The Three-Dimensional Stress Analysis of the Short Rectangular Prism*

(An analogous case of a cantilever problem)

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In this paper, the three-dimensional problems of the short rectangular prism, which are fixed at one end and partially loaded vertically on the upper surface near the opposite end, namely, which are analogous to a cantilever, are analyzed by the use of J.Boussinesq's or an extended H.Neuber's approach. From the results of numerical calculations the following conclusions are obtained.

Both longitudinal normal and shear stress distributions vary irregularly in the neighborhoods of the fixed and the loaded ends. In the regions far from those ends the former distributions are nearly linear and the latter ones almost parabolic.

1. Introduction

Long ago, one of the authors [6] presented a method of the three-dimensional elasticity solution for the short rectangular prism by using the approach given by J.Boussinesq or H.Neuber. Taking simple forms of the double Fourier series for the displacement and stress components that can be obtained using the properties of his series and simpler than ones expressed by the general double Fourier series, this method of solution appears to be very convenient to analyze the problem of the stress distributions in a short rectangular prism whose surfaces are free from the tangential tractions.

It is the purpose of this paper to analyze the three-dimensional problems of the short rectangular prism, which is fixed at one end and partially loaded vertically on the upper surface near the opposite end, namely, which is analogous to a short cantilever.

Although A.E.H.Love in his book [8] gave a full detail of the approximate solution of the three-dimensional analysis of bending of a cantilever, being a practically important one in engineering, the problem of a short cantilever treated in this paper can't be analyzed using his approximate solution. It would seem to us that few theoretical investigations of the problem in question have been carried out because of complication and numerous analytical difficulties of the three-dimen-

2. Basic Equations of Three-Dimensional Analysis

2.1 Boundary Conditions

It is assumed that the prism is made of a homogeneous, isotropic and elastic material. The dimensions of the prism and coordinate system used are shown in Fig.1, where the origin is located at the junction of the centerline of the upper surface (yz-plane) and fixed end (xy-plane). A uniformly distributed load of intensity p is applied over a small semicircular area of the upper surface, whose centre is at a point (0,0,2h) and whose radius is c. In this paper analysis and numerical calculations have been done for convenience sake about the body under the action of a tensile load, though Fig.1 and the figures of distributions of the displacement and stress in the subsequent section are given for the body under the action of a compressible load. The boundary conditions can be written as follows:

\[ \sigma_{x,x} = f(x,y) = \sum \frac{\cos \beta_0 \cos kx}{\beta_0} \]

\[ \sigma_{y,y} = f(y,z) = 0, \quad u_{x,y} = u_{y,y} = 0 \]

\[ w_{x,z} = w_{y,z} = 0, \quad w_{x,y} = w_{y,x} = 0 \]

\[ \sigma_{x,x} = r_{z,z} = r_{x,x} = 0 \]

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where
\[ \sigma_{xx} = \rho \alpha (1 - \mu^2) \frac{c(t)}{4bh_{xx}} + \frac{f}{(c(t))} \]
\[ = \frac{2\pi^2}{8G} \quad (n = z = 0) \]  
\[ \beta_n = \frac{4\pi}{h} \quad \eta_n = \frac{\pi}{2} \quad \lambda_n = \sqrt{\eta_n^2 + \beta_n^2} \]
\[ \alpha_n = \left\{ \begin{array}{ll} 1 & (n = 0) \\ 2 & (n \neq 0) \end{array} \right. \]
\[ \lambda_n = 2(n + 1) \]
\[ J(\lambda_n) \text{ is a Bessel function of the first kind. The mathematical manipulation to obtain the expressions for } \sigma_{xx} \text{ is given in Appendix.} \]

2.2 Solution to the Problems of Given Surface Tractions or Displacements for a Rectangular Prism

Using J. Boussinesq's or an extended H. Neuber's approach \(^{(1)} \), a general solution of the governing equations of the theory of elasticity in the absence of body forces can be expressed as follows:

\[ 2Gw = -\nabla \cdot \mathbf{F} + \mathbf{t} \cdot \mathbf{\Phi} + 2\mathbf{r} \otimes \mathbf{\Phi} \]
\[ \mathbf{u} = (x, y, z), \quad \mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1, \quad \mathbf{\Phi} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1, \]
\[ \mathbf{\Phi} = (\Phi_x, \Phi_y, \Phi_z), \quad r = (x, y, z) \]

\[ J(\lambda) \text{ in Eq. (1) is a Bessel function of the first kind.} \]

in which \( \Phi_i (i = 0, 3) \) and \( \beta_i (i = 1, 3) \) are harmonic functions and \( \mathbf{\nabla} \) is Laplace's operator, i.e., \( \mathbf{\nabla}^2 \Phi = \partial^2 \Phi / \partial x^2 + \partial^2 \Phi / \partial y^2 + \partial^2 \Phi / \partial z^2 \). \( G \) and \( \nu \) are shear modulus and Poisson's ratio, respectively. For a short rectangular prism, the basic harmonic functions are expressed in double Fourier series as follows:

\[ \Phi_{1,0,0} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} A_{10} \sinh \lambda_n + B_{10} \cosh \lambda_n \]
\[ \Phi_{0,1,0} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} A_{01} \sinh \lambda_n + B_{01} \cosh \lambda_n \]
\[ \Phi_{0,0,1} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} A_{00} \sinh \lambda_n + B_{00} \cosh \lambda_n \]

\[ \lambda_n = \sqrt{\eta_n^2 + \beta_n^2} \]

where \( \lambda_n = \pi / h \), \( \eta_n = \pi / 2h \), and \( \lambda_n = \pi / 2h \). In the expressions (3), it is taken into consideration that the prism is subjected to the symmetrical loading about the plane \( y = 0 \). Now, the displacements and stresses obtained from the solutions will be referred to as \( (n,s) \)-, \( (n,r) \)- and \( (r,s) \)-displacements and stresses respectively.

Substituting expressions (3) into Eq. (2), the displacements are written as follows:

\[ (n,s) \text{-displacements} \]
\[ 2Gw_{10} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{10} + (3 - \nu) B_{10}) \sinh \lambda_n + (-\nu \beta_n A_{10} + (3 - \nu) \beta_n B_{10}) \cosh \lambda_n \right] \]
\[ 2Gw_{01} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{01} + (3 - \nu) B_{01}) \sinh \lambda_n + (-\nu \beta_n A_{01} + (3 - \nu) \beta_n B_{01}) \cosh \lambda_n \right] \]
\[ 2Gw_{00} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{00} + (3 - \nu) B_{00}) \sinh \lambda_n + (-\nu \beta_n A_{00} + (3 - \nu) \beta_n B_{00}) \cosh \lambda_n \right] \]

\[ (n,r) \text{-displacements} \]

\[ 2Gw_{10} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{10} + (3 - \nu) B_{10}) \sinh \lambda_n + \nu \beta_n A_{10} \sinh \lambda_n \right] \]
\[ 2Gw_{01} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{01} + (3 - \nu) B_{01}) \sinh \lambda_n + \nu \beta_n A_{01} \sinh \lambda_n \right] \]
\[ 2Gw_{00} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (-\eta_n A_{00} + (3 - \nu) B_{00}) \sinh \lambda_n + \nu \beta_n A_{00} \sinh \lambda_n \right] \]

and \( (r,s) \text{-displacements} \)

\[ 2Gw_{10} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (3 - \nu) A_{10} \sinh \lambda_n + \nu \beta_n A_{10} \sinh \lambda_n \right] \]
\[ 2Gw_{01} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (3 - \nu) A_{01} \sinh \lambda_n + \nu \beta_n A_{01} \sinh \lambda_n \right] \]
\[ 2Gw_{00} = \sum \frac{2\pi \cos \lambda_n \cos \beta_n}{\eta_n^2 + \beta_n^2} \left[ (3 - \nu) A_{00} \sinh \lambda_n + \nu \beta_n A_{00} \sinh \lambda_n \right] \]

The expressions for the stress components can be obtained from Eqs. (4)-(6) by using following stress-strain relationships.

\[ \sigma_{xx} = \lambda t + 2Gt_{xx}, \quad \sigma_{yy} = \lambda t + 2Gt_{yy}, \quad \sigma_{zz} = \lambda t + 2Gt_{zz}, \quad \tau_{xy} = Gt_{xy}, \quad \tau_{yz} = Gt_{yz}, \quad \tau_{zx} = Gt_{zx} \]

in which \( \lambda, \mu, \text{ and } E \) are Lamé constant and Young's modulus respectively, i.e., \( J = E(1+\nu)(1-2\nu) \). Therefore, the stress components are written as follows:

\( (n,s) \text{-stresses} \)
\[
\begin{align*}
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right)
\end{align*}
\]

\[\text{(7)}\]

\[
\begin{align*}
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right)
\end{align*}
\]

\[\text{(8)}\]

and \((r,s)\)-stresses:

\[
\begin{align*}
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\sigma_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right) \\
\tau_{rr,0} &= \sum_{m=0}^{\infty} \cos k_1 \cos k_2 \beta [2(1-v)A_{m,0}^{(ii)} + 2A_{m,0}^{(ii)}] \sinh \left( k_1 x + k_2 y \right) + \beta \sin \left( A_{m,0}^{(ii)} \sinh k_1 x + k_2 y \right)
\end{align*}
\]

\[\text{(9)}\]

2.3 A Particular Solution to Satisfy the Normal Stress Boundary Conditions on the Surfaces \(x=0\) and \(x=2a\)

In order to satisfy the normal stress boundary conditions on the surfaces \(x=0\) and \(x=2a\), a particular solution of the problem of a prism subjected to uniform compression at \(x=0\) and free from tractions at \(x=2a\), which can be derived using A.E.H. Love's moderately thick plate theory \(4\), will have to be superposed over the solutions described before.

When the surface \(x=0\) is subjected to uniform compression, we have \(\sigma_{rr}=0\) everywhere, \(\partial \sigma_{rr}/\partial x=0\) at \(x=0\) and \(x=2a\), \(\sigma_{rr}=\text{constant}(\neq 0)\) at \(x=0\), \(\sigma_{rr}=0\) at \(x=2a\). A particular solution satisfying these conditions is

\[\sigma_{rr}=A_{x}(x-2a)^{(2)}(x+a)\]

To determine \(\theta_{rr}\), \(\tau_{rr}\), etc., we have the equations

\[\tau_{rr}=0, \quad A_{y}(x-2a)^{(2)}(x+a)\]
\[
\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} = 6u(x-a) + a(x-a) \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0
\]  
(10)

\[
\sigma_y + \sigma_x = A(x-y) + (2+\alpha)(x^2+2x+y^2) + 3(1+\nu)(x-y)^2 - 4(\nu^2 + \alpha^2)
\]  
(11)

\[
F_{x} = -6u(x-a), \quad F_{y} = 0, \quad F_{z} = 0
\]  
(12)

To satisfy Eq. (10) we take \( \sigma_x, \sigma_y \), and \( \tau_{xy} \) to have the forms

\[
\sigma_x = 3A(x-y)^2 + \frac{\partial^2 u}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 u}{\partial x \partial y}, \quad \tau_{xy} = -\frac{\partial^2 u}{\partial x^2}
\]  
(13)

where \( \chi \) must satisfy the following equation

\[
\frac{\partial^2 \chi}{\partial y^2} + (2+\alpha)(\frac{\partial^2 \chi}{\partial x \partial y} + 3(1+\nu)\frac{\partial^2 \chi}{\partial x^2}) = 4A(x-y) \quad \text{and} \quad f(x,y) = 4(\alpha^2 + \nu^2)
\]  
(14)

and from Eqs. (12) and (13) we have the following equation.

\[
\frac{\partial^2 \chi}{\partial y^2} + 6A(x-y)^2 = 0
\]  

Using Eq. (14) and the above equation \( \chi \) can be written as follows:

\[
\chi = A(x-y) \left( \frac{(2+\alpha)^2}{6} (x-y)^2 + (\frac{1}{2} a^2 + \frac{3}{2} \beta^2 y^2) - \frac{1}{4} (a^2 - 3\beta^2) x^2 + 2\alpha x^2 + \frac{1}{4} \beta^2 y^2 - \frac{1}{4} \beta^2 y^2 \right)
\]  

Furthermore, in order to satisfy the remaining normal stress boundary conditions, it is necessary to use additional stress functions such as \( \Phi = A(x-y) \), \( \Phi = B(x-y) \), and \( \Phi = C(x-y) \). Thus, we have the following expressions for the additional displacement and stress components.

\[
\begin{align*}
A_{x} &= \left( 1 + \nu \right) (x-y)^2 + 2(1-2\nu)B(x-y) \quad \text{and} \quad A_{y} = \left( 1 + \nu \right) (x-y)^2 + 2(1-2\nu)B(x-y) \\
A_{x} &= \left( 1 + \nu \right) (x-y)^2 + 2(1-2\nu)C(x-y) \quad \text{and} \quad A_{y} = \left( 1 + \nu \right) (x-y)^2 + 2(1-2\nu)C(x-y)
\end{align*}
\]  

Consequently, the resulting displacement and stress components are expressed as follows:

\[
\begin{align*}
A_{x} &= 2\nu A_{x} + 2A_{x} + 2A_{x} + 2(1-2\nu)B(x-y) \quad \text{and} \quad A_{y} = 2\nu A_{y} + 2A_{y} + 2A_{y} + 2(1-2\nu)B(x-y) \\
A_{x} &= 2\nu A_{x} + 2A_{x} + 2(1-2\nu)C(x-y) \quad \text{and} \quad A_{y} = 2\nu A_{y} + 2A_{y} + 2(1-2\nu)C(x-y)
\end{align*}
\]  

\[
\begin{align*}
\tau_{xy} &= A_{x} (x-y)^2 + 2(1-2\nu)B_2(x-y) \\
\tau_{xy} &= A_{x} (x-y)^2 + 2(1-2\nu)C_2(x-y)
\end{align*}
\]  

2.4 Derivation of Relations among Fourier Coefficients

From the boundary conditions \( r_{x, y = 0} = 0 \) and \( r_{t,y = 0} = 0 \), we readily obtain the relations for \( A_{x,y}(1)+A_{x,y}(1-1-3) \).

\[
A_{x,y} = \sinh 2a_{1}, \quad A_{x,y} = \sinh 2a_{1}, \quad A_{x,y} = \sinh 2a_{1}, \quad A_{x,y} = \sinh 2a_{1}
\]  

Likewise, from the boundary conditions \( r_{x, y = 0} = 0 \), \( r_{t,y = 0} = 0 \), and \( \Phi_{x,y} = 0 \), \( \Phi_{x,y} = 0 \), we obtain the following relations for \( B_{x,y}(1-1-3) \) and \( C_{x,y}(1-1-3) \).

\[
B_{x,y} = 0, \quad C_{x,y} = 0
\]  

\[
\begin{align*}
C_{x,y} &= \left( \sinh 2a_{1} \cosh 2a_{1} \right) \quad \text{and} \quad C_{x,y} = \left( \sinh 2a_{1} \cosh 2a_{1} \right) \\
C_{x,y} &= \left( \sinh 2a_{1} \cosh 2a_{1} \right) \quad \text{and} \quad C_{x,y} = \left( \sinh 2a_{1} \cosh 2a_{1} \right)
\end{align*}
\]  

Where \( \delta \) and \( \delta \) in Eq. (18) are given as follows:

\[
\delta = \begin{cases} 
1 (r = 0) \\
0 (r = 0) 
\end{cases}
\]  

Therefore, six coefficients \( A_{x,y}, A_{x,y}, B_{x,y}, C_{x,y}, B_{x,y}, \) and \( C_{x,y} \) are independent of each other. Then, for brevity's sake, we change the coefficients by putting as \( A_{x,y} = A_{x,y}, B_{x,y} = B_{x,y}, C_{x,y} = C_{x,y} \) and \( A_{x,y} = A_{x,y} \).

From the remaining boundary conditions, after some mathematical manipulations with the aid of the orthogonality of trigonometric functions, we finally obtain the following relation formulae.

\[
A_{x,y} = a_{2}(\text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y})) + a_{2}(\text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y}))
\]  

in which

\[
\begin{align*}
G_{x,y} &= \text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y}) \\
F_{x,y} &= \text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y}) \\
I_{x,y} &= \text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y}) \\
J_{x,y} &= \text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y}) \\
N_{x,y} &= \text{Im}(B_{x,y} + B_{x,y} + C_{x,y} + C_{x,y})
\end{align*}
\]  

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\[ Q_u = e^{(1-\frac{a}{b} + \alpha)}(\cos \theta)_{\frac{2}{3}} + a \sin \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]  
\[ Q_\theta = e^{(1-\frac{a}{h} + \alpha)}(\cos \theta)_{\frac{2}{3}} + a \sin \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]  
\[ Q_\phi = e^{(1-\frac{a}{b} + \alpha)}(\cos \theta)_{\frac{2}{3}} + a \sin \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]  
\[ Q_\nu = e^{(1-\frac{a}{h} + \alpha)}(\cos \theta)_{\frac{2}{3}} + a \sin \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]

and

\[ 2A_\alpha + u_\alpha A_\alpha + (1-\psi)B_\alpha + C_\alpha = 0 \]
\[ H_{\alpha B} + B = \frac{\sqrt{1}}{4} \left[ \left( T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} \right) \right] + M_{\alpha B} \]

in which

\[ H_{\alpha B} = \alpha A_\alpha \cos \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]
\[ L_{\alpha B} = 3A_\alpha \cos \theta_{\frac{2}{3}} \]
\[ M_{\alpha B} = (1-\psi) \frac{\sqrt{1}}{4} \left( T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} \right) \]
\[ T_{\alpha B} = \frac{\sqrt{1}}{4} \left( T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} + T_{\alpha B} \right) \]

and

\[ (x-2)A_\alpha + u_\alpha A_\alpha + B_\alpha + B_\alpha = 0 \]
\[ K_{\alpha C} = -K_{\alpha C} - \frac{\sqrt{1}}{4} \left( T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} \right) \]
\[ \frac{\sqrt{1}}{4} \left( T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} + T_{\alpha C} \right) \]

in which

\[ K_{\alpha C} = \alpha \left( z_\alpha \cos \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \right) \]
\[ K_{\alpha C} = \alpha \left( z_\alpha \cos \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \right) \]
\[ z_\alpha = 2 \alpha \theta_{\frac{2}{3}} \cos \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]
\[ \theta_{\alpha C} = \alpha \theta_{\frac{2}{3}} \cos \theta_{\frac{2}{3}} + b \sin \theta_{\frac{4}{3}} \]

The set of equations (20)-(26) is an infinite set of simultaneous equations and can be solved for the unknown Fourier coefficients \( A_u, A_\theta, A_\phi, C_\mu \) and \( D_\alpha \). Substituting the values of Fourier coefficients into Eqs. (4)-(9) and Eqs. (15), (16), the displacement and stress components can be evaluated.

3. One Method of Improving Numerical Results

Supposing that the Fourier coefficients are truncated up to \( n \), \( n = N \) and \( n = N \), an infinite set of simultaneous equations (20)-(26) is truncated also up to \( n = N \), \( n = N \) and \( n = N \). For a prescribed value of \( N \) the unknown Fourier coefficients \( A_u, A_\theta, A_\phi, C_\mu \) and \( D_\alpha \) can then be solved by the method of S.O.R. (Successive Overrelaxation). But in such asymmetrical problem as mentioned above, we have some difficulties of numerical calculation, e.g., cancelling, overflow in numerical calculation by an electronic digital computer etc. To get over these difficulties, the unknown Fourier coefficients are changed as follows:

\[ A_u = A_u' + A_u'' \]
\[ A_\theta = A_\theta' + A_\theta'' \]
\[ C_\mu = C_\mu' + C_\mu'' \]
\[ D_\alpha = D_\alpha' + D_\alpha'' \]

in which \( A_u', A_\theta', C_\mu', A_u'' \) and \( D_\alpha'' \) are new unknown coefficients after changing.

In the case of a cube \( a = b = h = 1 \)
subjected to a uniform load on the semicircular area of radius \( c/\pi = 0.2 \), the mean value of \( \sigma \) on the upper surface \((x=0)\) calculated by taking \( N=20 \) is

\[ \int_0^\pi x f(s) \cos(s) ds = 0.456 \times 10^{-4} \]

and the mean value of loading function on the upper surface is

\[ \int_0^\pi f(s) \cos(s) ds = 0.157 \times 10^{-4} \]

The mean value of \( \sigma \) on the lower surface \((x=2a)\) is

\[ \int_0^\pi x f(s) \cos(s) ds = 0.145 \times 10^{-4} \]

and the mean value of loading function on the lower surface is

\[ \int_0^\pi f(s) \cos(s) ds = 0.0 \]

Furthermore, in the case of \( N=30 \), the mean value of \( \sigma \) on the upper surface is

\[ \int_0^\pi x f(s) \cos(s) ds = 0.450 \times 10^{-4} \]

and that of \( \sigma \) on the lower surface is

\[ \int_0^\pi f(s) \cos(s) ds = 0.143 \times 10^{-4} \]

Namely, \( \sigma_0/\sigma \) becomes nearly \( 0.015 \) in the domain of \( \sigma=0 \) on the upper surface and becomes nearly \( 0.015 \) over the lower surface. A main cause for these errors is truncation of Fourier series and the calculated erroneous values of normal stress on the upper and lower surfaces have a conspicuous effect on the distributions of displacements and stresses in the analogous case of a cantilevered beam. In order to obtain more correct results we have to calculate using a sufficient number of terms in the Fourier series, but then its calculation involves great difficulties in regard to the capacity of an electronic digital computer.

In this paper, approximations or improvements are made only in satisfying the normal stress boundary conditions prescribed on the upper and lower surfaces. In the first place, we replace \( \Delta w_{\text{up}} \) and \( \Delta w_{\text{down}} \) with \( \Delta w_{\text{up}}(x+y/z)^{2/3} \) and \( \Delta w_{\text{down}}(x+y/z)^{2/3} \) respectively and write for \( \varphi_w \) from Eqs. (21) and (22) as follows:

\[ \varphi_w = \Delta w_{\text{up}} - \Delta w_{\text{down}} + \sum_{n=1}^{\infty} \left[ \varphi_n \text{sin}(nx) + \varphi_n \text{cos}(nx) \right] \]

In the second place, the sums of the squares of errors of the normal stress over the upper and lower surfaces are denoted by \( F^0 \) and \( F^0 \) respectively. Namely, \( F^0 \) and \( F^0 \) are

\[ F^0 = \sum_{n=1}^{\infty} \left( (\sigma_n(x,y) - \sigma_n(x,y)) \right)^2 dy \]

in which

\[ (\sigma_n(x,y))_{\text{up}} = \Delta w_{\text{up}} + \sum_{n=1}^{\infty} \left[ \varphi_n \text{sin}(nx) + \varphi_n \text{cos}(nx) \right] \]

Provided that under the constrained conditions \( \varphi_n = \psi_n = 0 \) \((n, z = 0, N)\) without \( n = 0 \), the functionals \( F^0 \) and \( F^0 \) are minimized with respect to the unknown coefficients \( \Delta w_{\text{up}}(x,y,z=0, N), \Delta w_{\text{down}}(x,y,z=0, N) \) which are to satisfy Eqs. (23)–(26), the errors of the normal stress over the upper and lower surfaces may be reduced. Instead of the functionals \( F^0 \) and \( F^0 \), new functionals \( Q^0 \) and \( Q^0 \) are defined by using Lagrange multipliers \( \lambda_i \) and \( \mu_i \) as follows:

\[ Q^0 = F^0 + \sum_{i=0}^{N-1} \frac{1}{2} \left( \Delta w_{\text{up}} - \Delta w_{\text{down}} \right) \lambda_i \mu_i \]

Therefore, the minimum conditions of \( F^0 \) and \( F^0 \) can be expressed as follows:

\[ \frac{\partial Q^0}{\partial \Delta w_{\text{up}}} = \frac{\partial Q^0}{\partial \Delta w_{\text{down}}} = 0 \]

\[ \frac{\partial Q^0}{\partial \lambda_i} = \frac{\partial Q^0}{\partial \mu_i} = \frac{\partial Q^0}{\partial \lambda_i} = \frac{\partial Q^0}{\partial \mu_i} = 0 \]

\[ (n, i = 0, N, i = 1, 2) \]

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\[
\begin{align*}
\frac{\partial Q_{1n}}{\partial C_{n}} &= 1 \frac{\partial F_{1n}}{\partial C_{n}}, \\
\frac{\partial Q_{2n}}{\partial C_{n}} &= 1 \frac{\partial F_{2n}}{\partial C_{n}}, \\
\frac{\partial Q_{3n}}{\partial C_{n}} &= 1 \frac{\partial F_{3n}}{\partial C_{n}}, (r,s=\pi,N, t=1,2)
\end{align*}
\]

From Eq. (27) the following equations are obtained:

\[
\begin{align*}
45z(4)A_{m} + sA_{m} - s^2(1-\gamma)(a/b) \sum_{r,s=\pi,N} \frac{\partial \delta C_{n}}{\partial D_{n}} &= 0 \\
\end{align*}
\]

where \( \zeta(z) \) is Riemann's \( \zeta \)-function and \( \zeta(4)=\pi^4/90 \). On the other hand, Eqs. (28) and (29) are satisfied obviously. This method means that the uniformly distributed loads are superposed over the loads applied to the upper and lower surfaces of the short rectangular prism so as to reduce the errors caused by truncation of the series.

Now, in the case of a cube \((a=b=h=1)\) subjected to a uniform load on the semi-circular area of radius \( c/b=0.2 \) by calculating through Eqs. (21) \~ (26) and (30), taking \( N=30 \), the values of \( \sigma_{y}/\sigma \) come to be nearly \( \pm 0.05 \) in the domain of \( \sigma_{y} \neq 0 \) of the upper surface and the mean value of \( \sigma_{y}/\sigma \) on the upper surface is

\[
\int \int_{s} \sigma_{y} dA_{y} / (2\pi b) = 0.1527 \times 10^{-3}.
\]

\( \sigma_{y}/\sigma \) becomes nearly \( \pm 0.005 \) on the lower surface and the mean value of \( \sigma_{y}/\sigma \) on the lower surface is

\[
\int \int_{s} \sigma_{y} dA_{y} / (2\pi b) = -0.1439 \times 10^{-3}.
\]

By comparing these results with the previous results obtained from Eqs. (20) \~ (26), it is evident that the numerical results can be improved to some extent by using this method.

4. Numerical Results

4.1 The Case of a Cube

The numerical calculation is done by taking \( a=b=h=1, c/b=0.2, \gamma=0.33 \) and \( N=30 \), with 46 iterations with the aid of the acceleration parameter \( \gamma = 0.55 \). Although analysis and the numerical calculation have been done for the body under the action of a tensile load in the previous sections, the figures of distributions of displacements and stresses in this section are given for the body under a compressible load. In the figures the coordinates \((x, y, z)\) mean the dimensionless coordinates \((x/b, y/b, z/b)\).

![Fig.2 Deformation of the cross-section at y=0](image1)

![Fig.3 Deformations of the cross-section at z=1.0 and 2.0](image2)

![Fig.4 Distributions of \( \sigma_{z} \) along the lines x=0, 0.2, ... , and 1.0 in the cross-section y=0](image3)

![Fig.5 Distributions of \( \sigma_{z} \) along the lines z=1.0, 1.4, ... , and 2.0 in the cross-section y=0](image4)

![Fig.6 Distributions of \( \sigma_{z} \) along the lines z=1.0, 1.4, ... , and 2.0 in the cross-section y=0.5](image5)

![Fig.7 Distributions of \( \sigma_{z} \) along the lines x=0, 0.4, ... , and 2.0 in the cross-section y=0](image6)
Figure 2 shows the deformation of the cross-section at $y=0$. The right hand side of Fig.3 shows the deformation of the cross-section at $z=1.0$ and the opposite side is one at $z=2.0$. In Figs.2 and 3 the deformations are expressed in $u$, $v$, and $w$, respectively. From Figs.2 and 3 it may be found that the deformation becomes uniform at large distance from the loaded area. Figure 4 shows the distribution of the normal stresses $\sigma_x$ along the lines $x=0$, $0.2$, $0.4$, $y=0$, and $1.0$ in the cross-section $y=0$, and also Figs.5, 6 show ones along the lines $z=0.1$, $0.4$, $y=0$, and $1.0$ in the cross-sections $y=0$ and $y=0.5$ respectively. In Figs.4 and 5 the magnitude of the normal stress $\sigma_x$ decreases rapidly as it goes away from the loaded area.

The distributions of the longitudinal normal stresses $\sigma_y$, along the lines $x=0$, $0.2$, $0.4$, $0.6$, and $1.0$ in the sections $y=0$ and $y=0.5$ are shown in Figs.7 and 8 respectively, and those along the lines $z=0$, $0.4$, $0.8$, $1.0$ in the section $y=0$ are shown in Fig.9. From Fig.9 it may be found that the longitudinal normal stress distributions in the sections parallel to the $xy$-plane, which are far from the fixed and loaded ends, are almost linear.

The distributions of the shear stresses $\tau_{xy}$ along the lines $z=0.6$, $1.0$, $1.8$, and $1.0$ in the section $y=0$ are shown in Fig.10. In Fig.10 the peak of the magnitude of the shear stress $\tau_{xy}$ occurs at the neighborhood of the loaded area.

4.2 The Case of Decreasing Depth

The numerical calculation is done by taking $a/b=0.5$, $b/h=1.0$, $c/h=2$, $y=0.33$ and $N=30$, with 54 iterations with the aid of the acceleration parameter $\omega=0.16$.

Fig.11 shows the deformation of the cross-section at $y=0$ and the right and the left side of Fig.12 show the deformations of the cross-section at $z=1.0$ and $z=2.0$, respectively. In Figs.11 and 12 the deformations are described using $2u/(5pb)$, $2v/(5pb)$ and $2w/(5pb)$ instead of $u$, $v$, and $w$. It may be found from Fig.12 that the deformation of the cross-section at $z=1.0$ except the neighborhood of its corners shows the tendency of the anti-clastic deformation.

In Fig.13 the distributions of the shear stresses $\tau_{xy}$ along the lines $z=0.2$, $0.6$, $0.8$, $1.0$, and $1.8$ in the sections $y=0.5$ are shown. The distributions of the shear stresses $\tau_{xy}$ in the sections parallel to the $xy$-plane are almost parabolic as shown in Fig.13.

5. Conclusions

In this paper, the three-dimensional stress problems of the short rectangular prism, which is fixed at one end and par-
tially loaded on the upper surface near the opposite end, namely, which are anal-
ogous to a cantilever, are analyzed using J. Boussinesq's or an extended H. Neuber's ap-
proach.

The solution of such asymmetrical problems being expressed in the form of
double Fourier series, the cancelling may occur in carrying out the numerical cal-
ulation. We can overcome the difficulty of the cancelling by means of changing the
unknown Fourier coefficients. From the results of numerical calculations the follow-
ing conclusions are obtained.

The errors of normal stress on the free boundaries of the upper and lower
surfaces, which come from the effect of truncation of double Fourier series, have
a remarkable effect on the distributions of displacements and stresses in the short
rectangular prism. In the present paper, one method by which these errors can be
reduced is presented and it is found that the distributions of displacements and
stresses in the prism are obtained with accuracy by using this method.

The distributions of the longitudinal normal stresses \( a_n \) and the shear stresses \( a_s \) vary irregularly in the neighborhood of the fixed and loaded ends. On the other
hand, they are nearly linear and nearly parabolic respectively in the regions
which are far from the fixed and loaded ends.

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Appendix

The function \( f_1(y, z) \) is p (=constant)
over a semicircular area, whose centre is
situated at a point \((0, 0, 2h)\) and whose ra-
dius is \( c \), and zero outside its area. Now,
we expand the function \( f_1(y, z) \) into the form of
double Fourier series in terms of \( \cos \beta x \cos \gamma y \)

\[
\begin{aligned}
\{a_n\} = \sum_{n=1}^{\infty} a_n \cos \beta_n x \cos \gamma_n y
\end{aligned}
\]

From the following relation

\[
\begin{aligned}
\int_0^1 \int_0^{2\pi} \cos \beta x \cos \gamma y (x - 2b)(\cos \beta y) \, dy \, dx = \frac{(a_{n-1})^2}{\beta n (1 - a_{n-1})^2}
\end{aligned}
\]

\( a_n \) is written as follows:

\[
\begin{aligned}
a_n = \frac{(a_{n-1})^2}{\beta n (1 - a_{n-1})^2} \int_0^1 \int_0^{2\pi} f_1(y, z) \cos \beta x (x - 2b) \\
\times \cos \gamma y \, dy \, dx = \frac{(a_{n-1})^2}{\beta n (1 - a_{n-1})^2} \int_0^{2\pi} \int_0^1 \cos \Gamma \cos \theta \, d\theta \\
\times \cos \gamma \theta \, d\theta \int_0^{2\pi} \int_0^1 \cos \Gamma \cos \theta \, d\theta
\end{aligned}
\]

in which \( \lambda = \pi r/(2h) \), \( \beta = \pi r/\beta \), \( a_n = 1(n=0) \), \( a_n = 2(n \geq 1) \). From Eq. (31) \( a_n \) becomes as follows.

(1) \( n=0 \)

\[
\begin{aligned}
a_n = \frac{p}{2b} \int_0^{2\pi} \int_0^1 \cos \theta \, d\theta = \frac{p}{2b} \pi r^2
\end{aligned}
\]

(2) \( n \neq 0 \) or \( n=0 \)

in Eq. (31) \( \int_0^{2\pi} \int_0^1 \cos \Gamma \cos \theta \cos \beta \cos \gamma \theta \, d\theta \, d\theta \) can be expanded into

\[
\begin{aligned}
\int_0^{2\pi} \int_0^1 \cos \Gamma \cos \theta \cos \beta \cos \gamma \theta \, d\theta \, d\theta = \frac{1}{4} J_0(\lambda) J_0(\beta) - \frac{1}{2} J_1(\lambda) J_1(\beta)\]

\[
J_n(\lambda) J_n(\beta) + \frac{1}{4}
\]

in which \( J_n(\lambda) \), \( J_0(\lambda) \), \( J_1(\lambda) \), ... are Bessel func-
tions of the first kind. Using Gegenbauer's addition theorem \( a_n \) becomes as follows:

\[
\begin{aligned}
a_n = \frac{\pi \rho e^0}{4\lambda} \int_0^{2\pi} J_n(\lambda) J_n(\beta) \, d\theta \int_0^1 \cos \Gamma \cos \theta \, d\theta
\end{aligned}
\]

From Eqs. (32) and (33) \( a_n \) becomes as follows:

\[
\begin{aligned}
a_n = \frac{\pi \rho e^0}{4\lambda} \int_0^{2\pi} J_n(\lambda) J_n(\beta) \, d\theta \int_0^1 \cos \Gamma \cos \theta \, d\theta
\end{aligned}
\]

in which \( \rho e = \sqrt{\lambda^2 + \beta^2} \). The above expansion was presented by S. Woinowsky-Krieger (7).