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Motion of Pistons in Piston-Type Hydraulic Machines*

(3rd. Report : Exponential Function-Type Piston)

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To stabilize the motion of pistons, it is possible to use such a piston that the film thickness between the cylinder and the piston varies exponentially with the longitudinal position along the piston.

For piston-type hydraulic pumps and motors the numerical results indicate that in such a piston in which the film is thicker on the high pressure side, both stabilization of the motion and creation of film in the clearance are expectable. It is shown that the load capacity increases by expansion of cylinder due to pressure, while it reduces by projection of pistons out of a cylinder block. The leakage flow rate in the clearance and the frictional force acting on pistons are also shown.

1. Introduction

It has been reported that the motion of pistons in oil-hydraulic machines, which are widely used as oil-hydraulic equipment, is generally unstable⁽¹⁾⁽²⁾. To stabilize the motion, it is possible to use such a piston that the clearance between the cylinder bore and the piston varies exponentially with the longitudinal position along the piston. In this paper, the motion of these pistons, which are here called the exponential function-type pistons, is theoretically analyzed. The analysis is approximately applicable to tapered pistons.

Expansion of the cylinder bore due to the pressure is considered, and the leakage flow rate in the clearance as well as the frictional force acting on the piston is also estimated.

Though it is well-known that the tapered pistons can effectively prevent hydraulic-lock, there is no study on the motion of pistons, except one⁽³⁾ which estimated the statical locking force and analyzed the lateral vibration.

2. Nomenclature

We use the symbols used in the 1st. report. The following are the additional ones used in this report.

- C_0 : radial clearance at low pressure end of piston
- p_0 : pressure acting on low pressure end of piston
- p_1 : pressure acting on high pressure end of piston
- p_{bmax} : maximum value of p_b

- Q_z : leakage flow rate in z direction
- R : piston radius at low pressure end
- R_1 : parameter for projecting length of piston from cylinder-block end
- z_1 : z coordinate at low pressure end of piston
- z_2 : z coordinate at high pressure end of piston
- λ : parameter to express clearance (Eq. (1))
- λ_0 : a part in λ decided by manufacturing (Eq.(21))
- λ_p : a part in λ corresponding to expansion of cylinder bore (Eq.(21))

Dimensionless quantities :

$$\bar{R}_1 = R_1/R$$

In definition of dimensionless quantities, we use C_0 instead of C used in 1st.report.

3. Fluid force acting on piston

The following assumptions are made:

- 1) Assumptions in the theory of hydrodynamic lubrication are valid for the flow in the clearance.
- 2) Fluid is incompressible and its properties such as μ are constant.
- 3) ω_0 and ω are given.
- 4) Expansion of a cylinder bore is proportional to the pressure and there is no axial interference.⁽⁴⁾

We set xyz coordinate system shown in Fig.1, where z axis coincides with the center line of the cylinder bore. In Fig. 1, O' is the center of the piston and O_z is an arbitrary point on the center line of the piston z' . Referring to Fig.1, we obtain

$$\bar{h} = \exp \left\{ \lambda \left(\frac{L}{2R} + z \right) \right\} \\ - \bar{\alpha} [\varepsilon \cos(\varphi - \gamma) + \bar{z} \cos(\varphi - \beta)] \dots \dots (1)$$

As h is very small compared with piston diameter, we can ignore the curvature effects in the flow. Let $\hat{y}$ denote the coordinate which is perpendicular to the cylinder wall, $\hat{x}$ the peripheral direction and $\hat{z}(=z)$ the axial direction. Then the boundary conditions are given as follows:

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$$\left. \begin{aligned} \dot{y} &= 0 \text{ (cylinder), } u=v=w=0 \\ \dot{y} &= h \text{ (piston), } u=R\omega, \quad v=\partial h/\partial t, \\ w &= w_0 \end{aligned} \right\} \quad \dots\dots\dots (2)$$

Reynolds equation satisfying Eq.(2) is

$$\frac{\partial}{\partial \varphi} \left(\bar{h}^3 \frac{\partial \bar{p}}{\partial \varphi} \right) + \frac{\partial}{\partial \bar{z}} \left(\bar{h}^3 \frac{\partial \bar{p}}{\partial \bar{z}} \right) = \bar{\omega} \frac{\partial \bar{h}}{\partial \varphi} - \bar{w}_0 \frac{\partial \bar{h}}{\partial \varphi} + 2 \frac{RQ}{V} \frac{\partial \bar{h}}{\partial \tau} \quad \dots\dots\dots (3)$$

where $\chi = R\varphi$. The boundary conditions for $\bar{p}(\varphi, \bar{z}, \tau)$ are

$$\left. \begin{aligned} \bar{p}(0, \bar{z}, \tau) &= \bar{p}(2\pi, \bar{z}, \tau) \\ \frac{\partial}{\partial \varphi} \bar{p}(0, \bar{z}, \tau) &= \frac{\partial}{\partial \varphi} \bar{p}(2\pi, \bar{z}, \tau) \\ \bar{p}(\varphi, \bar{z}_1, \tau) &= \bar{p}_a(\tau) \\ \bar{p}(\varphi, \bar{z}_2, \tau) &= \bar{p}_b(\tau) \end{aligned} \right\} \quad \dots\dots\dots (4)$$

where $\bar{z}_2 = L/2R$ and generally $\bar{z}_1 = \bar{z}_1(\tau)$. When the piston reciprocates completely within the cylinder bore, $\bar{z}_1 = -L/2R$.

As $\bar{\alpha}$ is usually small enough, we may express $\bar{p}$ as follows:

$$\bar{p} = \bar{p}_0 + \bar{\alpha} \bar{p}_1 + \bar{\alpha}^2 \bar{p}_2 + \dots\dots\dots (5)$$

At first, the equation of $\bar{p}_0$ is

$$\frac{\partial^2 \bar{p}_0}{\partial \varphi^2} + \frac{\partial^2 \bar{p}_0}{\partial \bar{z}^2} + 3\lambda \frac{\partial \bar{p}_0}{\partial \bar{z}} = -\bar{w}_0 \lambda \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} + 2 \frac{RQ}{V} \left(\frac{L}{2R} + \bar{z}\right) \frac{d\lambda}{d\tau} \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \quad \dots\dots\dots (6)$$

$$\begin{aligned} a_0 &= \left\{ \bar{w}_0 \lambda - \frac{RQ}{V} \left(\frac{L}{R} - \frac{1}{\lambda} \right) \frac{d\lambda}{d\tau} \right\} / 2\lambda^2, \quad a_1 = -(RQ/\lambda^2 V) d\lambda/d\tau \\ C_1 &= \frac{\bar{p}_b - \bar{p}_a + (a_0 + a_1 \bar{z}_1) \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}_1\right)\right\} - (a_0 + a_1 \bar{z}_2) \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}_2\right)\right\}}{\exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}_2\right)\right\} + \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}_1\right)\right\}} \\ C_2 &= \bar{p}_b - (a_0 + a_1 \bar{z}_2) \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}_2\right)\right\} - C_1 \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}_2\right)\right\} \end{aligned}$$

Referring to $\partial \bar{p}_0 / \partial \varphi = 0$, we obtain the following equation for $\bar{p}_1$.

$$\frac{\partial^2 \bar{p}_1}{\partial \varphi^2} + \frac{\partial^2 \bar{p}_1}{\partial \bar{z}^2} + 3\lambda \frac{\partial \bar{p}_1}{\partial \bar{z}} = b_1 \cos(\varphi - \beta) + b_2 \cos(\varphi - \gamma) + b_3 \sin(\varphi - \beta) + b_4 \sin(\varphi - \gamma) \quad \dots\dots\dots (9)$$

where

$$\begin{aligned} b_1 &= \left\{ \bar{w}_0 - 6\lambda a_0 + 3a_1 - 12\lambda \bar{z} a_1 + 2 \frac{RQ}{V} \frac{d\bar{\alpha}}{d\tau} \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} - 9\lambda C_1 (1 - \lambda \bar{z}) \exp\left\{-4\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \\ b_2 &= -\left\{ 6\epsilon \lambda a_1 + 2 \frac{RQ}{V} \left(1 + \frac{\epsilon}{\bar{\alpha}}\right) \frac{d\epsilon}{d\tau} \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} + 9\epsilon \lambda^2 C_1 \exp\left\{-4\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \\ b_3 &= \bar{z} \left\{ \bar{w}_0 - 2 \frac{RQ}{V} \frac{d\beta}{d\tau} \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\}, \quad b_4 = \epsilon \left\{ \bar{w}_0 - 2 \frac{RQ}{V} \frac{d\gamma}{d\tau} \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \end{aligned}$$

From Eq.(4) the boundary conditions are

$$\bar{p}_1(0, \bar{z}, \tau) = \bar{p}_1(2\pi, \bar{z}, \tau), \quad \frac{\partial}{\partial \varphi} \bar{p}_1(0, \bar{z}, \tau) = \frac{\partial}{\partial \varphi} \bar{p}_1(2\pi, \bar{z}, \tau), \quad \bar{p}_1(\varphi, \bar{z}_1, \tau) = \bar{p}_1(\varphi, \bar{z}_2, \tau) = 0 \quad \dots\dots\dots (10)$$

Thus,

$$\begin{aligned} \bar{\alpha} \bar{p}_1 &= \left[A_1 \exp\left\{\left(-\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}\right\} + B_1 \exp\left\{\left(-\frac{3}{2}\lambda - \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}\right\} + G \right] \cos \varphi \\ &+ \left[D_1 \exp\left\{\left(-\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}\right\} + E_1 \exp\left\{\left(-\frac{3}{2}\lambda - \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}\right\} + H \right] \sin \varphi \quad \dots\dots\dots (11) \end{aligned}$$

where G and H are the particular solutions.

$$\begin{aligned} G(\bar{z}, \tau) &= \left\{ 2 \frac{RQ}{V} \frac{d\bar{X}}{d\tau} + 6\lambda a_1 \bar{X} + \bar{\omega} \bar{Y} - 3\lambda \bar{\omega} \bar{\phi}_X - 6\lambda \frac{RQ}{V} \frac{d\bar{\phi}_Y}{d\tau} - (\bar{w}_0 - 6\lambda a_0 + 3a_1 + 36\lambda^2 a_1) \bar{\phi}_Y \right. \\ &+ \bar{z} \left\{ \bar{\omega} \bar{\phi}_X + 2 \frac{RQ}{V} \frac{d\bar{\phi}_Y}{d\tau} + 12\lambda a_1 \bar{\phi}_Y \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} + \frac{9\lambda C_1}{4\lambda^2 - 1} \left\{ \lambda \bar{X} \right. \\ &\left. - \left(1 - \frac{5\lambda^2}{4\lambda^2 - 1}\right) \bar{\phi}_Y + \bar{z} \lambda \bar{\phi}_Y \right\} \exp\left\{-4\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \\ H(\bar{z}, \tau) &= \left\{ -\bar{\omega} \bar{X} + 2 \frac{RQ}{V} \frac{d\bar{Y}}{d\tau} + 6\lambda a_1 \bar{Y} - 6\lambda \frac{RQ}{V} \frac{d\bar{\phi}_X}{d\tau} - (\bar{w}_0 - 6\lambda a_0 + 3a_1 + 36\lambda^2 a_1) \bar{\phi}_X + 3\lambda \bar{\omega} \bar{\phi}_Y \right. \\ &+ \bar{z} \left\{ 2 \frac{RQ}{V} \frac{d\bar{\phi}_X}{d\tau} + 12\lambda a_1 \bar{\phi}_X - \bar{\omega} \bar{\phi}_Y \right\} \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \\ &\left. + \frac{9\lambda C_1}{4\lambda^2 - 1} \left\{ \lambda \bar{Y} - \left(1 - \frac{5\lambda^2}{4\lambda^2 - 1}\right) \bar{\phi}_X + \bar{z} \lambda \bar{\phi}_X \right\} \exp\left\{-4\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \right\} \end{aligned}$$

Fig. 1 Schematic representation of piston and cylinder bore

where $d\lambda/d\tau$ shows the change in $\bar{h}$ due to the cylinder pressure. The boundary conditions are derived from Eq.(4). That is,

$$\left. \begin{aligned} \bar{p}_0(0, \bar{z}, \tau) &= \bar{p}_0(2\pi, \bar{z}, \tau) \\ \frac{\partial}{\partial \varphi} \bar{p}_0(0, \bar{z}, \tau) &= \frac{\partial}{\partial \varphi} \bar{p}_0(2\pi, \bar{z}, \tau) \\ \bar{p}_0(\varphi, \bar{z}_1, \tau) &= \bar{p}_a(\tau) \\ \bar{p}_0(\varphi, \bar{z}_2, \tau) &= \bar{p}_b(\tau) \end{aligned} \right\} \quad \dots\dots\dots (7)$$

Thus,

$$\begin{aligned} \bar{p}_0 &= C_1 \exp\left\{-3\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} + C_2 \\ &+ (a_0 + a_1 \bar{z}) \exp\left\{-2\lambda\left(\frac{L}{2R} + \bar{z}\right)\right\} \quad \dots\dots\dots (8) \end{aligned}$$

where if we assume $\lambda \neq 0$,

$\bar{X}, \bar{Y}$, etc. are related to $\bar{X}, \bar{Y}$, etc. as follows⁽¹⁾;

$$\bar{X} = \bar{a} \varepsilon \cos \gamma, \quad \bar{Y} = \bar{a} \varepsilon \sin \gamma, \quad \bar{\phi}_X = \bar{a} \sin \beta, \quad \bar{\phi}_Y = \bar{a} \cos \beta \quad (12)$$

Integral constants A_1 and B_1 are decided by applying the third condition of Eq.(10).

$$A_1 = \frac{G(\bar{z}_2, \tau) \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1\right\} - G(\bar{z}_1, \tau) \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\}}{\left[\exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1 - \left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\}\right] - \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1 - \left(\frac{3}{2}\lambda - \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\}} \\ B_1 = \frac{G(\bar{z}_1, \tau) \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\} - G(\bar{z}_2, \tau) \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1\right\}}{\left[\exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1 - \left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\}\right] - \exp\left\{-\left(\frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_1 - \left(\frac{3}{2}\lambda - \frac{1}{2}\sqrt{9\lambda^2 + 4}\right)\bar{z}_2\right\}}$$

In the same way, D_1 and E_1 are respectively given by the above equations if $\bar{\phi}$ is replaced by H .

$\bar{P}_x, \bar{P}_y, \bar{M}_x$ and $\bar{M}_y$ are,

$$\bar{P}_x = -\frac{R}{2L} \int_{\bar{z}_1}^{\bar{z}_2} \int_0^{2\pi} \bar{p} \cos \varphi d\varphi d\bar{z}, \quad \bar{P}_y = -\frac{R}{2L} \int_{\bar{z}_1}^{\bar{z}_2} \int_0^{2\pi} \bar{p} \sin \varphi d\varphi d\bar{z} \quad (13)$$

$$\bar{M}_x = \frac{1}{2} \left(\frac{R}{L}\right)^2 \int_{\bar{z}_1}^{\bar{z}_2} \int_0^{2\pi} \bar{z} \bar{p} \sin \varphi d\varphi d\bar{z}, \quad \bar{M}_y = -\frac{1}{2} \left(\frac{R}{L}\right)^2 \int_{\bar{z}_1}^{\bar{z}_2} \int_0^{2\pi} \bar{z} \bar{p} \cos \varphi d\varphi d\bar{z} \quad (14)$$

Substituting Eq.(11) into the above expressions, we obtain

$$\bar{P}_x = -\frac{\pi R}{2L} \left[A_{11} \frac{d\bar{X}}{d\tau} + A_{12} \bar{X} + A_{13} \bar{Y} + A_{14} \frac{d\bar{\phi}_Y}{d\tau} + A_{15} \bar{\phi}_X + A_{16} \bar{\phi}_Y \right] \quad (15)$$

$$\bar{P}_y = -\frac{\pi R}{2L} \left[A_{21} \frac{d\bar{Y}}{d\tau} + A_{22} \bar{X} + A_{23} \bar{Y} + A_{24} \frac{d\bar{\phi}_X}{d\tau} + A_{25} \bar{\phi}_X + A_{26} \bar{\phi}_Y \right] \quad (16)$$

$$\bar{M}_x = \frac{\pi}{2} \left(\frac{R}{L}\right)^2 \left[A_{31} \frac{d\bar{Y}}{d\tau} + A_{32} \bar{X} + A_{33} \bar{Y} + A_{34} \frac{d\bar{\phi}_X}{d\tau} + A_{35} \bar{\phi}_X + A_{36} \bar{\phi}_Y \right] \quad (17)$$

$$\bar{M}_y = -\frac{\pi}{2} \left(\frac{R}{L}\right)^2 \left[A_{41} \frac{d\bar{X}}{d\tau} + A_{42} \bar{X} + A_{43} \bar{Y} + A_{44} \frac{d\bar{\phi}_Y}{d\tau} + A_{45} \bar{\phi}_X + A_{46} \bar{\phi}_Y \right] \quad (18)$$

Constants A_{11}, A_{12} , etc. are shown in the Appendix.

If $\bar{\omega}_0, \bar{P}_b - \bar{P}_a$, etc. are constant and $\bar{\omega} = 0$, for example, the steady characteristics $\bar{P}_x$ and $\bar{M}_y$ are

Table 1 Values used for examples

$L/R=5$	$RQ/V=1$
$\lambda_0=0.05$	$\bar{\omega}=0$
$\lambda_p=0$	$\bar{a}=0$
$\bar{p}_{b\max}=1$	$\bar{m}=0$
$\bar{W}_x=0$	$\bar{I}=0$
$\bar{W}_y=0.001\bar{p}_b$	$\bar{F}_x=\bar{F}_y=0$
$\bar{R}_1=0$	$\bar{T}_x=\bar{T}_y=0$

Fig.2 Steady characteristics
(1... A_{12} , 2... A_{16} , 3... A_{42} , 4... A_{46})

Fig.3 Effect of λ on steady characteristics

Fig.4 Stability condition

Fig.5 Pressure variation used for numerical examples

$$\bar{P}_s = -\frac{\pi R}{2L} [A_{12}\bar{X} + A_{16}\bar{\phi}_Y]$$

$$\bar{M}_v = -\frac{\pi}{2} \left(\frac{R}{L} \right)^2 [A_{12}\bar{X} + A_{16}\bar{\phi}_Y]$$

Fig.2 shows the numerical examples of A_{12} , etc. The dotted lines indicate the case when the value is negative. In case of $\bar{w}_0 = -1$, for example, the sign A_{12} and so on change at certain pressure. Fig.3 shows the effect of λ on A_{12} and A_{16} . Here, except in the case of $\lambda=1$, A_{12} increases with λ , while A_{16} decreases. As A_{12} and others contain terms which have $4\lambda^2 - 1$ in the denominator, there is some difference between the curves for $\lambda=1$ and others in this figure. Unlike the cylindrical piston, in the present case of the exponential function-type piston, since A_{12} , A_{23} , A_{35} and A_{46} are not always equal to zero, we can expect the action of restoring force.

4. Motion of piston

The equations of motion are given as follows:

$$\left. \begin{aligned} m \left(\frac{RQ}{V} \right)^2 \frac{d^2 \bar{X}}{d\tau^2} &= \bar{P}_s + \bar{W}_s + \bar{F}_s \\ m \left(\frac{RQ}{V} \right)^2 \frac{d^2 \bar{Y}}{d\tau^2} &= \bar{P}_v + \bar{W}_v + \bar{F}_v \end{aligned} \right\} \dots\dots\dots (19)$$

$$\left. \begin{aligned} \left(\bar{I} + \bar{a}^2 m \frac{R}{L} \right) \left(\frac{RQ}{V} \right)^2 \frac{d^2 \bar{\phi}_X}{d\tau^2} &= -\bar{M}_s - \bar{a} \frac{R}{L} \bar{P}_v - \bar{T}_s \\ \left(\bar{I} + \bar{a}^2 m \frac{R}{L} \right) \left(\frac{RQ}{V} \right)^2 \frac{d^2 \bar{\phi}_Y}{d\tau^2} &= \bar{M}_v - \bar{a} \frac{R}{L} \bar{P}_s + \bar{T}_v \end{aligned} \right\} \dots\dots\dots (20)$$

At first, let us discuss roughly about the stability of the motion. Fig.4 shows an example of the stability condition obtained from Eqs.(19) and (20) for a case of $\bar{w}_0 = -1$, where both $\bar{I}$ and $\bar{m}$ are negligible while $\bar{w}_0$ and $\bar{P}_b - \bar{P}_a$ are constant. In a case of $\bar{w}_0 = 1$, if $\lambda > 0$, the motion is stable.

When the parameters in Eqs.(19) and (20) are given by such periodic functions as expected in hydraulic pumps or motors, it may be concluded that the motion is stable if $\lambda > 0$. In this case, therefore, the magnitude of λ can be decided from the load carrying capacity as a bearing rather than the stability condition.

In the following numerical examples we assume that $\bar{P}_b - \bar{P}_a$ changes as in Fig.5. Then, $\bar{w}_0 = \cos \tau$ corresponds to the pump and $\bar{w}_0 = -\cos \tau$ the motor. And we use numerical values shown in Table 1 unless otherwise stated. In cases of piston-type pumps and motors, $\bar{I}$ and $\bar{m}$ are so small that the inertia terms in Eqs.(19) and (20) are negligible.

Fig.6 shows the effect of λ_0 on the motion for a motor. As λ_0 increases, $\bar{Y}$ decreases but $\bar{\phi}_X$ increases. $\bar{h}_{min}$ is given from Eqs.(1) and (12) as follows:

at $\bar{z} = -(L/2R)$

$$\bar{h}_{min} = 1 - \left\{ \left(\bar{X} - \frac{L}{2R} \bar{\phi}_Y \right)^2 + \left(\bar{Y} - \frac{L}{2R} \bar{\phi}_X \right)^2 \right\}^{1/2}$$

at $\bar{z} = L/2R$

$$\bar{h}_{min} = \exp(\lambda L/R) - \left\{ \left(\bar{X} + \frac{L}{2R} \bar{\phi}_Y \right)^2 + \left(\bar{Y} + \frac{L}{2R} \bar{\phi}_X \right)^2 \right\}^{1/2}$$

Then, the increase of $\bar{\phi}_X$ in the figure does not matter. When $\lambda_0 = 0.01$, $\bar{h}_{min}$ becomes negative, that is, the piston directly contacts the cylinder wall. In this case

$\bar{X}$ and $\bar{\phi}_Y$ are zero, for $\bar{w}$ and $\bar{w}_X$ are zero.

Fig.7 shows the effect of λ_0 and λ_p in a pump. Here, $\bar{P}_a = 0$ and the following relationship is assumed

$$\lambda = \lambda_0 + \lambda_p (\bar{P}_b / \bar{P}_{bmax}) \dots\dots\dots (21)$$

In case of $\lambda_p \neq 0$, both $\bar{Y}$ and $\bar{\phi}_X$ change with $\bar{P}_b$. To reduce $\bar{Y}$ and $\bar{\phi}_X$, that is, to increase the load carrying capacity, it is effective to increase λ_p .

Fig.6 Effect of λ_0 on radial displacement of piston (motor, $\lambda_p = 0$)

Fig.7 Effect of λ_0 and λ_p on radial displacement of piston (pump)

Fig.8 Effect of $\bar{P}_{bmax}$ on radial displacement of piston (pump)

Fig.8 shows the effect of $\bar{P}_{bmax}$ in the pump. As $\bar{P}_{bmax}$ increases, $\bar{\phi}_x$ has a tendency to decrease, but $\bar{Y}$ rather complicatedly changes. This is because parameters A_{25} and A_{35} depend on both $\bar{P}_b$ and $\bar{W}_0$. On the contrary, in case of the motor, the motion is almost independent of $\bar{P}_{bmax}$ within the range of $\bar{P}_{bmax}$.

Fig.9 shows the effect of projection of the piston from the cylinder-block end in case of the pump. In this example, we assume

$$z_1 = -(L/2R) + \bar{R}_t(1 - \sin \tau) \dots (22)$$

, then the projecting length is zero at the top dead center and equal to $2\bar{R}_t$ at the bottom dead center. Since at $\bar{R}_t \neq 0$ the effective length of the piston tends to reduce, the position of the fulcrum is likely to shift. In this figure, therefore, $\bar{\alpha}$ is also chosen as a parameter. $\bar{R}_t$ affects the motion remarkably and it is effective to change $\bar{\alpha}$ when $\bar{R}_t$ is large. If $\bar{R}_t=1$ and $\bar{\alpha}=0$, the piston contacts the wall.

For axial piston pumps $\bar{W}$ reaches $0.1\bar{P}_b$ and also its direction alternates periodically. And $\bar{Y}$ and $\bar{\phi}_x$ are in proportion to the ratio $\bar{W}/\bar{P}_b$ in these examples.

The effect of the change in the direction is shown in Fig.10 for the case of a swash plate type axial piston pump where

$$\bar{W}_x = k\bar{P}_b \cos \tau, \quad \bar{W}_y = -k\bar{P}_b \sin \tau$$

The constant k is decided by the swash plate angle, etc., and in this figure $k=0.01$ and $\bar{P}_b=0$ are assumed. The magnitude of $\bar{w}$ ranges $-1 \sim 0$ depending on the lubricating condition. This figure shows that the load carrying capacity does not increase even if λ_0 increases, and that the effect of $\bar{w}$ is remarkable.

Fig.9 Effect of $\bar{R}_t$ on radial displacement of piston (pump)

Fig.10 Radial displacement of piston in swash plate type piston pump

5. Leakage flow rate and frictional force

From the Navier-Stokes equations with no small terms and from the boundary conditions Eq.(2), we obtain

$$w = \frac{1}{2\mu} \frac{\partial p}{\partial z} (y-h)y + \frac{w_0}{h} y \quad \dots\dots\dots(23)$$

Therefore, if we take $z=0$ as a representative position, the leakage flow rate is given by

$$\bar{Q}_z = \frac{1}{4\pi} \int_0^{2\pi} \left(-h^3 \frac{\partial \bar{p}}{\partial z} + h \bar{w}_0 \right) d\varphi \Big|_{z=0} \quad \dots\dots\dots(24)$$

The z component of shearing stress acting on the piston is given by

$$\tau_z = \mu \frac{\partial w}{\partial y} \Big|_{y=h}$$

The resultant force, that is, the frictional force is given by

$$\bar{F}_z = -\frac{C_0}{12L} \int_{z_1}^{z_2} \int_0^{2\pi} \left(3h \frac{\partial \bar{p}}{\partial z} + \bar{w}_0 \right) d\varphi dz \quad \dots\dots\dots(25)$$

Fig.11 shows $\bar{Q}_z$. $\bar{\gamma}$ and $\bar{\phi}_x$ corresponding to curves 1 and 2 are given in Fig.7. Curve 3 corresponds to the case where $\bar{\gamma}$ and $\bar{\phi}_x$ are twenty times as large as those

Fig.11 Leakage flow rate

Fig.12 Frictional force (Numbers in figure correspond to those in Fig.11)

for curve 2. Positive $\bar{Q}_z$ means that the fluid flows into the cylinder, so in this figure the flow rate due to the reciprocating motion of the piston is larger than that due to the pressure difference, while the latter is comparable to the former, when $\bar{p}_b = 4$. The singular changes appearing on curve 1 in the vicinity of $\tau = \pi/2$ and $3\pi/2$ are related to changes of both the reciprocating velocity and the pressure. Expansion or contraction of the cylinder bore and a sudden change in $\bar{\gamma}$ and $\bar{\phi}_x$ are reflected in the curves 2 and 3. The difference between curves 2 and 3 in high pressure range comes from the piston eccentricity, and the difference between curves 1 and 2 does from the eccentricity as well as the gap width.

Now, let $\bar{Q}_{z0}$ be the leakage flow rate, when $\bar{\gamma}$, $\bar{\phi}_x$, etc. are zero, i.e. the piston and the cylinder bore are completely concentric. The ratio $\bar{Q}_z/\bar{Q}_{z0}$ for curve 3 is 0.68 at $\tau=0$ and 1.01 at $\tau=\pi$, but $\bar{Q}_z/\bar{Q}_{z0}$ for curve 2 is nearly equal to unity at each τ .

Fig. 12 shows $\bar{F}_z$ for $C_0/R = 0.001$. In this case both the pressure and the reciprocating velocity give almost the same effect on the frictional force. Let $\bar{F}_{z0}$ be the frictional force when $\bar{\gamma}$, $\bar{\phi}_x$, etc. are zero. The ratio $\bar{F}_z/\bar{F}_{z0}$ for curve 3 is 1.17 at $\tau=0$ and 1.19 at $\tau=\pi$, but for curve 2 the ratio is nearly unity at each τ .

6. Conclusions

The motion of "exponential function-type piston" is analyzed theoretically. From the numerical examples for piston-type oil hydraulic pumps and motors, the following conclusions are obtained:

- (1) The motion of a piston can be stabilized by distributing the clearance between the piston and the cylinder bore exponentially from the low pressure and to the high pressure end.
- (2) The exponential function-type piston builds up a fluid film in the clearance space.
- (3) Expansion of the cylinder bore due to the pressure results in an increase in the load carrying capacity.
- (4) Reduction in the load carrying capacity due to the piston projection is effectively prevented by the shift of the piston fulcrum.

The leakage flow rate through this clearance and the longitudinal frictional force acting on the piston are also derived.

(The numerical calculations have been done in the Computer Center, University of Tokyo.)

Appendix

$$\begin{aligned} A_{11} &= -2(RQ/V)G_1 \\ A_{12} &= -6\lambda a_1 G_1 + J_3 G_2 \\ A_{13} &= -\bar{\omega} G_1 \\ A_{14} &= 2(RQ/V)(3\lambda G_1 - G_3) \\ A_{15} &= \bar{\omega}(3\lambda G_1 - G_3) \\ A_{16} &= J_3 G_1 - J_4 G_2 - 12\lambda a_1 G_3 + J_5 G_4 \\ A_{21} &= A_{11} \end{aligned}$$

$$\begin{aligned}
A_{22} &= -A_{13} \\
A_{23} &= A_{12} \\
A_{24} &= A_{14} \\
A_{25} &= A_{16} \\
A_{26} &= -A_{15} \\
A_{31} &= 2(RQ/V)G_5 \\
A_{32} &= -\bar{\omega}G_5 \\
A_{33} &= 6\lambda a_1 G_5 + J_2 G_6 \\
A_{34} &= -2(RQ/V)(3\lambda G_5 - G_7) \\
A_{35} &= -J_2 G_5 - J_4 G_6 + 12\lambda a_1 G_7 + J_8 G_8 \\
A_{36} &= \bar{\omega}(3\lambda G_5 - G_7) \\
A_{41} &= A_{31} \\
A_{42} &= A_{33} \\
A_{43} &= -A_{32} \\
A_{44} &= A_{34} \\
A_{45} &= -A_{36} \\
A_{46} &= A_{35} \\
G_1 &= H_1 F_1 + H_2 F_3 + (1/3\lambda)N_1 \\
G_2 &= -H_1 F_3 - H_2 F_7 - (1/4\lambda)N_2 \\
G_3 &= H_1 F_2 + H_2 F_6 + H_5 \\
G_4 &= -H_1 F_4 - H_2 F_8 - H_7 \\
G_5 &= -H_3 F_1 - H_4 F_5 - H_5 \\
G_6 &= -H_3 F_3 - H_4 F_7 - H_7 \\
G_7 &= -H_3 F_2 - H_4 F_6 - H_6 \\
G_8 &= -H_3 F_4 - H_4 F_8 - H_8 \\
F_1 &= F_{22}E_{31} - F_{21}E_{32} \\
F_2 &= \bar{z}_1 F_{22}E_{31} - \bar{z}_2 F_{21}E_{32} \\
F_3 &= F_{22}E_{41} - F_{21}E_{42} \\
F_4 &= \bar{z}_1 F_{22}E_{41} - \bar{z}_2 F_{21}E_{42} \\
F_5 &= F_{11}E_{32} - F_{12}E_{31} \\
F_6 &= \bar{z}_2 F_{11}E_{32} - \bar{z}_1 F_{12}E_{31} \\
F_7 &= F_{11}E_{42} - F_{12}E_{41} \\
F_8 &= \bar{z}_2 F_{11}E_{42} - \bar{z}_1 F_{12}E_{41} \\
F_{11} &= \exp(\bar{z}_1 J_1/2) \\
F_{12} &= \exp(\bar{z}_2 J_1/2) \\
F_{21} &= \exp(-\bar{z}_1 J_2/2) \\
F_{22} &= \exp(-\bar{z}_2 J_2/2)
\end{aligned}$$

$$\begin{aligned}
F_{30} &= 1/(F_{11}F_{22} - F_{12}F_{21}) \\
F_{31} &= (F_{12} - F_{11})F_{30} \\
F_{32} &= (F_{22} - F_{21})F_{30} \\
H_1 &= 2F_{31}/J_1 \\
H_2 &= -2F_{32}/J_2 \\
H_3 &= -2F_{30}[(2 - \bar{z}_2 J_1)F_{12} - (2 - \bar{z}_1 J_1)F_{11}]/J_1^2 \\
H_4 &= -2F_{30}[(2 + \bar{z}_2 J_2)F_{22} - (2 + \bar{z}_1 J_2)F_{21}]/J_2^2 \\
H_5 &= (1/9\lambda^2)(3\lambda N_3 + N_1) \\
H_6 &= (1/27\lambda^3)(9\lambda^2 N_3 + 6\lambda N_8 + 2N_1) \\
H_7 &= (1/16\lambda^2)(4\lambda N_4 + N_2) \\
H_8 &= (1/32\lambda^3)(8\lambda^2 N_6 + 4\lambda N_4 + N_2) \\
J_1 &= -3\lambda + \sqrt{9\lambda^2 + 4} \\
J_2 &= 3\lambda + \sqrt{9\lambda^2 + 4} \\
J_3 &= 9\lambda^2 C_1/(4\lambda^2 - 1) \\
J_4 &= [9\lambda C_1/(4\lambda^2 - 1)][1 - (5\lambda^2/(4\lambda^2 - 1))] \\
J_5 &= \bar{\omega}_0 - 6\lambda a_0 + 3a_1(1 + 12\lambda^2) \\
E_{31} &= \exp[-3\lambda\{(L/2R) + \bar{z}_1\}] \\
E_{32} &= \exp[-3\lambda\{(L/2R) + \bar{z}_2\}] \\
E_{41} &= \exp[-4\lambda\{(L/2R) + \bar{z}_1\}] \\
E_{42} &= \exp[-4\lambda\{(L/2R) + \bar{z}_2\}] \\
N_1 &= E_{32} - E_{31} \\
N_2 &= E_{42} - E_{41} \\
N_3 &= \bar{z}_2 E_{32} - \bar{z}_1 E_{31} \\
N_4 &= \bar{z}_2 E_{42} - \bar{z}_1 E_{41} \\
N_5 &= \bar{z}_2^2 E_{32} - \bar{z}_1^2 E_{31} \\
N_6 &= \bar{z}_2^2 E_{42} - \bar{z}_1^2 E_{41}
\end{aligned}$$

References

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