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Flow of Liquid Metals with a Transversely Applied Magnetic Field*
(6th Report, Turbulent Flow in the Entrance Region)

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The present paper is a theoretical investigation on the MHD entrance region of turbulent flow. In the first place, an approximate solution of the velocity profile for a fully developed turbulent flow between parallel planes is obtained by assuming the solution of the series as the velocity profile. Next a linear combination of the velocity profile for a hydrodynamically developed flow and that for a magnetohydrodynamically developed one is used as the velocity profile in the entrance region and a momentum-integral method is applied.

From the comparison between the results of the present analysis and those of Maciulaitis et al., it is found that the present analytical results are applicable only to the region of relatively small Hartmann numbers where the analysis of Maciulaitis et al. can not be applied. In addition, a few examples of the results involving the variation of the frictional shearing stress and the pressure in the entrance region are shown.

1. Introduction

Investigations on the entrance flow of liquid metals in channels with a transversely applied magnetic field are necessary for the basic design of MHD devices and are interesting from a standpoint of engineering as well.

The entrance flow problem of MHD channel consists of the following two cases ; (i) one of the channel entry found in usual hydrodynamics, (ii) another of the flow coming into a magnetic field with a non-magnetically fully developed velocity profile. Up to date, there have been many analytical investigations on laminar flow but the authors can not find any studies on turbulent flow except those done by Moffatt⁽¹⁾ and Maciulaitis et al.⁽²⁾ using with a momentum-integral method.

Moffatt obtained the solution in the case of (i) by assuming that flow was separated into a free stream and a boundary layer one and that the velocity profile was similar in the boundary layer and by applying a momentum-integral method. In his analysis, however, he assumed that the velocity of the free stream was constant. Maciulaitis et al. modified Moffatt's assumption for the velocity profile of the free stream and analyzed the cases of (i) and (ii) by the same method as Moffatt used. In the analysis, they adopted a one-seventh law as the velocity profile of the boundary layer and used Blasius' law of friction as the friction stress at walls. Though the analytical result of Maciulaitis

et al. can offer the solution of the turbulent entrance flow in the region of large Hartmann numbers, it is not applicable to the region of small Hartmann numbers. In addition, the transition to laminar flow due to a magnetic field is not taken into account. Judging from the above description, it seems fully significant in engineering that the authors obtain a solution applicable to the region of small Hartmann numbers where the analytical method of Maciulaitis et al. can not be applied and that they elucidate the behavior of turbulent flow over a wide region by using their analytical solution together with the result of Maciulaitis et al.

The authors⁽³⁾ analyzed previously the problem of a laminar entrance flow through parallel planes by employing a momentum-integral method and showed that the results were in good agreement with the numerical solutions obtained by Hwang et al.⁽⁴⁾ with a finite difference method. In the present report, the analytical method for laminar flow is applied to turbulent one and the problem of the entrance flow coming into a magnetic field is investigated analytically. However, the present results have restriction imposed on Hartmann number because of the viscous stress being neglected in comparison with the Reynolds stress and are not applicable to the region of large Hartmann numbers contrary to the results of Maciulaitis et al.

2. Nomenclature

a : half-height of a channel
 B : magnetic flux density $= \mu H$
 E : electric field strength
 f : coefficient of the wall friction stress
 H : magnetic field strength
 J : electric current density

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M : Hartmann number $= aB_0\sqrt{\sigma/\eta}$
 p : pressure
 $R_1 = \sigma\mu au_m$
 $R_2 = \rho au_m/\eta$
 $R_3 = B_0^2/\mu\rho u_m^2$
 R_e : Reynolds number $= 4\rho au_m/\eta$
 t : time
 u_m : mean velocity
 $\mathbf{v}$: velocity
 x, y, z : coordinates
 x_e : length of the entrance region
 γ : function of x
 $\delta = 4M^2/R_e$
 η : fluid viscosity
 κ, κ_* : proportional constant
 μ : magnetic permeability
 ν : kinematic viscosity
 ρ : density
 σ : electric conductivity
 τ : shearing stress
 τ_w : wall friction stress
 x^* : nondimensional expression of x
 $\bar{x}$: time-mean component of x
 x' : fluctuation component of x

3. Theoretical analysis

3-1. Basic equations Basic equations for flow of liquid metals under a magnetic field are as follows. Maxwell's equation:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (1)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (4)$$

Ohm's law:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mu \mathbf{H}) \quad (5)$$

The continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad (6)$$

The modified Navier-Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \frac{1}{\rho} (\mathbf{J} \times \mu \mathbf{H}) \quad (7)$$

The following nondimensional expressions are introduced:

$$\left. \begin{aligned} \mathbf{V}^* &= \frac{\mathbf{v}}{u_m}, & \mathbf{E}^* &= \frac{\mathbf{E}}{u_m B_0}, & \mathbf{B}^* &= \frac{\mathbf{B}}{B_0} \\ \mathbf{J}^* &= \frac{\mathbf{J}}{\sigma u_m B_0}, & p^* &= \frac{p}{\rho u_m^2}, & \nabla^* &= a \nabla \\ t^* &= \frac{u_m t}{a}, & x^* &= \frac{x}{a}, & y^* &= \frac{y}{a} \end{aligned} \right\} \quad (8)$$

Using Eq. (8) reduces Eqs. (1) ~ (7) to

$$\nabla^* \times \mathbf{B}^* = \mathbf{J}^* \quad (9)$$

$$\nabla^* \times \mathbf{E}^* = -\frac{\partial \mathbf{B}^*}{\partial t^*} \quad (10)$$

$$\nabla^* \cdot \mathbf{J}^* = 0 \quad (11)$$

$$\nabla^* \cdot \mathbf{B}^* = 0 \quad (12)$$

$$\mathbf{J}^* = \mathbf{E}^* + \mathbf{V}^* \times \mathbf{B}^* \quad (13)$$

$$\nabla^* \cdot \mathbf{V}^* = 0 \quad (14)$$

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + (\mathbf{V}^* \cdot \nabla^*) \mathbf{V}^* = -\nabla^* p^* + \frac{1}{R_2} \nabla^{*2} \mathbf{V}^* + R_1 R_3 (\mathbf{J}^* \times \mathbf{B}^*) \quad (15)$$

where

$$R_1 = \sigma \mu a u_m, \quad R_2 = \frac{\rho a u_m}{\eta}, \quad R_3 = \frac{B_0^2}{\mu \rho u_m^2} \quad (16)$$

3-2. Fully developed flow Let's consider a fully developed turbulent flow in parallel planes shown in Fig.1. The plates are electrically insulated and a uniformly distributed magnetic field B_0 is applied perpendicular to the plates.

The pressure and each component of the velocity and magnetic field vectors can be split into time-mean and fluctuation components as follows:

$$\left. \begin{aligned} (\mathbf{V}^*)_i &= \bar{v}_i^* + v_i'^* \\ (\mathbf{B}^*)_i &= \bar{B}_i^* + B_i'^* \\ p^* &= \bar{p}^* + p'^* \end{aligned} \right\} \quad (17)$$

where

$$\bar{v}_i'^* = \bar{B}_i'^* = 0 \quad (18)$$

From Eqs. (12), we get (5)

$$\bar{B}_y^* = 1 \quad (19)$$

The x-component of the modified Navier-Stokes equation Eq. (15) is

$$\frac{1}{R_2} \frac{d^2 \bar{v}_x^*}{dy^{*2}} + R_3 \frac{d \bar{B}_x^*}{dy^*} - \frac{d}{dy^*} [\bar{v}_x'^* \bar{v}_y'^* - R_3 \bar{B}_x'^* \bar{B}_y'^*] = \frac{\partial \bar{p}^*}{\partial x^*} \quad (20)$$

The expression in the square bracket of the third term in the left-hand side of Eq. (20) corresponds to the Reynolds stress in hydrodynamics but the following relation seems to be generally valid in flow of liquid metals (5):

$$R_3 \bar{B}_x'^* \bar{B}_y'^* \ll \bar{v}_x'^* \bar{v}_y'^* \quad (21)$$

Therefore, we can neglect the stress due to fluctuation of a magnetic field.

The boundary conditions for the velocity and magnetic field are

$$y^* = \pm 1: \bar{v}_x^* = \bar{v}_y^* = 0 \quad (22)$$

$$y^* = \pm 1: \bar{B}_x^* = \bar{B}_y^* = 0 \quad (23)$$

Integrating Eq. (20) over the sectional area of the channel and using Eqs. (22), (23), we can get

$$\frac{\partial \bar{p}^*}{\partial x^*} = -\frac{\tau_w}{\rho u_m^2} = -f \quad (24)$$

Accordingly, Eq. (20) reduces to

$$\frac{d}{dy^*} \left\{ -\bar{v}_x'^* \bar{v}_y'^* + \frac{1}{R_2} \frac{d \bar{v}_x^*}{dy^*} \right\} + R_3 \frac{d \bar{B}_x^*}{dy^*} + f = 0 \quad (25)$$

From Eqs. (9), (13), the following equation is induced (5):

$$-\frac{d \bar{B}_x^*}{dy^*} = R_1 \bar{J}_z^* = R_1 (\bar{E}_x^* + \bar{v}_z^*) \quad (26)$$

Fig.1

In the case of the side walls being electrically nonconductive, the induced electric current circulates in liquid metal and the total current in the cross section becomes zero. Therefore, Eq. (27) is obtained from Eq. (26):

$$\int_{-1}^1 (\bar{E}_z^* + \bar{v}_z^*) dy^* = 0 \quad (27)$$

As $\bar{E}_z^*$ is regarded as constant in the cross section from Eq. (10), the following relation is derived from Eq. (27):

$$\bar{E}_z^* = -1 \quad (28)$$

Accordingly, Eq. (26) reduces to

$$\frac{d\bar{B}_z^*}{dy^*} = -R_1(\bar{v}_z^* - 1) \quad (29)$$

Substituting Eq. (29) into Eq. (25) results in

$$\frac{d}{dy^*} \left\{ -\bar{v}_z^* \bar{v}_y^* + \frac{1}{R_2} \frac{d\bar{v}_z^*}{dy^*} \right\} - R_1 R_2 (\bar{v}_z^* - 1) + f = 0 \quad (30)$$

We define Reynolds and Hartmann number as follows:

$$R_1 = \frac{4\rho a u_m}{\eta} = 4R_2, \quad M = \sqrt{\frac{\sigma}{\eta}} a B_0 \quad (31)$$

Substituting Eq. (31) into Eq. (30), we can obtain the following momentum equation for a developed turbulent flow of liquid metals through parallel planes under a transversely applied magnetic field:

$$\frac{d}{dy^*} \left\{ -\bar{v}_z^* \bar{v}_y^* + \frac{4}{R_2} \frac{d\bar{v}_z^*}{dy^*} \right\} - \frac{4M^2}{R_2} (\bar{v}_z^* - 1) + f = 0 \quad (32)$$

Now it seems reasonable in the region of small Hartmann numbers to neglect the viscous stress as compared with that due to fluctuation of velocity. Therefore, Eq. (32) is simplified to

$$\frac{d}{dy^*} \left\{ -\bar{v}_z^* \bar{v}_y^* \right\} - \frac{4M^2}{R_2} (\bar{v}_z^* - 1) + f = 0 \quad (33)$$

We introduce Prandtl's mixing length concept and apply the momentum transfer theory to the Reynolds stress. Exchanging the coordinate and rewriting $(y+a)$ to y , Eq. (34) is obtained:

$$-\bar{v}_z^* \bar{v}_y^* = \left(\frac{l}{a} \right)^2 \left(\frac{d\bar{v}_z^*}{dy^*} \right)^2 = \kappa^2 y^{*2} \left(\frac{d\bar{v}_z^*}{dy^*} \right)^2 \quad (34)$$

Substituting Eq. (34) into Eq. (33) yields

$$\frac{d}{dy^*} \left\{ \kappa^2 y^{*2} \left(\frac{d\bar{v}_z^*}{dy^*} \right)^2 \right\} - \partial (\bar{v}_z^* - 1) + f = 0 \quad (35)$$

where

$$\partial = \frac{4M^2}{R_2} \quad (36)$$

It is assumed that the velocity and the coefficient of friction stress can be expressed in the form of the following series:

$$\left. \begin{aligned} \bar{v}_z^* &= v_0 + \partial v_1 + \partial^2 v_2 + \dots \\ f &= f_0 + \partial f_1 + \partial^2 f_2 + \dots \end{aligned} \right\} \quad (37)$$

Substituting Eq. (37) into Eq. (35) and collecting the terms of the same power of ∂ yields the following equations:

$$\frac{d}{dy^*} \left\{ \kappa^2 y^{*2} \left(\frac{dv_0}{dy^*} \right)^2 \right\} = -f_0 \quad (38)$$

$$2 \frac{d}{dy^*} \left\{ \kappa^2 y^{*2} \frac{dv_0}{dy^*} \frac{dv_1}{dy^*} \right\} = (v_0 - 1) - f_1 \quad (39)$$

$$\frac{d}{dy^*} \left\{ \kappa^2 y^{*2} \left(\frac{dv_0}{dy^*} \frac{dv_2}{dy^*} + \frac{dv_1}{dy^*} \frac{dv_1}{dy^*} \right) \right\} = v_1 - f_2 \quad (40)$$

Accordingly, the velocity profile is obtained by solving Eqs. (38), (39), ... under the following boundary and continuity conditions.

The boundary condition is

$$y^* = 1:$$

$$\frac{dv_0}{dy^*} = \frac{dv_1}{dy^*} = \frac{dv_2}{dy^*} = \dots = 0 \quad (41)$$

and the continuity condition is

$$\int_0^1 v_0 dy^* = 1, \quad \int_0^1 v_1 dy^* = \int_0^1 v_2 dy^* = \dots = 0 \quad (42)$$

The following equation is obtained as the zeroth approximation from Eq. (38):

$$\begin{aligned} v_0 &= \frac{2\sqrt{f_0}}{\kappa} \sqrt{1-y^*} + \frac{\sqrt{f_0}}{\kappa} \log_* \frac{(1-\sqrt{1-y^*})}{(1+\sqrt{1-y^*})} \\ &+ \left(1 + \frac{2}{3} \frac{\sqrt{f_0}}{\kappa} \right) \end{aligned} \quad (43)$$

In solving Eqs. (39), (40), Eq. (43) should be used for v_0 in these equations but the following equations are substituted in order to simplify the analysis, namely

$$\frac{dv_0}{dy^*} = \frac{\sqrt{f_0}}{\kappa_* y^*} \quad (44)$$

$$v_0 = \frac{\sqrt{f_0}}{\kappa_*} \log_* y^* + \left(1 + \frac{\sqrt{f_0}}{\kappa_*} \right) \quad (45)$$

The above equations are obtained by assuming that the viscous stress is negligible as compared with the Reynolds stress and the shearing stress in fluid is equal to the wall shearing stress near the walls. Eq. (44) does not satisfy the boundary condition Eq. (41) but it will be useful over the whole region as the approximate expression for Eq. (43).

Substituting Eqs. (44), (45) into Eq. (39) and solving it results in

$$\begin{aligned} v_1 &= \frac{1}{2\kappa_*} y^* \log_* y^* - \frac{1}{2\kappa_* \sqrt{f_0}} \left(f_1 + \frac{\sqrt{f_0}}{\kappa_*} \right) y^* \\ &+ \frac{f_1}{2\kappa_* \sqrt{f_0}} \log_* y^* + \frac{3}{8\kappa_* \sqrt{f_0}} \left(2f_1 + \frac{\sqrt{f_0}}{\kappa_*} \right) \end{aligned} \quad (46)$$

From Eq. (40), v_2 is obtained in the same manner as v_1 was derived:

$$\begin{aligned} v_2 &= -\frac{1}{16\kappa_*^2 \sqrt{f_0}} (y^* \log_* y^*)^2 \\ &+ \frac{1}{8\kappa_*^2 \sqrt{f_0}} \left(\frac{1}{\kappa_*} + \frac{f_1}{\sqrt{f_0}} \right) y^{*2} \log_* y^* \\ &+ \frac{1}{2\kappa_* \sqrt{f_0}} \left(f_2 - \frac{f_1^2}{4f_0} \right) \log_* y^* \\ &- \frac{1}{32\kappa_* \sqrt{f_0}} \left(\frac{5}{\kappa_*^2} + \frac{4f_1}{\kappa_* \sqrt{f_0}} + \frac{2f_1^2}{f_0} \right) y^{*2} \\ &+ \frac{1}{2\kappa_* \sqrt{f_0}} \left(\frac{3}{2} f_2 - \frac{11f_1^2}{24f_0} - \frac{1}{72\kappa_* \sqrt{f_0}} - \frac{5}{108\kappa_*^2} \right) \\ &+ \frac{1}{16\kappa_* \sqrt{f_0}} \left(\frac{3}{\kappa_*^2} + \frac{4f_1^2}{f_0} - 8f_2 + \frac{2f_1}{\kappa_* \sqrt{f_0}} \right) y^* \end{aligned} \quad (47)$$

Therefore, the velocity profile of the turbulent flow of liquid metals through parallel planes under a transversely applied magnetic field with a uniform distribution is as follows.

The zeroth approximation:

$$v^{(0)} = v_0 \quad (48)$$

The first approximation:

$$v^{(1)} = v_0 + \partial v_1 \quad (49)$$

The second approximation:

$$v^{(2)} = v_0 + \partial v_1 + \partial^2 v_2 \quad (50)$$

Though κ and κ_* are regarded as constant independently of Hartmann number in the analysis, it seems approximately good to use κ and κ_* without a magnetic field because the analysis is confined to the region of small Hartmann numbers.

Choosing κ in Eq. (43) to give agreement at $y^* = 0.3$ with the experimental re-

sults of Dönch ($R_e \div 4 \times 10^5$) and Nikuradse ($R_e \div 2.4 \times 10^5$) yields (6)

$$\kappa = 0.23 \dots \dots \dots (51)$$

On the other hand, 0.4 is usually adopted as κ_* but it appears better to choose 0.35 as κ_* in order to obtain the results in good agreement with experimental values, (7) namely

$$\kappa_* = 0.35 \dots \dots \dots (52)$$

An example of the relations between Murgatroyd's experimental results of the friction stress (8) and δ is shown in Fig.2. In this figure, real lines express the relations between the coefficient of the friction stress f and δ approximated with a quadratic equation and they agree well with the experimental values up to $\delta = 0.07$. The results of the velocity profile calculated for $R_e = 1.01 \times 10^5$ with f_0, f_1, f_2 i.e., the coefficients of the quadratic equation in Fig.2 are shown in Fig.3. Though the comparison between the analytical results and experimental ones can not be undertaken satisfactorily for lack of appropriate experimental values, the comparison between the experimental results ($R_e = 1.01 \times 10^5$, aspect ratio of a channel 5 : 1) of Brouillette et al. (9) and the second approximation is made in Fig.4. Quantitative discussions are impossible because of Hartmann number and aspect ratio of the channel being different in both results but the trend of equalization of velocities due to a magnetic field (Hartmann effect) is recognized well.

3-3. Entrance flow Let us consider the entrance problem concerned with a turbulent flow coming in a magnetic field with a non-magnetically fully developed velocity profile. In the analysis, the following assumption are made:

- (1) The flow is a steady one.
- (2) All fluid properties are constant.
- (3) The flow does not vary in z-direction, i.e., the flow is two-dimensional.
- (4) The induced magnetic field in x-direction is negligible in comparison

with the applied field in y-direction.

- (5) The stress due to fluctuation of a magnetic field is negligible in comparison with that due to fluctuation of velocity.
- (6) The pressure is constant over the cross section.

$$(7) \quad \frac{\partial^2 \bar{v}_x}{\partial y^2} \gg \frac{\partial^2 \bar{v}_x}{\partial x^2}, \quad \frac{\partial \bar{v}_x' \bar{v}_y'}{\partial y} \gg \frac{\partial \bar{v}_x'^2}{\partial x}$$

- (8) The walls are electrically insulated.

Basic equations are written on the basis of the above mentioned assumptions and Eqs.(6),(7) as follows:

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} = 0 \dots \dots \dots (53)$$

$$\rho \left(\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} \right) = - \frac{d\bar{p}}{dx} + \frac{\partial \bar{\tau}}{\partial y} + B_0 \frac{\partial \bar{H}_x}{\partial y} \dots \dots \dots (54)$$

where $\bar{\tau}$ is a time-mean component of the shearing stress in fluid. From Eq.(53), we get

$$\bar{v}_y = - \int_0^y \frac{\partial \bar{v}_x}{\partial x} dy \dots \dots \dots (55)$$

Fig.3

Fig.2

Fig.4

Integrating Eq. (54) with respect to y from 0 to a and using Eq. (55) results in

$$2\rho \int_0^a \bar{v}_z \frac{\partial \bar{v}_z}{\partial x} dy - \rho \left[\bar{v}_z \int_0^y \frac{\partial \bar{v}_z}{\partial x} dy \right]_0^a = -a \frac{d\bar{p}}{dx} - \tau_w \quad (56)$$

where τ_w is the wall friction stress. At the center plane of the channel, Eq. (54) becomes

$$\rho \bar{V}_x \frac{d\bar{V}_x}{dx} = -\frac{d\bar{p}}{dx} + \left[\frac{\partial \bar{\tau}}{\partial y} \right]_{y=0} + B_0 \left[\frac{\partial \bar{H}_z}{\partial y} \right]_{y=0} \quad (57)$$

where $\bar{V}_x$ is the velocity at the center plane of the channel. The following equation is derived from Eqs. (56), (57):

$$2 \int_0^a \bar{v}_z \frac{\partial \bar{v}_z}{\partial x} dy - \left[\bar{v}_z \int_0^y \frac{\partial \bar{v}_z}{\partial x} dy \right]_0^a - a \bar{V}_x \frac{d\bar{V}_x}{dx} = -\frac{\tau_w}{\rho} - \frac{a}{\rho} \left[\frac{\partial \bar{\tau}}{\partial y} \right]_{y=0} - \frac{a B_0}{\rho} \left[\frac{\partial \bar{H}_z}{\partial y} \right]_{y=0} \quad (58)$$

It is assumed that the velocity profile $\bar{v}_x(x, y)$ at an arbitrary position in the entrance region is expressed in the form of linear connection of $\bar{v}_{xi}(y)$ and $\bar{v}_{xf}(y)$ as follows:

$$\bar{v}_x(x, y) = \bar{v}_{xi}(y)\gamma(x) + \bar{v}_{xf}(y)[1-\gamma(x)] \quad (59)$$

where $\bar{v}_{xi}(y)$, $\bar{v}_{xf}(y)$ are a non-magnetically developed velocity profile and a magnetically developed one respectively and $\gamma(x)$ is a function of x which must satisfy the following conditions:

$$\gamma(0)=1, \quad \gamma(\infty)=0, \quad \left[\frac{\partial \gamma}{\partial x} \right]_{x=\infty}=0 \quad (60)$$

The wall friction stress in the entrance region is expressed on the basis of the velocity profile Eq. (59) as follows:

$$\tau_w = \eta \left[\frac{\partial \bar{v}_z}{\partial y} \right]_{y=0} = \eta \left[\gamma \frac{\partial \bar{v}_{xi}}{\partial y} + (1-\gamma) \frac{\partial \bar{v}_{xf}}{\partial y} \right]_{y=0} = \gamma \tau_{wi} + (1-\gamma) \tau_{wf} \quad (61)$$

On the other hand, the following relations are valid in the magnetically developed flow:

$$0 = -a \frac{d\bar{p}}{dx} - \tau_w, \quad 0 = -y \frac{d\bar{p}}{dx} + \bar{\tau} - \tau_w + B_0 \bar{H}_z \quad (62)$$

and the shearing stress $\bar{\tau}$ in fluid is written as

$$\bar{\tau} = \tau_w [1 - (y/a)] - B_0 \bar{H}_z \quad (63)$$

Accordingly, the following relations are induced in the non-magnetically developed flow and the magnetically developed one respectively:

$$\bar{\tau}_i = \tau_{wi} [1 - (y/a)], \quad \bar{\tau}_f = \tau_{wf} [1 - (y/a)] - B_0 \bar{H}_{zf} \quad (64)$$

The shearing stress in the entrance region seems to be expressed by Eqs. (61), (64) as follows:

$$-2 \int_0^1 \{v_0 v_1 + \delta v_0 v_2 + (1-\gamma) \delta(v_0 v_1 + 2\delta v_1 v_2)\} \frac{d\gamma}{dx^*} dy^* + \left\{ [v_0 + (1-\gamma) \delta(v_0 + \delta v_2)] \int_0^{y^*} (v_1 + \delta v_2) dy^* \right\} \frac{d\gamma}{dx^*} + \{v_0 v_1 + \delta v_0 v_2 + (1-\gamma) \delta(v_0 v_1 + 2\delta v_1 v_2)\} y^* = 1 \frac{d\gamma}{dx^*} = \gamma(v_0 - 1) y^* = 1 \quad (74)$$

Omitting the calculating process and writing only the result, the following differential equation of $\gamma(x^*)$ is obtained:

$$\left\{ \frac{1}{D_0} (\delta \beta_0 + \delta^2 \beta_1) - \frac{1}{D_1} (\alpha_0 + \delta \alpha_1 + \delta^2 \alpha_2) \right\} \frac{1}{\gamma} d\gamma = dx^* \quad (75)$$

Solving the above equation results in

$$\frac{1}{D_0} (\delta \beta_0 + \delta^2 \beta_1) (\gamma - 1) - \frac{1}{D_1} (\alpha_0 + \delta \alpha_1 + \delta^2 \alpha_2) \log_e \gamma = x^* \quad (76)$$

$$\begin{aligned} \bar{\tau}(x, y) &= \bar{\tau}_i(y)\gamma(x) + \bar{\tau}_f(y)[1-\gamma(x)] \\ &= \tau_{wi} \left(1 - \frac{y}{a}\right) \gamma + \left[\tau_{wf} \left(1 - \frac{y}{a}\right) - B_0 \bar{H}_{zf} \right] (1-\gamma) \end{aligned} \quad (65)$$

As $\bar{H}_{xf} = 0$ at $y = 0$ in Eq. (65), it reduces to

$$\tau_w = \tau_{wi} \gamma + \tau_{wf} (1-\gamma)$$

and agrees with Eq. (61). In addition, Eq. (65) satisfies the condition of the shearing stress being zero at the center plane of the channel because of $\bar{H}_{xf}(a) = 0$ at $y = a$. From Eq. (65), Eq. (66) is obtained:

$$\left[\frac{\partial \bar{\tau}}{\partial y} \right]_{y=0} = -\frac{\tau_{wi}}{a} \gamma - \left(\frac{\tau_{wf}}{a} + B_0 \frac{d\bar{H}_{zf}}{dy} \right)_{y=0} (1-\gamma) \quad (66)$$

Substituting Eqs. (61), (66) into Eq. (58) results in

$$2 \int_0^a \bar{v}_z \frac{\partial \bar{v}_z}{\partial x} dy - \left[\bar{v}_z \int_0^y \frac{\partial \bar{v}_z}{\partial x} dy \right]_0^a - a \bar{V}_x \frac{d\bar{V}_x}{dx} = \frac{a B_0}{\rho} \left\{ \left[\frac{d\bar{H}_{zf}}{dy} (1-\gamma) - \frac{\partial \bar{H}_z}{\partial y} \right]_{y=0} \right\} \quad (67)$$

On the other hand, the following equation is derived from Eqs. (1), (5):

$$\frac{\partial \bar{H}_z}{\partial y} = -j_z = -\sigma (\bar{E}_z + \bar{v}_z B_0) \quad (68)$$

and from Eq. (28), Eq. (69) is obtained:

$$\bar{E}_z = -B_0 u_m \quad (69)$$

Therefore, Eq. (70) is induced on the basis of Eqs. (68), (69):

$$\frac{\partial \bar{H}_z}{\partial y} = -\sigma B_0 (\bar{v}_z - u_m) \quad (70)$$

Substituting Eq. (70) into Eq. (67) yields

$$2 \int_0^a \bar{v}_z \frac{\partial \bar{v}_z}{\partial x} dy - \left[\bar{v}_z \int_0^y \frac{\partial \bar{v}_z}{\partial x} dy \right]_0^a - a \bar{V}_x \frac{d\bar{V}_x}{dx} = \frac{a \sigma B_0^2}{\rho} \gamma (\bar{V}_x - u_m) \quad (71)$$

Making Eq. (71) nondimensional with Eq. (8), dropping the overscore which means the time-mean component in order to avoid complication, and rewriting it, we obtain the following equation:

$$2 \int_0^1 v_z^* \frac{\partial v_z^*}{\partial x^*} dy^* - \left[v_z^* \int_0^{y^*} \frac{\partial v_z^*}{\partial x^*} dy^* \right]_0^{y^*} - V_x^* \frac{dV_x^*}{dx^*} = \frac{4M^2}{R_e} \gamma (V_x^* - 1) = \delta \gamma (V_x^* - 1) \quad (72)$$

It is impossible to obtain the velocity profile v_{xi} , v_{xf} of the turbulent channel flow, including the flow of the viscous sub-layer and therefore we substitute the velocity profiles [Eqs. (43), (46), (47)] obtained in the preceding section for v_{xi} , v_{xf} , namely

$$v_{xi}^* = v_0, \quad v_{xf}^* = v_0 + \delta v_1 + \delta^2 v_2 \quad (73)$$

Substituting the above relations into Eq. (59) which expresses the velocity profile in the entrance region and applying this result to Eq. (72) yields

where

$$\begin{aligned}
 \alpha_0 &= 2A_0 - A_1 - A_2, & \alpha_1 &= 2B_0 + 2C_0 - B_1 - B_2, & \alpha_2 &= 4D_0 - C_1 - C_2, \\
 \beta_0 &= 2C_0 - E_1 - E_2, & \beta_1 &= 4D_0 - F_1 - F_2 = \alpha_2 \\
 A_0 &= c_0 \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right) - \frac{\sqrt{f_0}}{\kappa_*} \left(-\frac{a_1}{4} - 2b_1 + \frac{c_1}{4} + d_1 \right), & A_1 &= c_0 \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right), & A_2 &= c_0 (c_1 + d_1) \\
 B_0 &= c_0 \left(\frac{2}{27} a_2 - \frac{b_2}{9} - c_2 + \frac{d_2}{3} + \frac{e_2}{2} + g_2 \right) - \frac{\sqrt{f_0}}{\kappa_*} \left(\frac{2}{27} a_2 - \frac{2}{27} b_2 - 2c_2 + \frac{d_2}{9} + \frac{e_2}{4} + g_2 \right) \\
 B_1 &= c_0 \left(\frac{2}{27} a_2 - \frac{2}{9} b_2 - c_2 + \frac{d_2}{3} + \frac{e_2}{2} + g_2 \right) + (c_1 + d_1) \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right), & B_2 &= c_0 (d_2 + e_2 + g_2) + (c_1 + d_1)^2 \\
 C_0 &= a_1 \left(\frac{2}{27} a_1 + \frac{1}{2} b_1 - \frac{2}{9} c_1 - \frac{d_1}{2} \right) + b_1 \left(2b_1 - \frac{1}{2} c_1 - 2d_1 \right) + c_1 \left(\frac{1}{3} c_1 + d_1 \right) + d_1^2 \\
 C_1 &= (d_2 + e_2 + g_2) \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right) + (c_1 + d_1) \left(\frac{2}{27} a_2 - \frac{2}{9} b_2 - c_2 + \frac{d_2}{3} + \frac{e_2}{2} + g_2 \right) \\
 C_2 &= 2(c_1 + d_1)(d_2 + e_2 + g_2) \\
 D_0 &= a_2 \left(-\frac{3}{128} a_1 - \frac{2}{27} b_1 + \frac{c_1}{32} + \frac{2}{27} d_1 \right) + b_2 \left(\frac{a_1}{32} + \frac{2}{27} b_1 - \frac{c_1}{16} - \frac{d_1}{9} \right) + c_2 \left(\frac{a_1}{4} + 2b_1 - \frac{c_1}{4} - d_1 \right) \\
 &+ d_2 \left(-\frac{a_1}{16} - \frac{b_1}{9} + \frac{c_1}{4} + \frac{d_1}{3} \right) + e_2 \left(-\frac{a_1}{9} - \frac{b_1}{4} + \frac{c_1}{3} + \frac{d_1}{2} \right) + g_2 \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right) \\
 D_3 &= (c_0 - 1), & E_1 &= (c_1 + d_1) \left(-\frac{a_1}{4} - b_1 + \frac{c_1}{2} + d_1 \right), & E_2 &= (c_1 + d_1)^2, & F_1 &= C_1, & F_2 &= C_2 \\
 a_0 &= \frac{2\sqrt{f_0}}{\kappa}, & b_0 &= \frac{\sqrt{f_0}}{\kappa}, & c_0 &= \left(1 + \frac{2}{3} \frac{\sqrt{f_0}}{\kappa} \right), & a_1 &= \frac{1}{2\kappa_*}, & b_1 &= \frac{f_1}{2\kappa_* \sqrt{f_0}} \\
 c_1 &= -\frac{1}{2\kappa_* \sqrt{f_0}} \left(f_1 + \frac{\sqrt{f_0}}{\kappa_*} \right), & d_1 &= \frac{3}{8\kappa_* \sqrt{f_0}} \left(2f_1 + \frac{\sqrt{f_0}}{\kappa_*} \right), & a_2 &= -\frac{1}{16\kappa_*^2 \sqrt{f_0}} \\
 b_2 &= \frac{1}{8\kappa_*^2 \sqrt{f_0}} \left(\frac{1}{\kappa_*} + \frac{f_1}{\sqrt{f_0}} \right), & c_2 &= \frac{1}{2\kappa_* \sqrt{f_0}} \left(f_2 - \frac{f_1^2}{4f_0} \right), & d_2 &= -\frac{1}{32\kappa_* \sqrt{f_0}} \left(\frac{5}{\kappa_*^2} + \frac{4f_1}{\kappa_* \sqrt{f_0}} + \frac{2f_1^2}{f_0} \right) \\
 e_2 &= \frac{1}{16\kappa_* \sqrt{f_0}} \left(\frac{3}{\kappa_*^2} + \frac{4f_1^2}{f_0} - 8f_2 + \frac{2f_1}{\kappa_* \sqrt{f_0}} \right), & g_2 &= \frac{1}{2\kappa_* \sqrt{f_0}} \left(\frac{3}{2} f_2 - \frac{11f_1^2}{24f_0} - \frac{1}{72\kappa_* \sqrt{f_0}} - \frac{5}{108\kappa_*^2} \right)
 \end{aligned}$$

From Eqs. (61), (56), the coefficient of the wall friction stress, the pressure gradient, and the pressure drop in the entrance region can be expressed as follows:

$$\tau_w^* = \frac{\tau_w}{\rho u_m^2} = f_0 + (1-\gamma) \delta (f + \delta f_2) \quad (77)$$

$$-\frac{dp^*}{dx^*} = 2 \int_0^1 v_z^* \frac{\partial v_z^*}{\partial x^*} dy^* - \left[v_z^* \int_0^1 \frac{\partial v_z^*}{\partial x^*} dy^* \right]_0^1 + \tau_w^* \quad (78)$$

$$\begin{aligned}
 p_0^* - p_z^* &= \left[\frac{E_1}{2} - C_0 + \frac{1}{2D_3} f_1 \beta_0 \right] \delta^2 (1-\gamma^2) - \left[A_1 - 2A_0 + \frac{1}{D_3} \alpha_0 f_1 \right] \delta (1-\gamma) \\
 &- \left[B_1 - 2B_0 - 2C_0 + \frac{1}{D_3} (\alpha_1 f_1 + \alpha_0 f_2) \right] \delta^2 (1-\gamma) + (f_0 + \delta f_1 + \delta^2 f_2) x^* \quad (79)
 \end{aligned}$$

3-4. Results and discussion The entrance length x_e is defined as the distance required for the velocity of center of the channel to come within 1% of fully developed value. The results of x_e/aR_e for $R_e = 7 \times 10^4$, 1.01×10^5 are shown in Fig. 5 and the solution of laminar flow⁽³⁾ is described for reference. In order to compare between the results of the present analysis and those of Maciulaitis et al.⁽²⁾, the wall friction stress is employed for the definition of the entrance length instead of the velocity at the center plane. On the basis of Murgatroyd's experimental data, the coefficient of the wall friction stress f becomes 1.11 times as large as that f_0 of $M = 0$, namely $f_0 = 0.9f$, at $M \approx 39$ in the case of $R = 1.01 \times 10^5$. Therefore, employing the same definition as Maciulaitis' (the distance required for the wall friction stress to come within 10% of fully developed value) results in $x_e = 0$ in $M < 39$. So the entrance length is defined as the length required for the wall friction stress to come within 1 or 5% of fully developed value. The calculating results on the basis of the above definition are shown in Fig. 6. The direct comparison between the results of the present analysis and those of Maciulaitis et al. (the dis-

tance required for the wall friction stress to come within 10% of fully developed value) is impossible because of the above reason but for reference the results of Maciulaitis et al. are shown in the same figure.

From the results of Maciulaitis et al., it is found that the entrance length has the maximum value at a certain Hartmann number and decreases as Hartmann number increases or decreases from it. According-

Fig. 5

ly, it seems that the present analysis is valid in the region of relatively small Hartmann numbers below the Hartmann number at which the entrance length attains the maximum value.

A few examples of the results concerning the wall friction stress and the variation of the pressure are shown in Figs. 7-9. $M = 18.7, 28.1, 32.4$ at $Re = 7 \times 10^4$ correspond to $\delta = 0.02, 0.045, 0.06$ (see Fig. 2) and the curves expressing the coefficient of the wall friction stress and the variation of the pressure take different shapes according to the value of M (that is, δ).

4. Conclusions

The theoretical analysis has been for the MHD turbulent flow of an electrically conducting fluid entering a magnetic field from a region without a magnetic field. In the first place, the velocity profile of a fully developed flow is obtained in the form of the power series of $\delta (= 4M^2/Re)$ and next a momentum-integral method is applied to the analysis of the entrance flow in which the velocity profile at the arbitrary section is assumed to be expressed in the form of linear combination of the velocity profiles of a non-magnetically fully developed flow and of a magnetically fully developed one. The present results are subject to restriction of Hartmann number because of the viscous stress being neglected and the second approximation being employed as the velocity profile but they seem to be valid in the region of relatively small Hartmann numbers judging from the comparison with the results of Maciulaitis et al. In the region where the present analysis can not be applied, it is considered that a kind of the transition region is formed because of the effect of turbulent suppression due to a magnetic field being remarkable. Therefore, it is impossible to regard the flow in that region as a fully turbulent one and it seems necessary to make the analysis in consideration of the viscous stress.

References

- (1) Moffatt, W.C., Massachusetts Inst. Tech. Magnetogasdynamics Lab. Rep., 61-4 (1961-5).
- (2) Maciulaitis, A. and Loeffler, A.L. Jr., AIAA J., 2-12 (1964), 2100.
- (3) Sudou, K. et al., Bull. JSME, 16-92 (1973), 282.
- (4) Hwang, C.L. et al., Phys. Fluids, 9-6 (1966), 1134.
- (5) Harris, L.P., Hydromagnetic Channel Flows, (1960), John Wiley & Sons.
- (6) Goldstein, S., Modern Developments in Fluid Dynamics, Vol. 11 (1965), Dover.
- (7) Davies, J.T., Turbulence Phenomena, (1972), Academic Press.
- (8) Murgatroyd, W., Phil. Mag., 44 (1953), 1348.
- (9) Brouillette, E.C. and Lykoudis, P.S., Phys. Fluids, 10-5 (1967), 995.

Fig. 6

Fig. 7

Fig. 8

Fig. 9