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Theory of Inviscid Shear Flow over Bluff Bodies

Attached to a Plane Wall*

(Part 2, Application to Semielliptical Projection and Arbitrary-Shaped Body)

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As a continuation to the preceding paper in which a general method of analysis of the separated flows past bluff bodies attached to a plane wall was presented within the framework of the free-streamline theory, the present paper applies the method to the cases of semielliptical projection and bluff bodies of arbitrary shapes. The theoretical pressure distribution on the wetted surface of the semielliptical projection is found to show good agreement with an experimental measurement. An analytical solution for arbitrary bluff bodies is obtained by assuming an inverse power series for mapping functions. Numerical calculations showed that reasonable pressure distributions could be obtained for a few profiles which were arbitrarily chosen.

1. Introduction

In the preceding paper⁽¹⁾, a free-streamline theory for bluff bodies attached to a plane wall was presented by replacing a turbulent boundary layer along the wall by a hypothetical inviscid shear flow of constant vorticity. Solutions were worked out for a normal flat plate and a semicircular projection, with the result that the theoretical pressure distributions on the wetted surface of the two bodies agreed well with the experimental measurements.

In the first half of this paper, the theory is applied to the case of a semielliptical projection and a comparison with the theoretical and experimental results is given. Although the general theory presented in Part 1 is, in principle, applicable once the conformal transformation $z = f(\zeta)$ is given, the integration which is needed to obtain w_s can not be performed when the function $f(\zeta)$ happens to have a complicated form. Accordingly, it would be convenient from a practical point of view to construct analytical solutions for bluff bodies of various shapes in terms of approximate mapping functions which have not only simpler forms than the exact ones but also good accuracy. The last half of the present paper describes an attempt for this purpose.

2. Semielliptical projection

Let the four complex planes, i.e. $z = x + iy$, $z' = x' + iy'$, $\zeta = \xi + i\eta$ and $Z = X + iY$ be related by the transformations

$$z = r(z' + 2a^2 - 1) \pm \frac{1}{r(z' + 2a^2 - 1)} \quad \dots\dots\dots (1)$$

$$z' = Z - a^2 - a^2(1 - a^2)/(Z - a^2) \quad \dots\dots\dots (2)$$

$$\zeta = Z + a^2/Z \quad \dots\dots\dots (3)$$

where a is defined by

$$a = \cos \alpha \quad \dots\dots\dots (4)$$

α being real. As already shown in Part 1, the circular arc slit ASB in the z' -plane is mapped onto a segment of a line which lies on the real axis of the ζ -plane in the range $|\xi| \leq 2a$, and also onto a semicircle of radius a with its center at the origin in the Z -plane. The radius of the circular arc in the z' -plane is unity and

Fig. 1 Physical and transform planes for semielliptical projection

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its center is located at $(\Delta, 0)$, where $\Delta = 1 - 2a^2$, as shown in Fig. 1. The separation angle β'_S in the z' -plane is related to α by

$$\beta'_S = \pi - 2\alpha \quad (5)$$

Writing the equation of the circular arc slit in the z' -plane as

$$z' + 2a^2 - 1 = e^{i\theta'} \quad (5.a)$$

and substituting Eq. (5.a) into Eq. (1), one obtains

$$x = (r \pm 1/r) \cos \theta' \quad (6)$$

$$y = (r \mp 1/r) \sin \theta' \quad (7)$$

in which $\pi - \beta'_S \leq \theta \leq \pi$ or

$$2\alpha \leq \theta' \leq \pi \quad (8)$$

As demonstrated in Fig. 1, Eqs. (6) and (7) describe an elliptical arc slit ASB which represents the wetted surface of the semi-elliptical projection in the z -plane with the major axis $r + (1/r)$ and the minor axis $r - (1/r)$. If the upper signs of Eqs. (6) and (7) are employed, the major axis will be on the x -axis and the minor axis on the y -axis, and vice versa. Writing the eccentricity of ellipse as E , one has

$$r = E^{-1} \{1 + (1 - E^2)^{1/2}\} \quad (9)$$

Equations (1), (2) and (3) yield

$$z \sim r\zeta + O(\zeta^{-1})$$

as $|\zeta| \rightarrow \infty$. On comparing this relation with Eq. (I.6)*, one obtains $k_1 = r$. Therefore, from Eq. (I.8), W_u is determined as

$$W_u = U_0 r \zeta = U_0 r (Z + a^2/Z) \quad (10)$$

The complex flow potential W_s will now be obtained. Substituting the expression $Z = ae^{i\theta}$ ($0 \leq \theta \leq \pi$) into Eqs. (1) and (2) and rearranging the terms, one gets

$$y = 2a \left(r \mp \frac{1}{r} \right) \frac{\sin \theta (1 - a \cos \theta)}{1 - 2a \cos \theta + a^2}$$

Substitution of the relation $\cos \theta = \xi/(2a)$ into the above equation yields

$$y^2(\xi, 0) = \frac{1}{4} \left(r \mp \frac{1}{r} \right)^2 \frac{(\xi - 2)^2 (4a^2 - \xi^2)}{(\xi - a^2 - 1)^2}$$

Since the factors of this equation which contain the variable ξ are the same as the right-hand side of Eq. (I.41), W_s is easily determined as

$$W_s = \frac{K}{8\pi} \left(r \mp \frac{1}{r} \right)^2 \left\{ 4aJ_1 - J_2 \log \frac{\zeta - 2a}{\zeta + 2a} + J_3 \log \left(\frac{1-a}{1+a} \right)^2 + J_4 \frac{4a}{(a^2 - 1)^2} \right\} \quad (11)$$

where $J_1 \sim J_4$ are the functions of ζ and their forms are given in Part 1. When the same quantities as in Part 1 appear in what follows, they will have the same meaning as in Part 1, unless otherwise stated.

* When the equations in Part 1 are referred to in this paper, they will be distinguished by the letter "I".

The expressions for W_{Qu} and W_{Qs} are the same as Eq. (I.25).

The separation point S corresponds to

$$Z_s = ae^{i\alpha}, \quad \zeta_s = 2a^2, \quad z' = 2ai \sin \alpha$$

in the transform planes, respectively. Since

$$z' + 2a^2 - 1 = e^{2i\alpha}$$

at the point S , the variable θ' in Eq. (5.a) takes the value of $\theta' = \theta'_S = 2\alpha$ at the same point. Substituting $\theta'_S = 2\alpha$ into Eqs. (6) and (7), one obtains the coordinate (x_S, y_S) of the separation point in the physical z -plane as

$$x_S = (r \pm 1/r)(2a^2 - 1), \quad y_S = (r \mp 1/r)2a \sin \alpha$$

Moreover, the separation angles in the z - and z' -planes are equal to each other, i.e.

$$\beta_S = \beta'_S$$

The complex velocity in the z -plane is given by

$$\begin{aligned} \bar{w} &= Ky + \frac{d}{dz} \{ (W_u + W_{Qu}) + (W_s + W_{Qs}) \} \\ &= Ky + \left(\frac{dW_u}{d\zeta} + \frac{dW_{Qu}}{d\zeta} \right) \frac{d\zeta}{dZ} \frac{dZ}{dx'} \frac{dx'}{dz} \\ &\quad + \left(\frac{dW_s}{d\zeta} + \frac{dW_{Qs}}{d\zeta} \right) \frac{d\zeta}{dZ} \frac{dZ}{dx'} \frac{dx'}{dz} \quad (12) \end{aligned}$$

Although $d\zeta/dZ$ and dz'/dz take finite non-zero values at the point S , the derivative dZ/dz' , i.e.

$$\frac{dZ}{dz'} = \frac{Z - a^2}{Z^2 - 2a^2Z + a^2}$$

becomes infinitely large at the same point, since $Z_S = ae^{i\alpha}$. Therefore, the application of Eq. (I.15) yields

$$Q_u = -2\pi ar U_0 (a - \cosh \beta) \quad (13)$$

$$Q_s = -K(r \mp 1/r)^2 a (a - \cosh \beta)$$

$$\times \{6a + (3a^2 - 1) \log(|1-a|/|1+a|)\} \quad (14)$$

Substituting Eqs. (13) and (14), together with Eqs. (10), (11) and (I.25), into Eq. (12) and removing the factor $Z^2 - 2a^2Z + a^2$ which becomes zero at the point S from the denominator, one obtains

Fig. 2 Surface pressure distribution for semielliptical projection. h = height of ellipse, δ = boundary-layer thickness, u_τ = friction velocity, U_∞ = velocity outside boundary layer, ν = kinematic viscosity, β_m = point of maximum pressure.

$$\bar{w} = Ky + U_0 r \frac{dz'}{dz} \frac{(Z^2 - a^2)(Z - a^2)^2}{Z^2(Z^2 - 2aZ \cosh \beta + a^2)} + \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 \frac{dz'}{dz} \frac{Z^2 - a^2}{Z^2(Z - 1)^2} \\ \times \left\{ \frac{a(L_1 + 2aL_2 \cosh \beta)}{Z^2 - 2aZ \cosh \beta + a^2} + \frac{1}{4} \log \left(\frac{Z - a}{Z + a} \right)^2 \frac{L_3(Z^2 - 2Z + a^2)}{Z(Z - 1)(Z - a^2)} \right. \\ \left. + \frac{1}{2} \log \left(\frac{1 - a}{1 + a} \right) \frac{Z(L_4 + 2aL_5 \cosh \beta)}{(Z - 1)(Z - a^2)(Z^2 - 2aZ \cosh \beta + a^2)} \right\} \quad (15)$$

The complex velocity on the wetted surface is obtained by substituting $Z = ae^{i\theta}$ ($\alpha \leq \theta \leq \pi$) into Eq. (15). The result is

$$\bar{w}_f = K \left(r \mp \frac{1}{r} \right) \frac{2a \sin \Theta (1 - a \cos \Theta)}{1 - 2a \cos \Theta + a^2} + U_0 r \left(\frac{dz'}{dz} \right)_f (M_1 + iM_2)N \\ + \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 \left(\frac{dz'}{dz} \right)_f \left\{ \left(\frac{M_1 + iM_2}{M_3} \right) a(L_1^* + 2aL_2^* \cosh \beta)N + \left(\frac{M_4 + iM_5}{M_3^4} \right) \right. \\ \left. \times \left(\log \tan^2 \frac{\Theta}{2} + i\pi \right) 2aL_3^* \sin \Theta (a \cos \Theta - 1) + \frac{1}{2} \log \left(\frac{1 - a}{1 + a} \right) \left(\frac{M_4 + iM_5}{M_3^4} \right) (L_4^* + 2aL_5^* \cosh \beta)N \right\} \quad (16)$$

in which $(dz'/dz)_f$ means the value of dz'/dz for $Z = ae^{i\theta}$ and is expressed as

$$(dz'/dz)_f = r \mp (1/r) e^{-2i\theta} \quad (17)$$

The complex velocity at the separation point S is calculated by putting $\theta = \alpha$ into Eq. (16). The result can be written as

$$\bar{w}_S = ie^{-2i\alpha} (dz'/dz)_S R_S \quad (18)$$

where

$$(dz'/dz)_S = r \mp (1/r) e^{-4i\alpha}$$

and

$$R_S = U_0 r (a^2 - 1) \sin \alpha / (a - \cosh \beta) + \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 \left\{ 3a(a^2 - 1)(1 - 2a \cosh \beta + a^2) \frac{\sin \alpha}{a - \cosh \beta} \right. \\ \left. + 2a(2a^2 - 1) \log \left(\frac{1 - a}{1 + a} \right) + \left(\frac{r^2 \pm 1}{r^2 \mp 1} \right) 4\pi a^2 (1 - a^2) - \frac{1}{2} \log \left(\frac{1 - a}{1 + a} \right) \right. \\ \left. \times [11a^4 - 2a^2 - 1 - 2a(7a^2 - 3) \cosh \beta] \left(\frac{\sin \alpha}{a - \cosh \beta} \right) \right\}$$

Accordingly, the magnitude of velocity at the separation point is given by

$$q_S = |(dz'/dz)_S| R_S \quad (19)$$

where, from Eq. (1.18), q_S should be

$$q_S = (U_0^2 - c_{pb})^{1/2} \quad (20)$$

The constants U_0 , K , β and α can be determined by the method described in detail in Part 1. Since U_0 and α are easily obtained, the equations for K and β will be given here. From the condition of Eq. (1.18), one obtains

$$\cosh \beta = \frac{\pm a q_S |(dz'/dz)_S| - U_0 r A_1 - \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 (A_2 + a A_3^* - A_4)}{\pm q_S |(dz'/dz)_S| + \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 (B_2 - A_3^* + B_4)} \quad (21)$$

where the sign $\pm$ corresponds to $R_S \gtrless 0$, and

$$A_3^* = 2a(2a^2 - 1) \sin \alpha \log \left(\frac{1 - a}{1 + a} \right) + \left(\frac{r^2 \pm 1}{r^2 \mp 1} \right) 4\pi a^2 (1 - a^2)$$

Moreover, the condition of Eq. (1.20) yields

$$\cosh \beta = - \frac{U_0 r m_1 + \frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 (m_2 + m_3^* \cos \Theta_m + m_4)}{\frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 (n_2 - m_3^* + n_4)} \quad (22)$$

where

$$m_3^* = m_3 + \Re \left\{ \frac{\pi r}{r^2 \mp 1} \left(\frac{dz}{dz'} \right)_m \frac{2a \sin \Theta_m (1 - a \cos \Theta_m)}{1 - 2a \cos \Theta_m + a^2} \right\} \quad (23)$$

Eliminating β between Eqs. (21) and (22), one obtains an expression for K as follows:

$$\frac{K}{\pi} \left(r \mp \frac{1}{r} \right)^2 = \frac{1}{2b_1} \{ b_2 \pm (b_2^2 + 4b_1b_3)^{1/2} \} \dots (24)$$

where

$$\begin{aligned} b_1 &= l_2^* - k_2 l_4^*, & b_2 &= l_1^* + r \tilde{k}_1 l_4^* + k_2 l_3^* \\ b_3 &= r \tilde{k}_1 l_3^*, & l_1^* &= \pm a q_s |(dz/dz')_s| - U_0 r A_1 \\ l_2^* &= A_2 + a A_3^* - A_4, & l_3^* &= \pm a q_s |(dz/dz')_s| \\ l_4^* &= B_2 - A_3^* - B_4 \end{aligned}$$

The proper sign of the right-hand side of Eq. (24) should be chosen such that the value of K becomes positive.

If the constants thus determined are substituted into Eq. (16), one obtains the velocity components, i.e. u_f and v_f , on the wetted surface of the semielliptical projection and hence the surface pressure coefficient as

$$c_{p,f} = U_0^2 - u_f^2 - v_f^2 \dots (25)$$

Figure 2 compares the theoretical pressure distribution for the semielliptical projection of the eccentricity $E = \sqrt{3}/2$ with the experiment performed by

Funahashi⁽²⁾. In this case, the minus sign on the right-hand side of Eq. (24) has been employed. The present theory is seen to show good agreement with the experimental measurement.

3. Application to arbitrary-shaped body

In view of the foregoing applications, the free-streamline theory developed in Part 1 has been shown to be effective in predicting the surface pressure loading for typical shapes of the two-dimensional bluff bodies in contact with the plane wall. Accordingly, it may naturally be conjectured that the present theory can be applied to bluff bodies of a variety of shapes. In this chapter, an attempt is made to construct an analytical solution for arbitrary-shaped bluff bodies in terms of an approximate mapping function.

3.1 Mapping Functions

The transform planes Z and ζ are the same as those introduced in Chapter 2. The problem to transform the arc slit ASB in the z -plane, where S is the separation point and AS is the wetted surface, into a semicircle in the Z -plane corresponds to a special case of zero thickness in the two-dimensional theory of arbitrary wing sections. Let the conformal transformation which maps the arc slit into the semicircle of radius a in the Z -plane be expressed as

$$z = A_{-1}Z + A_0 + \sum_{n=1}^{\infty} A_n Z^{-n} \dots (26)$$

where the constants $A_{-1}, A_0, A_n (n = 1, 2, \dots)$ depend on the particular shape of the slit ASB . These constants must be real to make it possible to regard the real axis on the z -plane as a solid wall.

The method developed by Moriya⁽³⁾ will be used in this analysis to determine the constants A_n in terms of the given shape of the slit ASB . As the zeroth order approximation to $f(Z)$, one employs

$$z = Z - a^2/Z \dots (27)$$

Accordingly, one has

$$A_{-1}^{(0)} = 1, \quad A_0^{(0)} = 0, \quad A_1^{(0)} = -a^2$$

$$A_n^{(0)} = 0 \quad (n = 2, 3, \dots)$$

From Eq. (27), the shape of the slit in the zeroth order approximation is expressed by

$$x_f^{(0)} = 0, \quad y_f^{(0)} = 2a \cos \theta$$

which is the normal plate of height $2a$.

When the exact x -coordinate of the slit which corresponds to $y_f^{(0)}$ is again written as $x_f^{(0)}, x_f^{(0)}$ can be regarded as a parametric function of θ . Therefore, $x_f^{(0)}$ can be expanded into a Fourier series of the form

$$x_f^{(0)} = \kappa_0^{(0)} + \sum_{n=1}^{\infty} \kappa_n^{(0)} \cos n\theta$$

where

$$\kappa_0^{(0)} = \frac{1}{\pi} \int_0^\pi x_f^{(0)} d\theta$$

$$\kappa_n^{(0)} = \frac{2}{\pi} \int_0^\pi x_f^{(0)} \cos n\theta d\theta \quad (n = 1, 2, \dots)$$

In terms of $\kappa_n^{(0)}$, the mapping function $f(Z)$ in the first approximation becomes

$$z = A_{-1}^{(1)}Z + A_0^{(1)} + \sum_{n=1}^{\infty} A_n^{(1)}Z^{-n} \dots (28)$$

in which

$$A_{-1}^{(1)} = 1 + \left(-\frac{1}{2a} \right) \sum_{n=1}^{\infty} n \kappa_n^{(0)}, \quad A_0^{(1)} = \kappa_0^{(0)}$$

$$A_1^{(1)} = a \kappa_1^{(0)} - a^2 - \left(\frac{a}{2} \right) \sum_{n=1}^{\infty} n \kappa_n^{(0)}$$

$$A_n^{(1)} = a^2 \kappa_n^{(0)} \quad (n = 2, 3, \dots)$$

In general, if the coefficients of the p -th order approximation is denoted by $A_n^{(p)}$, the $(p+1)$ -th approximate coefficients $A_n^{(p+1)}$

can be calculated from

$$\left. \begin{aligned} A_{-1}^{(p+1)} &= 1 + \left(-\frac{1}{2a} \right) \sum_{n=1}^{\infty} n \kappa_n^{(p)} \\ A_0^{(p+1)} &= \kappa_0^{(p)} \\ A_1^{(p+1)} &= a \kappa_1^{(p)} - a^2 - \left(\frac{a}{2} \right) \sum_{n=1}^{\infty} n \kappa_n^{(p)} \\ A_n^{(p+1)} &= a^2 \kappa_n^{(p)} \quad (n = 2, 3, \dots) \end{aligned} \right\} \dots (29)$$

This process of approximation will be repeated until the approximate shape of the

slit approaches the exact one within a desired accuracy. When the shape of the slit is largely different from the normal plate, the $(p+1)$ -th approximation to the coefficients should be calculated from

$$A_n^{(p+1)} = A_n^{(p)} + \varepsilon(A_n^{*(p+1)} - A_n^{(p)})$$

where $A_n^{*(p+1)}$ means the coefficients given by Eq. (29) and ε is a parameter for convergence in the range $0 \leq \varepsilon \leq 1$. The constant α can be put equal to unity without loss of generality. If another value is chosen for α , the slit ASB will change only in scale and its shape remains always similar. For later reference, it is assumed that the mapping function thus determined locates the separation point S at $\theta = \delta$, the ordinate of the separation point being $y = y_S$.

Figure 4 shows the mapping function for the arc slit defined by $y/y_S = (1 + \lambda x/y_S)^{1/m}$, with $m = 4$ and $\lambda = 2$, which was obtained by the present method by retaining the first fifteen terms of Eq. (26). Moreover, Table 1 summarizes the values of coefficients for this case.

3.2 Determination of W_u and W_S

Since substitution of Eq. (3) into Eq. (26) yields

$$z \sim A_{-1}\zeta + A_0 + O(\zeta^{-1})$$

as $|\zeta| \rightarrow \infty$, one has $k_1 = A_{-1}$. Hence, the complex flow potential W_u is determined as

$$W_* = U_0 A_{-1} \zeta = U_0 A_{-1} (Z + a^2/Z) \quad (30)$$

The expressions for W_{Q_u} and W_{Q_S} are the same as those given by Eq. (I.25).

For the purpose of obtaining W_S , one substitutes $Z = ae^{i\theta}$ ($0 \leq \theta \leq \pi$) into Eq. (26) to obtain the coordinate of the arc slit as

Fig. 3 Physical and transform planes for arbitrary bluff bodies

$$x = A_{-1}a \cos \theta + A_0 + \sum_{n=1}^{\infty} A_n a^{-n} \cos n\theta$$

$$y = A_{-1}a \sin \theta - \sum_{n=1}^{\infty} A_n a^{-n} \sin n\theta$$

where y corresponds to $y(\xi, 0)$ in Eq. (I.9). Writing $y(\xi, 0)$ in the form

$$y(\xi, 0) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

where

$$B_1 = A_{-1}a - A_1/a, \quad B_n = -A_n a^n \quad (n=2, 3, \dots)$$

one gets

$$\begin{aligned} y^2(\xi, 0) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n B_m \sin n\theta \sin m\theta \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n B_m [\cos(n+m)\theta \\ &\quad - \cos|n-m|\theta] \dots \dots \dots (31) \end{aligned}$$

For an arbitrary positive integer N , $\cos N\theta$ can be expanded as

$$\cos N\theta = \sum_{r=0}^{[N/2]} \frac{(-1)^r N}{2(N-r)} \binom{N-r}{r} (2 \cos \theta)^{N-2r}$$

in which $[]$ stands for a maximum integer not exceeding $N/2$. Substituting this formula into Eq. (31) and the resulting equation into Eq. (I.9), one gets

$$\begin{aligned} W_s &= -\frac{K}{4\pi} \left\{ \sum_{n,m,r} S_{n+m-2r}(\zeta) \right. \\ &\quad \left. - \sum'_{n,m,r} S_{|n-m|-2r}(\zeta) \right\} \dots \dots \dots (32) \end{aligned}$$

n	A_n	n	A_n
-1	0.751 246	8	-0.177 043 $\times 10^{-3}$
0	-0.179 809	9	-0.632 261 $\times 10^{-4}$
1	-0.707 049	10	-0.319 250 $\times 10^{-4}$
2	-0.171 850	11	-0.293 297 $\times 10^{-5}$
3	-0.399 429 $\times 10^{-1}$	12	-0.118 489 $\times 10^{-4}$
4	-0.115 324 $\times 10^{-1}$	13	0.671 748 $\times 10^{-5}$
5	-0.372 900 $\times 10^{-2}$	14	-0.934 647 $\times 10^{-5}$
6	-0.128 499 $\times 10^{-2}$	15	0.784 332 $\times 10^{-5}$
7	-0.466 805 $\times 10^{-3}$		

Table 1 Coefficients of mapping function $[y/y_S = (1 + \lambda x/y_S)^{1/m}, \lambda = 4, m = 2 \text{ and } a = 1]$

Fig. 4 Example of mapping function

$$[y/y_S = (1 + \lambda x/y_S)^{1/m}, \lambda = 4, m = 2 \text{ and } a = 1]$$

where

$$\begin{aligned}\Sigma_{nmr} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{[(n+m)/2]} B_n B_m \frac{(-1)^r (n+m)}{2(n+m-r)} \\ &\times \binom{n+m-r}{r} a^{2r-(n+m)} \\ \Sigma'_{nmr} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n B_m \frac{(-1)^r |n-m|}{2(|n-m|-r)} \\ &\times \binom{|n-m|-r}{r} a^{2r-|n-m|}\end{aligned}$$

and

$$\begin{aligned}S_N(\zeta) &= \int_{-2a}^{2a} \frac{(\xi^*)^N}{\zeta - \xi^*} d\xi^* \\ &= \int_{2a-\zeta}^{-(2a+\zeta)} (t+\zeta)^N \frac{dt}{t} \dots\dots\dots(33)\end{aligned}$$

The integration of Eq. (33) can be performed by expanding $(t+\zeta)^N$ into the binomial series, the result being

$$\begin{aligned}S_N(\zeta) &= \sum_{k=0}^{N-1} \binom{N}{k} \frac{\zeta^k}{N-k} \{(-1)^{N-k} (2a+\zeta)^{N-k} \\ &- (2a-\zeta)^{N-k}\} + \zeta^N \log \frac{\zeta+2a}{\zeta-2a} \dots\dots\dots(34)\end{aligned}$$

3.3 Determination of flow field

The separation point S in the physical plane corresponds to

$$Z_S = ae^{i\delta}, \quad \zeta_S = 2a \cos \delta$$

in the Z - and ζ -planes, respectively. Since the derivative

$$dz/dZ = f'(Z) \dots\dots\dots(35)$$

will have a simple zero at $Z = Z_S$, application of the condition (I.15) yields

$$Q_s = -2\pi a U_0 A_{-1} (\cos \delta - \cosh \beta) \dots\dots\dots(36)$$

$$\begin{aligned}Q_s &= \frac{Ka}{2} (\cos \delta - \cosh \beta) \{ \Sigma_{nmr} S'_{n+m-2r}(\zeta_S) \\ &- \Sigma'_{nmr} S'_{|n-m|-2r}(\zeta_S) \} \dots\dots\dots(37)\end{aligned}$$

in which S' means $ds/d\zeta$.

The complex velocity in the physical plane is given by

$$\bar{w} = (u_1 - iv_1) + (u_2 - iv_2) + Ky \dots\dots\dots(37.a)$$

where

$$\begin{aligned}u_1 - iv_1 &= \left(\frac{dW_u}{d\zeta} + \frac{dW_{Qu}}{d\zeta} \right) \frac{d\zeta}{dZ} \frac{dZ}{dz} \\ u_2 - iv_2 &= \left(\frac{dW_s}{d\zeta} + \frac{dW_{Qs}}{d\zeta} \right) \frac{d\zeta}{dZ} \frac{dZ}{dz}\end{aligned}$$

Substituting the expressions for W_u , W_{Qu} , W_s and W_{Qs} into the right-hand sides of these equations, one obtains

$$\begin{aligned}u_1 - iv_1 &= U_0 A_{-1} (Z - Z_S) \left(1 - \frac{a^2}{ZZ_S} \right) \left(1 - \frac{a^2}{Z^2} \right) \\ &\times \{ (\zeta - 2a \cosh \beta) f'(Z) \}^{-1} \dots\dots\dots(38) \\ u_2 - iv_2 &= -\frac{K}{\pi} \left(1 - \frac{a^2}{Z^2} \right) \{ (\zeta - 2a \cosh \beta) f'(Z) \}^{-1} \\ &\times \{ \Sigma_{nmr} T_{n+m-2r}(\zeta) - \Sigma'_{nmr} T_{|n-m|-2r}(\zeta) \} \\ &\dots\dots\dots(39)\end{aligned}$$

Here, $T_N(\zeta)$ has been defined as

$$\begin{aligned}T_N(\zeta) &= \zeta S_N'(\zeta) - \zeta_S S_N'(\zeta_S) \\ &- 2a \cosh \beta \{ S_N'(\zeta) - S_N'(\zeta_S) \} \dots\dots\dots(40)\end{aligned}$$

Since $f'(Z_S) = 0$, the velocity components at the separation point cannot directly be calculated from Eqs. (38) and (39). It may be noted that, in the neighbourhood of the separation point, $f'(\zeta)$ and $T_N(\zeta)$ are expanded as

$$\begin{aligned}f'(Z) &= f'(Z_S) + (Z - Z_S) f''(Z_S) + \dots\dots \\ T_N(\zeta) &= T_N(\zeta_S) + (\zeta - \zeta_S) T_N'(\zeta_S) + \dots\dots\end{aligned}$$

Hence, substitution of these expressions into Eqs. (38) and (39) yields

$$(u_1 - iv_1)_S = \frac{U_0 A_{-1} (1 - a^2/Z_S^2)^2}{2a (\cos \delta - \cosh \beta) f''(Z_S)} \dots\dots(41)$$

$$\begin{aligned}(u_2 - iv_2)_S &= -\frac{K}{4\pi} \frac{(1 - a^2/Z_S^2)^2}{2a (\cos \delta - \cosh \beta) f''(Z_S)} \\ &\times \{ \Sigma_{nmr} T'_{n+m-2r}(\zeta_S) - \Sigma'_{nmr} T'_{|n-m|-2r}(\zeta_S) \} \\ &\dots\dots\dots(42)\end{aligned}$$

3.4 Determination of constants

The constants U_0 and δ can be obtained from Eqs. (I.19) and (26), respectively.

Substituting Eq. (37.a) together with Eqs. (41) and (42) into Eq. (I.18), one gets

$$\begin{aligned}\cos \delta - \cosh \beta &= \{ C_2 \pm [C_2^2 + C_1(U_0^2 \\ &- c_{ps} - C_3)]^{1/2} \} (U_0^2 - c_{ps} - C_3)^{-1} \dots\dots\dots(43)\end{aligned}$$

where

$$\begin{aligned}C_1 &= \frac{1}{4a^2} \left\{ \left(U_0 A_{-1} R_u - \frac{K}{4\pi} R_{u1} \right)^2 \right. \\ &\left. + \left(U_0 A_{-1} I_u - \frac{K}{4\pi} I_{u1} \right)^2 \right\} \\ C_2 &= \frac{1}{2a} \left\{ \left(Ky_S - \frac{K}{4\pi} R_{s2} \right) \left(U_0 A_{-1} R_u - \frac{K}{4\pi} R_{u1} \right) \right. \\ &\left. - \frac{K}{4\pi} I_{s2} \left(U_0 A_{-1} I_u - \frac{K}{4\pi} I_{u1} \right) \right\} \\ C_3 &= \left(Ky_S - \frac{K}{4\pi} R_{s2} \right)^2 + \left(\frac{K}{4\pi} I_{s2} \right)^2 \\ R_u &= \Re(H_u), \quad I_u = \Im(H_u) \\ H_u &= (1 - a^2/Z_S^2) / f''(Z_S) \\ R_{u1} &= \Re(H_{u1}), \quad I_{u1} = \Im(H_{u1}) \\ H_{u1} &= (1 - a^2/Z_S^2) \{ f''(Z_S) \}^{-1} \\ &\times \{ \Sigma_{nmr} S'_{n+m-2r}(\zeta_S) - \Sigma'_{nmr} S'_{|n-m|-2r}(\zeta_S) \} \\ R_{s2} &= \Re(H_{s2}), \quad I_{s2} = \Im(H_{s2}) \\ H_{s2} &= (1 - a^2/Z_S^2) \{ f''(Z_S) \}^{-1} \\ &\times \{ \Sigma_{nmr} S''_{n+m-2r}(\zeta_S) - \Sigma'_{nmr} S''_{|n-m|-2r}(\zeta_S) \}\end{aligned}$$

and $\Re$ and $\Im$ imply the real and imaginary parts, respectively. Moreover, substitution of Eqs. (37.a), (38) and (39) into Eqs. (I.20) which describes the condition at the location of a maximum pressure yields

$$\cosh \beta = \frac{\Re(E_1 \zeta_{\max} + E_2 + E_3 \zeta_S)}{2a \Re(E_1 + E_3)} \dots\dots\dots(44)$$

in which

Fig. 5 Shapes of wetted surfaces

$$E_1 = Ky_{\max} - \frac{K}{4\pi} \left(\frac{d\zeta}{dz} \right)_{\max} \times \{ \sum_{n,m} S'_{n+m-2r}(\zeta_{\max}) - \sum'_{n,m} S'_{|n-m|-2r}(\zeta_{\max}) \}$$

$$E_2 = U_0 A_{-1} (\zeta_{\max} - \zeta_s) \left(\frac{d\zeta}{dz} \right)_{\max}$$

$$E_3 = \frac{K}{4\pi} \left(\frac{d\zeta}{dz} \right)_{\max} \{ \sum_{n,m} S'_{n+m-2r}(\zeta_s) - \sum'_{n,m} S'_{|n-m|-2r}(\zeta_s) \}$$

Equations (43) and (44) can simultaneously be solved to obtain the two constants β and K .

For some sort of bluff-body shapes, it happens that the second derivative $f''(Z_S)$ which is contained in Eq. (43) has only poor accuracy when it is directly calculated by differentiating Eq. (26) twice with respect to Z . In such a case, an approximate procedure, which is just to be described below, may be useful. Generally, $f'(Z)$ and $f''(Z)$ can be written as

$$f'(Z) = \frac{\partial x}{\partial X} + i \frac{\partial y}{\partial X}, \quad f''(Z) = \frac{\partial^2 x}{\partial X^2} + i \frac{\partial^2 y}{\partial X^2}$$

Since $Z = Re^{i\theta}$, the differential operator $\partial/\partial X$ can be transformed into

$$\frac{\partial}{\partial X} = -\frac{\sin \theta}{R} \frac{\partial}{\partial \theta} + \cos \theta \frac{\partial}{\partial R} \quad \dots\dots\dots (45)$$

In view of the Cauchy-Riemann relation

$$\frac{\partial x}{\partial R} = \frac{1}{R} \frac{\partial y}{\partial \theta}, \quad \frac{1}{R} \frac{\partial x}{\partial \theta} = -\frac{\partial y}{\partial R}$$

Eq. (45) gives

$$\begin{aligned} \frac{\partial x}{\partial X} &= \frac{1}{R} \left(-\sin \theta \frac{\partial x}{\partial \theta} + \cos \theta \frac{\partial y}{\partial \theta} \right) \\ \frac{\partial y}{\partial X} &= -\frac{1}{R} \left(\sin \theta \frac{\partial y}{\partial \theta} + \cos \theta \frac{\partial x}{\partial \theta} \right) \end{aligned}$$

Fig. 6 Surface pressure distributions ($y_{\max}/y_s = 0.5$, $c_{pf\max} = 1.0$, $c_{pb} = -1.18$)

Similarly, one obtains

$$\frac{\partial^2 x}{\partial X^2} = -\frac{1}{R^2} \left(\cos 2\theta \frac{\partial^2 x}{\partial \theta^2} + \sin 2\theta \frac{\partial^2 y}{\partial \theta^2} + \cos^2 \theta \frac{\partial^2 y}{\partial \theta^2} \right) \dots\dots\dots (46)$$

$$\frac{\partial^2 y}{\partial X^2} = \frac{1}{R^2} \left(-\cos 2\theta \frac{\partial^2 y}{\partial \theta^2} + \sin 2\theta \frac{\partial^2 x}{\partial \theta^2} + \cos^2 \theta \frac{\partial^2 x}{\partial \theta^2} \right) \dots\dots\dots (47)$$

Since, in view of Eq. (26), the coordinates (x_f, y_f) of the arc slit ASB are described as parametric functions of θ , i.e. $\{x_f(\theta), y_f(\theta)\}$, they can be approximated in the vicinity of $\theta = \delta$ by polynomials of the N -th degree

$$x_f = \sum_{n=0}^N A_n \theta^n, \quad y_f = \sum_{n=0}^N B_n \theta^n \quad \dots\dots\dots (48)$$

in which the coefficients A_n and B_n will be determined by the method of least squares. Succeeding differentiation of Eq. (48) yields approximate values of $\partial x/\partial \theta$, $\partial y/\partial \theta$, $\partial^2 x/\partial \theta^2$, $\partial^2 y/\partial \theta^2$ at $\theta = \delta$, which may be used to evaluate $f''(Z_S)$ by means of Eqs. (46) and (47). This approximate procedure has successfully been used to evaluate the constants K and β , even when the value of $f''(Z_S)$ obtained from Eq. (26) failed to determine these two constants.

3.5 Examples of application

The theory described above was applied to a semicircular projection with $\beta_S = 40^\circ$ and a wetted surface of the form $y/y_s = (1 + \lambda x/y_s)^{1/m}$. The shapes of the wetted surfaces are shown in Fig. 5. The numerical calculation was performed by taking $a = 1$ and retaining the first fifteen terms of the mapping function (26). The surface pressure distributions which were obtained by arbitrarily choosing the parameters as $y_{\max}/y_s = 0.5$, $c_{pf\max} = 1.0$, $c_{pb} = -1.18$ are shown in Fig. 6. The pressure distribution for the semicircular projection agrees almost completely with that calculated from the solution of Part 1.

Although the present method would not necessarily be applicable to all shapes of bluff bodies, there will be a considerable number of bluff-body shapes which could be treated by the method.

4. Conclusions

In this paper, the free-streamline theory which describes the separated flows past two-dimensional bluff bodies attached to a plane wall has been applied to the semielliptical projection and bluff bodies of arbitrary shapes. The theoretical pressure distribution along the surface of the semielliptical projection was found to show good agreement with an experimental measurement. The solution for arbitrary shapes was applied to the wetted surface of the

form $y/y_S = (1 + \lambda x/y_S)^{1/m}$, where y_S , λ and m are constants, and the semicircular pro-

jection with the separation point being located at 40° from the forward stagnation point. Physically reasonable pressure distributions were calculated for these shapes by the present solution.

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