 43	94.226 30
2.165 85	18.763 53	2.982 81	97.368 50
2.241 96	21.915 66	2.998 67	100.510 67
2.308 05	25.065 37	3.014 04	103.652 80
2.366 46	28.213 40	3.028 95	106.794 90
2.418 77	31.360 25	3.043 43	109.936 98
2.466 15	34.506 21	3.057 51	113.079 03
2.509 43	37.651 50	3.071 20	116.221 06
2.549 27	40.796 27	3.084 52	119.363 07
2.586 18	43.940 61	3.097 50	122.505 05
2.620 55	47.084 62	3.110 15	125.647 02
2.652 72	50.228 34	3.122 49	128.788 97
2.682 95	53.371 83	3.134 53	131.930 91
2.711 45	56.515 13	3.146 29	135.072 83
2.738 42	59.658 26	3.157 77	138.214 74
2.764 01	62.801 25	3.169 00	141.356 63
2.788 36	65.944 11	3.179 99	144.498 51
2.811 57	69.086 87	3.190 74	147.640 38
2.833 76	72.229 54	3.201 26	150.782 24
2.855 00	75.372 12	3.211 56	153.924 09
2.875 38	78.514 62		

Table 2 Roots of $E_1(\zeta)=0$

real part	imaginary part	real part	imaginary part
	2.832 60	2.668 19	51.790 46
1.497 49	4.308 44		54.148 58
	6.696 12	2.697 52	54.934 40
1.739 31	7.606 81		57.292 59
	9.966 63	2.725 22	58.078 11
1.901 96	10.813 30		60.436 34
	13.169 17	2.751 48	61.221 60
2.024 67	13.991 48		63.579 88
	16.346 90	2.776 42	64.364 93
2.123 24	17.156 87		66.723 23
	19.512 55	2.800 19	67.508 10
2.205 60	20.315 27		69.866 43
	22.671 35	2.822 87	70.651 14
2.276 35	23.469 41		73.009 49
	25.825 88	2.844 57	73.794 06
2.338 35	26.620 74		76.152 42
	28.977 56	2.865 37	76.936 88
2.393 52	29.770 14		79.295 25
	32.127 25	2.885 34	80.079 61
2.443 23	32.918 12		82.437 99
	35.275 47	2.904 54	83.222 25
2.488 44	36.065 06		85.580 64
	38.422 61	2.923 04	86.364 83
2.529 92	39.211 18		88.723 22
	41.568 90	2.940 87	89.507 34
2.568 22	42.356 68		91.865 73
	44.714 52	2.958 09	92.649 79
2.603 81	45.501 66		95.008 17
	47.859 62	2.974 74	95.792 19
2.637 03	48.646 23		98.150 57
	51.004 28	2.990 85	98.934 53

$$\begin{aligned}
CAA_1^{(2)} &\equiv \{-4k^2\zeta^2 + (2-2k^2)\zeta^4\}I_1I_2 + \{(-12+24k^2)\zeta + (-12+12k^2)\zeta^3 + (-1+k^2)\zeta^5\}I_2^2 \\
CAA_2^{(2)} &\equiv 4k^2\zeta^2I_1^2 + (12-24k^2)\zeta I_1I_2 + \{(12-12k^2)\zeta^2 + (2-2k^2)\zeta^4\}I_2^2 \\
CAA_3^{(2)} &\equiv -2(1-k^2)\zeta^4I_1^2 + \{12(1-k^2)\zeta^3 + (1-k^2)\zeta^5\}I_1I_2 - \{12(1-k^2)\zeta^2 + 2(1-k^2)\zeta^4\}I_2^2 \\
E_2 &\equiv \{6\zeta^2 + (-2+2k^2)\zeta^4\}I_1^3 + \{-24\zeta + 21\zeta^3 + \zeta^5 - k^2(12\zeta + 20\zeta^3 + \zeta^5)\}I_1^2I_2 \\
&\quad + \{-60\zeta^2 - 2\zeta^4 + k^2(48+48\zeta^2+2\zeta^4)\}I_1I_2^2 + \{(12+12k^2)\zeta - (12-12k^2)\zeta^3 - (1-k^2)\zeta^5\}I_2^3
\end{aligned}$$

(9) $E_2(\zeta)=0$ has only one real root $\zeta=0$, innumerable imaginary roots $\pm ic$ and innumerable complex roots $\pm(a+ib)$.

(10) The first modes ($m=1$) of Fourier components of $n=0$ and 1 pass through the origin and the following approximate relations exist in the region of small $|\zeta|$.

$$m=1, n=0;$$

$$\begin{aligned}
Q_0^1(\zeta) &= \sqrt{2(1+\nu)}(\zeta - \nu^2\zeta^3/4) + O(\zeta^5) \\
&\sim \sqrt{2(1+\nu)}(\zeta - \nu^2\zeta^3/4)
\end{aligned}$$

$$m=1, n=1;$$

$$\begin{aligned}
Q_1^1(\zeta) &= \sqrt{(1+\nu)/2}\zeta^2 + O(\zeta^4) \\
&\sim \sqrt{(1+\nu)/2}\zeta^2
\end{aligned}$$

In the case of $n \geq 2$ the first modes do not pass through the origin. The slope of the first mode at $\zeta=0$, $(dQ_n^1/d\zeta)_{\zeta=0}$ are zero in general except in the case of $n=0$. Phase velocities of each first mode approach Rayleigh's surface velocity c_R asymptotically with the increase of $|\zeta|$. Phase velocities of the higher modes ($m \geq 2$) of all components of n approach shear wave velocity c_s asymptotically with the increase of $|\zeta|$. The slopes at $\zeta=0$, $(dQ_n^m/d\zeta)_{\zeta=0}$ are zero in general.

(11) After the time T elapses considerably, Fourier integrals in Eq. (27) can be expressed as follows.

(a) in the case that $\chi(\zeta)$ is finite and $\psi'(\zeta) \neq 0$

$$I(T) = O(1/T)$$

(b) in the case that $\chi(\zeta)$ is finite, $\psi(\zeta_0) = \psi'(\zeta_0) = \dots = \psi^{(n)}(\zeta_0) = 0$ and $\psi^{(n+1)}(\zeta_0) \neq 0$

$$I(T) = O(1/T^{1/(n+1)})$$

If $\chi(\zeta)$ has a pole ζ_1 , the contribution from the residue at the pole should be added to the calculation. If ζ_1 is in the neighborhood of ζ_0 , Fourier integral should be calculated directly.

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