Critical Velocity of a Load Moving on a Beam
Supported by an Elastic Stratum

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The present paper investigates the characteristics of the critical velocity of a load moving on a beam supported by an elastic stratum. In the analysis it is assumed that the stratum is a continuum which is homogeneous, isotropic, and linearly elastic. Both welded and smooth contacts between beam and foundation are considered. The influence of the density and the thickness of a foundation on the critical velocities is studied. The results are compared with those obtained from the Winkler assumption. It is found that the critical velocities for smooth contact are not so different from those for welded contact. The critical velocities vary considerably with the values of the density and the thickness of the foundation. The velocity by the Winkler assumption and the one by the continuum theory are relatively close to each other when the foundation is a light and thin layer though they are generally different from each other.

1. Introduction

In several fields of engineering it has become of increasing interest to study the responses of beams and plates supported by flexible foundations to moving loads, and many papers have been published on the problem during the past years.\(^1\)\(^\text{11}\) It has been shown in several papers that the amplitudes of the deflection, the rotation and the bending moment in a beam or a plate become unbounded when the load velocity approaches a certain value, generally referred to as the critical velocity. The critical velocity has generally been obtained by assuming that the supporting media are represented by a Winkler foundation, i.e., a set of massless springs.\(^12\)-\(^\text{15}\) The Winkler foundation, however, has obvious deficiencies, because the restoring force from the foundation depends only on the local deflection of the mounting body, and because inertia effects in the foundation cannot be properly considered. Recently it has been shown that the responses of a beam or a plate on a real foundation such as an elastic continuum,\(^\text{15}\)-\(^\text{17}\),\(^\text{19}\) a set of independent elastic rods,\(^\text{15}\)-\(^\text{17}\) or a compressible liquid,\(^\text{15}\)-\(^\text{17}\),\(^\text{19}\) have many discrepancies compared to those of a beam or a plate on a Winkler foundation.

The present paper investigates the characteristics of the critical velocity of a load moving on a beam supported by an elastic stratum. In the analysis it is assumed that the stratum is a continuum and is homogeneous, isotropic, and linearly elastic. Both welded and smooth contacts between beam and foundation are considered.

The influence of the density and the thickness of the foundation on the critical velocity is studied. The velocity is compared with the one by the Winkler assumption.

2. Governing equations and stress components

A beam on an elastic stratum is shown in Fig. 1 in which the \(x\)-axis is taken along the interface between the stratum and the semi-infinite rigid base and the \(y\)-axis normal to the interface. We denote the displacements from the positions of equilibrium by \(u\) and \(v\) in the \(x\)- and the \(y\)-directions, respectively. The various quantities, in general, will be different between the beam and the stratum and will be distinguished from each other by the subscripts 1 and 2. In vibration the beam is subjected to an applied normal force \(p(x,t)\), a transverse restoring force \(q(x,t)\) from the foundation per unit length of the beam. For the displacement of the

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(a) Beam on an elastic stratum

(b) Beam on a Winkler foundation

Fig. 1
beam, we take the displacement at the mid plane of it. If we denote the rotation of the beam by \(\theta(x,t)\), the differential equations of the beam may be written in the form

\[
\begin{align*}
&1+3\nu G\frac{\partial^2 u}{\partial x^2} - \frac{1}{2} E G h_0 \left( 1 + \frac{\partial^2 u}{\partial x^2} \right) + \frac{1}{2} E G h_0 \left( 1 + \frac{\partial^2 u}{\partial x^2} \right)^2 = 0 \\
&\frac{\partial^2 \theta}{\partial x^2} - \frac{1}{2} E G h_0 \frac{\partial^2 \theta}{\partial x^2} = 0
\end{align*}
\]

where \(E\), \(G\), and \(\nu\) are the shear modulus, density and Poisson's ratio, and \(h_0\) and \(t\) are the shear coefficient, thickness and time, respectively. The sign of \(\theta\) is positive in the clockwise direction.

For the continuum, putting

\[
w_i = \frac{\partial w}{\partial x}, \quad n_i = \frac{\partial w}{\partial y}
\]

we have the equations of motion in the plane stress state

\[
\begin{align*}
&\frac{1}{1-\nu^2} \left( \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right) = \frac{E}{1-\nu^2} \frac{\partial^2 \sigma_{zz}}{\partial z^2} \\
&\frac{1}{1-\nu^2} \left( \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \right) = \frac{E}{1-\nu^2} \frac{\partial^2 \tau_{yz}}{\partial z \partial y}
\end{align*}
\]

The stress components are, in usual notations,

\[
\begin{align*}
\sigma_{xx} &= \frac{2G}{1-\nu} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
\sigma_{yy} &= \frac{2G}{1-\nu} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \\
\tau_{xy} &= G \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{align*}
\]

As an example, we assume that an applied load \(p(x,t)\) is a concentrated force of magnitude \(F\) and it moves with a constant velocity \(c\) in the positive \(x\)-direction. Then it may be expressed in the form

\[
p(x,t) = p(x-ct)
\]

where \(\delta\) is the Dirac delta function.

If we assume that the beam is fixed on the semi-infinite rigid body, we have at \(y = 0\)

\[
w_i = v_i = 0
\]

If we also assume that the beam is infinitely long and that steady state has been reached, the deformation pattern is time invariant relative to a coordinate system moving with the load. Therefore the following coordinate transformation may be used:

\[
x = x - ct
\]

By transforming Eqs. (1) – (11) to the moving coordinate and further by applying the complex Fourier transformation with respect to \(x\) which is defined as

\[
\begin{align*}
&f(\xi, t) = \int_{-\infty}^{\infty} f(x, t)e^{i\xi x} dx \\
&g(\xi, t) = \int_{-\infty}^{\infty} g(x, t)e^{i\xi x} dx
\end{align*}
\]

the following equations are obtained.

\[
\begin{align*}
&\left( \frac{\partial^2}{\partial t^2} - \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) g \left( \frac{\partial x}{\partial t} \right) \\
&= \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} + \frac{1}{4} \frac{\partial^2 g}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 \\
&= \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 \\
&= \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2
\end{align*}
\]

where \(\sigma_{11}\) and \(\sigma_{21}\) are the propagation velocities of longitudinal and shear waves in the continuum.

By obtaining \(\frac{\delta}{\partial x}\) and \(\frac{\delta}{\partial y}\) from Eqs. (16) and (17), and by substituting the results into Eqs. (15) and (18) (20), the general solutions of \(u_1, v_1, \sigma_{11}, \sigma_{21}\) and \(T_{eq}\) are obtained. Further, applying Eq. (22) to these equations, the following equations are obtained.

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= \frac{1}{c_{u_i}} \frac{\partial u_i}{\partial x} \\
\frac{\partial v_i}{\partial t} &= \frac{1}{c_{v_i}} \frac{\partial v_i}{\partial x} \\
\frac{\partial \sigma_{21}}{\partial t} &= \frac{1}{c_{\sigma_{21}}} \frac{\partial \sigma_{21}}{\partial x}
\end{align*}
\]

where \(c_{u_i}\) and \(c_{v_i}\) are the propagation velocities of longitudinal and shear waves in the continuum.

Using the procedure outlined, we now treat the cases of the interaction between the beam and the continuum in the state of smooth contact and welded contact.

3. Smooth contact

For smooth contact, there is no shearing force in the interface between the beam
and the stratum. The boundary conditions at
y = \( h_1 \) are shown in the form

\[ u = u_1 = 0 \]  
\[ \psi = 0 \]  
\[ \tau_{y z} = 0 \]  

From the condition \( \gamma = 0 \), we find that the
elongation of the mid-plane of the beam is
decoupled from the motion in other modes, and
we can omit Eq. (3) in this case.

By substituting Eqs. (23)~(27) into
transformed equations of Eqs. (28)~(30),
and by substituting the results into Eqs. (12)
and (13), a system of three equations
for the constants \( A_1 \), \( B_1 \) and \( \bar{B} \) is obtained,
from which the constants are determined as

\[ A_1 = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]  
\[ B_1 = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]  
\[ \bar{B} = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]

where

\[ A_1 = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]  
\[ B_1 = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]  
\[ \bar{B} = \frac{\text{d} \sinh \beta y}{\beta \text{d} y} \]

Proceeding in the same way as in
Chapter 3, we obtain a system of three
equations for the constants \( A_1 \), \( B_1 \) and \( \bar{B} \).
In this case, we have

\[ a_0 = \frac{(\alpha \beta \cosh \beta h_0)}{\beta \sinh \beta h_0} \]  
\[ a_0 = \frac{(1 - \beta^2) \beta \sinh \beta h_0}{\beta \sinh \beta h_0} \]  
\[ a_0 = \frac{(1 - \beta^2) \beta \sinh \beta h_0}{\beta \sinh \beta h_0} \]

5. Critical velocity of a moving load

The response of the system is de-
termined from the equations derived in
Chapter 3 and 4. For an example, the
placement of the beam is obtained as

\[ \nu_s = \frac{1}{2 \pi} \int [ (\alpha \sinh \beta h_0 + \beta \sinh \beta h_0) \alpha \bar{G} \frac{\beta \sinh \beta h_0}{\beta \sinh \beta h_0} + \alpha \cosh \beta h_0 + \beta \sinh \beta h_0 \frac{\bar{G}}{\beta \sinh \beta h_0} + \alpha \bar{G} \frac{\beta \sinh \beta h_0}{\beta \sinh \beta h_0} + \alpha \bar{G} \frac{\beta \sinh \beta h_0}{\beta \sinh \beta h_0} ] ds \]

where

\[ \nu_s = \frac{\bar{G}}{\beta \sinh \beta h_0} \]

The integral (38) is calculated by the
Jordan lemma and the residue theorem. But
when the integrand has poles of the second
order on the real axis, difficulties arise
in the evaluation of the integral. In
this case we have integral of the follow-
ing form

\[ \int \frac{f(x)}{(x-a)^n} \, dx, \quad n, a : \text{real} \]

which does not exist, not even in the sense of
a Cauchy principal value. It blows up,
and we encounter a resonance in the sense
that the amplitude of the displacement
becomes unbounded all along the beam.

Figures 2 and 3 indicates the variations of
the non-dimensional velocity of the
moving load \( \lambda \) versus the non-dimensional
integral variable \( s \), which were calculated
from the characteristic equations \( \Delta = 0 \).
The results are for the set of physical data listed below.

\[ G_{12}=0.1, \quad \rho_{12}=0.3, \quad k=6/\delta \]
\[ v_1=v_2=0.3, \quad h_{12}=3 \]

Figure 2 shows the case of smooth contact, and Fig. 3 the case of welded contact. These graphs are symmetrical with respect to \( s=0 \), and only the curves for real and positive values of \( s \) are shown.

The responses of a Timoshenko beam on a Winkler foundation as shown in Fig. 1(b) have been obtained by Achenbach and Sun and Crandall. The form of the characteristic equation in this case is shown as follows:

\[ a^2(1-\beta^2)(\lambda^2-\beta^2)\lambda^s+(\lambda^2-\beta^2)\lambda^s+1=0 \]  \hspace{1cm} (39)

where

\[ a^2=\frac{E h_1}{1+\nu \nu_0}, \quad \beta^2=1 \]
\[ \lambda^2-\beta^2, \quad \nu^2=\frac{\nu}{2\sqrt{3}} \]

In Fig. 2, the chained lines show the result obtained from Eq. (39). From Figs. 2 and 3, we can see that there are many differences between the results from the continuum theory and the one from the approximate theory. The critical velocities are given by the velocities at the maximum or minimum points of the curves in Figs. 2 and 3. At these load velocities the integrand has poles of the second order on the real axis. For the beam on the elastic stratum, there are many maximum and minimum points, and this type of resonance appears repeatedly both for smooth contact and for welded contact.

For the beam on the Winkler foundation, there is only one minimum point, and this type of resonance occurs once for all. Clearly from the Figures, the critical velocity for the beam on the Winkler foundation corresponds to the fundamental critical velocity for the beam on the stratum.

Hereinafter we will call the non-dimensional critical velocities the first critical velocity \( \sigma_{11} \), the second critical velocity \( \sigma_{22} \), the third critical velocity \( \sigma_{33} \), and so on from the lower one.

The dispersion equation, which is obtained in the study of free wave propagation in a beam supported by an elastic stratum, corresponds to the equation which is obtained by replacing both \( \lambda \) and \( \delta \) in the characteristic equation \( \delta=0 \) by the non-dimensional phase velocity and the non-dimensional wave number of free waves. Hence the critical velocities correspond to the phase velocities at which the dispersion curves show minima or maxima. It is of interest to note that at such points the group velocity equals the phase velocity.

The first critical velocity \( \sigma_{11} \) and the second critical velocity \( \sigma_{22} \) were calculated as functions of the density ratio \( \rho_{12} \) and the thickness ratio \( h_{12} \), and are shown in Figs. 4 and 5. Figure 4 shows the result for the beam in smooth contact, and Fig. 5 shows the one for the beam in welded contact. The dashed lines show the value of the first critical velocity \( \sigma_{11} \) for the beam on the Winkler foundation which was obtained from Eq. (39). Clearly from the Figures, the critical velocities vary considerably with the value of the density ratio and the thickness ratio. It is observed that the critical velocities decrease as the density ratio or the thickness ratio decreases, and take relatively constant values for \( h_{12}=0 \). The first and the second critical velocities have a tendency to approach each other as the density ratio increases. The first critical velocity by the Winkler assumption decreases monotonously as the thickness ratio increases. The velocity by the Winkler assumption and the one by the continuum theory take relatively close values when the foundation is a light and thin layer, though they are generally
different from each other. Then it should be noted that when the foundation is a heavy or a thick layer the actual critical velocity cannot be estimated from the theoretical value by the Winkler assumption.

For two values of the thickness ratio the first and the second critical velocities were calculated as a function of the density ratio. Figure 6 shows the case \( h_{12}=3 \) and Fig.7 the case \( h_{12}=0 \). In the Figures, the solid lines show the critical velocities for the beam in welded contact, and the chained lines show the velocities for the beam in smooth contact. The dashed line shows the first critical velocity for the beam on a Winkler foundation. It is observed from the figures that for the same values of the thickness ratio and the density ratio the critical velocity for welded contact is always higher than the corresponding velocity for smooth contact, but they are not so different from each other. Then it can be noted that the condition of contact between the beam and the foundation does not have so much effect upon the critical velocities. It is also clear from the figures that the actual critical velocity cannot generally be estimated from the theoretical value by the Winkler assumption.

6. Conclusions

The characteristics of the critical velocities of a load moving on a beam supported by an elastic stratum were investigated. In the analysis it is assumed that the stratum is a continuum and is homogeneous, isotropic, and linearly elastic. The influence of the density and the thickness of the foundation on the critical velocities was studied. The velocities were compared with those obtained from the Winkler assumption. The results are summarized as follows:

1. The critical velocities for the beam in smooth contact with the stratum are not so different from those for the beam in welded contact with the stratum.

2. For the beam on the elastic stratum, there are many critical velocities which are not estimated from the approximate theory by the Winkler assumption.

3. The critical velocities vary considerably with the values of the density and the thickness of the foundation.

4. The velocity by the Winkler assumption and the one by the continuum theory take relatively close values when the foundation is a light and thin layer. But it should be noted that when the foundation is a heavy or a thick layer the actual critical velocity cannot be estimated from the theoretical value by the Winkler assumption.

The numerical results were obtained by using the NEAC-2200 Model 700 computer of Tohoku University Computer Center.

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