Dynamical Stress Concentration due to Flexural Waves in Bars with Various Bends

By Yoshitaro HIRANO,** Kosuke NAGAYA,*** and Shinichi OKUDA****

This paper discusses the problems of flexural wave propagation in an infinite bar with a circular arc bend, in an infinite L-type bar with a sharp bend and in an infinite Z-type bar with two bends when excited by an incident wave. A new improved theory taking into account the effects of the extension, rotatory inertia and shearing deformation is advocated. The bending moment concentration factors for these types of bars at their important cross sections are calculated and the differences of the dynamical behaviors among these bars are clarified.

1. Introduction

For a long time the vibration problems in various structural systems composed of many bars rigidly joined with each other have attracted the attention of many researchers[1]. In this type of problem, since the dimension of the systems are usually finite, the main aim of research mostly seems to lie in obtaining their natural frequencies. However, when a bar is so long that the effect of the end conditions may be neglected, its motion is considered as due to a wave propagation along the bar. Therefore, in this case it must not be treated as a problem of steady vibration but as one of wave propagation. It is an important engineering problem that a stress concentration occurs at the discontinuity region in the elastic medium when the wave excited by a periodic load propagates through the medium. In order to increase the mechanical strength of machines and structures which, recently, have rapidly increased in their sizes, it is necessary to clarify the feature of the dynamical stress concentration. From this point of view, the problems of the plane wave propagation in an elastic medium with holes or inclusion have been discussed by Pak, Moy, Mente, Cheng, and others[2]; and similar problems of the flexural wave propagation in an elastic plate have been treated by Saito, Nagaya, and others[3].

This paper deals with the problems of the flexural wave propagation in three types of bars of infinite length, namely, a bar with a circular arc bend, an L-type bar with a sharp bend, and a Z-type bar with two sharp bends. Many studies by means of the elementary theory have already been published. There are, however, few reports which consider the effects of rotatory inertia and shearing deformation, and it is very far from clarifying the dynamical behaviors of structural system under high speed conditions. In the case of a curved bar, the governing equations have been given by Morley[4] in which the effect of the extension of the central line in addition to those of the rotatory inertia and shearing deformation on the flexural motion is considered. In this paper, in case of a circular arc bar, the equations of motion based on Morley's curved bar theory are used; and in case of a straight bar, those based on the improved theory (Timoshenko's beam theory[5]) of the flexural vibration, and those based on the elementary theory of the extensional vibration are used. In the numerical examples all constituent bars are assumed to be of the same material and of the same cross section. First, the results for circular arc bars are compared with those for L-type ones. Then, the difference in bending moment concentration is clarified between a circular bend bar with a round bend and an L-type bar with a sharp one. Lastly, the dynamical bending moment concentration factors of Z-type bars are calculated to reveal the change due to the variation in the wave frequency and the joint angle. Moreover, the results from the elementary theory are also given and compared with those from our new improved theory.

2. Wave propagation in an infinite bar with a circular arc bend

Figure 1 shows an infinite bar with a circular arc bend (briefly called a circular bend bar) in which a circular arc 2 of radius R joins together two straight bars 1 and 3 at the ends B and C, where central lines of all three bars lie in one plane. In this plane, the coordinates 1, 2, and 3 are measured...
along the central line of each bar as shown in the figure, and the axis $y$ is taken to be perpendicular to the element $dz$ of the central line. The displacement in the $x$ direction is denoted as $u$ and that in the $y$ direction as $v$. We consider only the motion in the plane containing the axes $x$ and $y$.

Taking into account the effects of the extension, rotary inertia and shearing deformation on the flexural vibration, we obtain the equations of motion as follows:

\[
\begin{align*}
\left( \frac{R^2}{V_1^2} \frac{\partial^2 w}{\partial t^2} - \frac{\partial u}{\partial x} \right) &= 0 \\
\left( \frac{R^2}{V_2^2} \frac{\partial^2 w}{\partial t^2} - \frac{\partial v}{\partial x} \right) &= 0 \\
\left( \frac{R^2}{V_1^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial x} \right) &= -G A R \frac{\partial w}{\partial x} - \phi
\end{align*}
\]

\[\ldots (1)\]

\[
\begin{align*}
\left( \frac{\partial^2 u}{\partial x^2} - \frac{E}{V_1} \frac{R^2}{(1+k^2)} \frac{\partial^2 u}{\partial z^2} \right) &= 0 \\
\left( \frac{\partial^2 v}{\partial x^2} - \frac{E}{V_2} \frac{R^2}{(1+k^2)} \frac{\partial^2 v}{\partial z^2} \right) &= 0 \\
\left( \frac{k^2 \partial^2 u}{\partial x^2} - \frac{E}{V_1} \frac{R^2}{(1+k^2)} \frac{\partial^2 \phi}{\partial z^2} \right) &= 0 \\
\left( \frac{\partial^2 \phi}{\partial x^2} - \frac{E}{V_2} \frac{R^2}{(1+k^2)} \frac{\partial^2 \phi}{\partial z^2} \right) &= 0 \\
\left( \frac{\partial^2 w}{\partial x^2} + \frac{E}{V_1} \frac{R^2}{(1+k^2)} \frac{\partial^2 w}{\partial z^2} \right) &= 0
\end{align*}
\]

\[\ldots (2)\]

where

\[
\begin{align*}
V_1 &= \frac{E}{A} \\
V_2 &= \frac{E}{A} \\
u &= \frac{u}{R} \\
v &= \frac{v}{R} \\
k &= \frac{1}{A} \int_a^b x (1-x) dA \\
h &= \frac{1}{A} \int_a^b x (1-x) dA
\end{align*}
\]

\[\ldots (3)\]

Besides, $A$ is the cross sectional area, $E$ the modulus of longitudinal elasticity, $G$ the shear modulus, $I$ the moment of inertia of the cross section, $K$ the shear coefficient, $t$ time, $s$ the nondimensional length in the $x$ direction, $V_1$ the velocity of longitudinal wave, $V_2$ the velocity of shear wave and $\phi$ the deflection slope due to the bending. Equations (1) are the equations of motion for a straight bar in which the first is the equation for extensional vibration and the second and third are those for bending vibration. Equations (2) are the equations of motion for a circular arc bar given by Morley. Equations (1) and (2) are expressed in dimensionless form by using the radius $R$ of the circular arc bar. The axial force $N$, the bending moment $M$ and the shearing force $Q$ are denoted in the forms

\[
\begin{align*}
N &= EA \left( \frac{\partial w}{\partial x} \right) \\
M &= -EI \left( \frac{\partial \phi}{\partial x} \right) \\
Q &= G A R \left( \frac{\partial w}{\partial x} - \phi \right)
\end{align*}
\]

\[\ldots (4)\]

for a straight bar, and

\[
\begin{align*}
N &= EA \left( \frac{\partial w}{\partial x} \right) \\
M &= -E A k \left( \frac{\partial \phi}{\partial x} \right) \\
Q &= G A R \left( \frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial x} \right)
\end{align*}
\]

\[\ldots (5)\]

for a circular arc bar.

Now, let us consider an incident flexural wave propagating in the direction of the increasing coordinate $x$ as represented by

\[
w = w_0 \exp \left( i(\beta x - \omega t) \right)
\]

\[\ldots (6)\]

where $w_0$ is the amplitude, $\omega$ the circular frequency, $m$ the wave number of the flexural wave $w_0$ propagating in the straight bar, and $m = \sqrt{1 - \frac{\omega^2}{c^2}}$. Hereafter, the subscript $m = 1, 2, 3$ is attached to the symbol of each bar to discriminate the one from the other. We solve Eqs. (1) and (2), and select the wave of Eq. (6) as an incident wave for determining the direction of wave propagation in each bar to get the displacements $u_n$, $w_n$, and the bending slope $\phi_n (n = 1, 2, 3)$. It is assumed that the cross sectional dimensions of bar 2 are small enough compared with the radius $R$, and the frequency $\omega$ is smaller than the shear frequency. Thus the displacements and the bending slope are expressed as follows:

\[\ldots (7)\]

\[\ldots (8)\]

* In Reference (4), the shear coefficient is designated as $k^*$, but in this paper as $k$ in accordance with the case of Timoshenko beam. For the second equation of Eqs. (15) in Reference (4) must be read

\[
\left[ \frac{\partial^2 \phi}{\partial x^2} - \frac{E}{V_2} \frac{R^2}{(1+k^*)^2} \right] \phi
\]

\[\ldots (9)\]

Fig. 1 Infinite bar with a circular arc bend
where

\[ \alpha = \frac{Ra}{V_{st}}, \quad \beta = \frac{Ra}{V_{st}}, \quad \epsilon = \frac{\beta a \alpha}{\beta a}, \quad \delta = \frac{V_{st}}{a}, \quad \xi = \frac{R}{\sqrt{1 + \delta^2}}, \quad \kappa = \frac{\sqrt{\delta^2 + 1}}{2} \]

\[ N_n = \delta^2 + 1, \quad a = \frac{2\delta^2 - 1}{\delta^2 + 1}, \quad (n = 1, 2, 3) \]

\[ p_i = \left\{ \frac{1}{2} \left( \delta^2 \kappa^2 + 1 + \delta^2 \kappa^2 - 1 \right) \right\} (d_i^2 - d_i) \]

\[ q_i = \lambda_i (d_i^2 + 1) \]

\[ \lambda_i = \sqrt{\delta^2} \quad (i = 1, 2, 3) \]

Furthermore, \( \lambda^2 \) is the root of the following cubic equation

\[ \lambda^3 + (m_1^2 + 2m_2^2 + 2 \lambda^2) \lambda^2 + (m_1^2 + 2 \lambda^2) \lambda + (2 \lambda^2 + m_1^2 + 2 \lambda^2) = 0 \]

and \( R \) denotes the radius of gyration about the central line of the bar 2, and there are the following relations

\[ k = \sqrt{e_1\alpha_1\beta_1} = \frac{1}{\xi}, \quad k_1 = \delta k, \quad k_2 = k \]

In Eqs. (7), (8) and (9) \( b_1, b_2, b_3, c_1, c_2, \ldots, c_4, d_1, d_2, d_3 \) are unknown constants to be determined from the equations of continuity between the bars 1 and 2, and between 2 and 3. As the discrepancy between the neutral axis and the central axis of the circular arc bar, considering the above-mentioned assumption, is negligibly small, the conditions of continuity are

\[ w_i(0) = w_i(0), \quad w_i(0) = w_i(0), \quad \phi_i(0) = \phi_i(0) \]

\[ w_i(\theta) = w_i(0), \quad w_i(\theta) = w_i(0), \quad \phi_i(\theta) = \phi_i(0) \]

\[ N_i(0) = N_i(0), \quad Q_i(0) = Q_i(0) \]

\[ M_1(0) = M_1(0), \quad N_1(\theta) = N_1(0) \]

\[ Q_1(\theta) = Q_1(0), \quad M_1(\theta) = M_1(0) \]

where

\[ w_i(0) = (w_i)_{x=0}, \quad w_i(\theta) = (w_i)_{x=\theta} \]

\[ w_i(\theta) = (w_i)_{x=\theta}, \quad w_i(0) = (w_i)_{x=0}, \ldots \]
Substituting Eqs. (7) and (9) into Eq. (13), a set of simultaneous equations to determine the unknown constants is obtained in the matrix form as the expression (14).

In Eq. (14), we have

\[
\begin{align*}
\epsilon_l &= \frac{E_A A_l}{E_{A_A} A_A}, \\
\epsilon_l &= \frac{E_A A_l}{A_{A_A} A_A}, \\
f_l &= \frac{E_A A_l}{E_{A_A} R^k}, \\
g_l &= \frac{E_A A_l}{E_{A_A} R^k}, \\
\beta_l &= \frac{b_l}{w_0}, \\
\gamma_l &= \frac{b_l}{w_0}.
\end{align*}
\]

...(15)

The maximum moment \( M_0 \) during one period of vibration in the bar 1 excited by the incident flexural wave is given by Eqs. (6) and (4) as

\[ M_0 = -\frac{E_A A_l w_{01} \beta_l}{R} \]  

...(16)

The axial force, the bending moment and the shearing force are obtained from Eqs. (7), (8), (9) and Eqs. (4) with (5). Now, calculating the maximum bending moment during one period of vibration in the bar 2 and introducing a dimensionless quantity by dividing it by \( M_0 \), we obtain the maximum nondimensional moment \( M^*_2 (= M_2 / M_0) \) in the following form

\[ M^*_2 = \frac{1}{\epsilon_l} \begin{pmatrix} q_1 \epsilon_l^{-1} \beta_l - q_2 \epsilon_l^{-1} \beta_l \\ + q_1 \epsilon_l^{-1} \epsilon_l \beta_l - q_2 \epsilon_l^{-1} \epsilon_l \beta_l + q_1 \epsilon_l^{-1} \epsilon_l \beta_l \end{pmatrix} / (i \sigma_{12} \beta_l) \] 

...(17)

In a similar manner the axial and the shearing forces are obtained. In Eq. (17) \( \epsilon_l \) through \( \epsilon_6 \) are found from Eq. (14), and the dynamical bending moment concentration factors at the rigid joints \( B_0 \) and \( C_0 \) and at the middle point \( D_0 \) of the circular arc bar 2 corresponding to \( s_2 = 0 \), \( s_2 = 6 \) and \( s_2 = 6/2 \) respectively are obtained.

3. Wave propagation in an infinite L-type bar

Let us consider wave propagation in a bar of infinite length which is composed of two straight bars joined into a shape L with a joint angle \( \theta' \) at the joint \( E \) as shown in

\[
\begin{pmatrix}
-1 & 0 & 0 & m_{11} & m_{12} & m_{13} \\
0 & -1 & -1 & m_{21} & m_{22} & m_{23} \\
0 & \epsilon_1 & -\epsilon_1 & 0 & 0 & 0 \\
\epsilon_1, m_{11}, \epsilon_2 & -\epsilon_1, \epsilon_1, \beta_1 & m_{11}, m_{13}, \beta_1 & 0 & 0 & 0 \\
\epsilon_1, m_{21}, \epsilon_2 & -\epsilon_1, \epsilon_1, \beta_1 & m_{21}, m_{23}, \beta_1 & 0 & 0 & 0 \\
0 & \beta_1 \epsilon_1, \beta_1 & \beta_1 \epsilon_1, \beta_1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5 \\
\delta_6
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

...(18)

The displacements, the deflection slope, the axial force, the bending moment and the shearing force must be continuous at the joint \( C_0 \) between the bars 1 and 2. Accordingly the conditions of continuity between the bars 1 and 2 are

\[
\begin{align*}
\omega_1(i) &= \omega_1(i) + \omega_2(i) \\
\omega_2(i) &= \omega_2(i) + \omega_1(i) \\
\phi_1(i) &= \phi_2(i) + \phi_1(i) \\
\phi_2(i) &= \phi_2(i) + \phi_1(i) \\
m_{11} N_1(i) + m_{12} Q_1(i) &= N_1(i) \\
m_{21} N_2(i) + m_{22} Q_2(i) &= N_2(i)
\end{align*}
\]

...(19)

where \( m_{11}, m_{12}, m_{21}, m_{22} \) are the direction cosines of the axes \( x_2, y_2 \) with respect to the axes \( x_1, y_1 \) and are in this case represented by

\[
\begin{align*}
m_{11} &= -\cos \theta', \\
m_{12} &= -\sin \theta', \\
m_{21} &= \sin \theta', \\
m_{22} &= \cos \theta'
\end{align*}
\]

...(20)

Substituting Eqs. (7) and (9) into Eq. (18), a set of simultaneous equations to determine the unknown constants is given as

\[
\begin{pmatrix}
-1 & 0 & 0 & m_{11} & m_{12} & m_{13} \\
0 & -1 & -1 & m_{21} & m_{22} & m_{23} \\
0 & \epsilon_1 & -\epsilon_1 & 0 & 0 & 0 \\
\epsilon_1, m_{11}, \epsilon_2 & -\epsilon_1, \epsilon_1, \beta_1 & m_{11}, m_{13}, \beta_1 & 0 & 0 & 0 \\
\epsilon_1, m_{21}, \epsilon_2 & -\epsilon_1, \epsilon_1, \beta_1 & m_{21}, m_{23}, \beta_1 & 0 & 0 & 0 \\
0 & \beta_1 \epsilon_1, \beta_1 & \beta_1 \epsilon_1, \beta_1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5 \\
\delta_6
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

...(21)

The bending moment concentration factor is obtained in a similar manner to that for the circular arc bend bar. Therefore, the maximum nondimensional moment \( M^*_2 (= M_2 / M_0) \) in the bar 1 is given as follows:

\[
M^*_2 = \frac{[i \beta_1 \epsilon_1, \beta_1 + i \beta_1 \epsilon_1 \beta_1 + \beta_1 \epsilon_1 \beta_1 \beta_1]}{(i \beta_1 \epsilon_1 \beta_1)}
\]

...(22)

Thus, by taking \( s_1 = 0 \) in Eq. (21) the bending moment concentration factor at the joint \( B_0 \) is obtained.
4. Wave propagation in an infinite Z-type bar

As shown in Fig. 3, let us consider the wave propagation in a Z-shape bar of infinite length (briefly called a Z-type bar) constructed by joining a bar of length \( z \), at its both ends \( B_2 \) and \( C_2 \), to two semi-infinite straight bars, where central lines of all three bars lie in one plane. In this case the joint angle between the bars 1 and 2 is \( \theta_1 \), and that between 2 and 3 is \( \theta_2 \), and the rectangular coordinates \( x_1, y_1 \), \( x_2, y_2 \) and \( x_3, y_3 \), respectively, are taken in the same way as in the circular bend bar. The equations of motion for this bar being expressed in dimensionless form by using the length \( l \) of bar \( 2 \), are given by substituting \( R \) for \( R \) in Eq. (1). If the bar is excited by the incident flexural wave (6), the displacements \( u_1 \) and \( u_2 \) and the bending slope \( \phi_1 \) of the bar 1 are given by Eq. (7), and \( u_2 \) and \( \phi_2 \) of the bar 2 by

\[
\begin{align*}
\ddot{u}_1 &= (c_{12}^2 \dot{x}_1^2 + c_{13}^2 \dot{x}_1 \dot{x}_3) e^{i \omega t}, \quad \ddot{u}_2 = (c_{23}^2 \dot{x}_2^2 + c_{21}^2 \dot{x}_2 \dot{x}_1) e^{i \omega t}, \\
\ddot{\phi}_1 &= (c_{12}^2 \dot{\theta}_1^2 + c_{13}^2 \dot{\theta}_1 \dot{\theta}_3) e^{i \omega t}, \quad \ddot{\phi}_2 = (c_{23}^2 \dot{\theta}_2^2 + c_{21}^2 \dot{\theta}_2 \dot{\theta}_1) e^{i \omega t}
\end{align*}
\]

and \( u_3 \), \( \dot{u}_3 \) and \( \phi_3 \) of the bar 3 by Eq. (9). In this case \( \alpha_1, \beta_1, \beta_2 \), ..., are given by the formulas in which \( \bar{R} \) is substituted for \( R \) in Eq. (10). In this section, similarly, we use the notation substituting \( \bar{R} \) for \( R \) in the symbols in Eqs. (1) through (16) which have been reduced to dimensionless forms by using \( R \). The conditions of continuity between the bars 1 and 2 and between 2 and 3 are given by

\[
\begin{align*}
\bar{u}_1(0) &= \bar{u}_2(0), \quad \bar{u}_1'(0) = \bar{u}_2'(0) = \bar{u}_3(0), \quad \bar{u}_1''(0) = \bar{u}_2''(0) = \bar{u}_3'(0), \\
\bar{\phi}_1(0) &= \bar{\phi}_2(0), \quad \bar{\phi}_1'(0) = \bar{\phi}_2'(0) = \bar{\phi}_3(0), \quad \bar{\phi}_1''(0) = \bar{\phi}_2''(0) = \bar{\phi}_3'(0), \\
\bar{M}_1(0) &= \bar{M}_2(0), \quad \bar{M}_1'(0) = \bar{M}_2'(0) = \bar{M}_3(0), \quad \bar{M}_1''(0) = \bar{M}_2''(0) = \bar{M}_3'(0)
\end{align*}
\]

where \( m_1 \) through \( m_2 \) and \( m_3 \) through \( m_4 \) are the direction cosines of the axes \( x_1, y_2 \) with respect to the axes \( x_1, y_1 \) and those of the axes \( x_2, y_3 \), respectively, and we have

\[
\begin{align*}
\bar{m}_1 &= -\sin \theta_1, \quad \bar{m}_1' = \cos \theta_1, \quad \bar{m}_1'' = -\sin \theta_1, \\
\bar{m}_2 &= -\cos \theta_1, \quad \bar{m}_2' = -\cos \theta_1, \quad \bar{m}_2'' = -\cos \theta_1
\end{align*}
\]

The substitution of Eqs. (7), (9) and (22) into Eqs. (23) yields a set of simultaneous equations Eq. (24) to determine the unknown constants.

In Eq. (24) \( \bar{e}_1, \bar{e}_2, \bar{f}_1 \) and \( \bar{f}_2 \) are given by Eq. (15), and \( \bar{h}_1, \bar{\eta}_1 \) and \( \bar{\gamma}_1 \) are denoted as Eq. (20), and moreover we have

\[
\begin{align*}
\bar{h}_1 &= \frac{1}{l}, \quad \bar{\tau}_1 = \frac{E_1 l}{E_1 l}
\end{align*}
\]

The bending moment is obtained from Eqs. (7), (9), (22) and (4). Accordingly, the maximum nondimensional bending moment \( M_2' = 1 = \frac{M_2}{M_1} \), \( N_0 = -E_1 l \omega^2 \bar{\theta}_1 \bar{\theta}_1', \bar{\tau}_1 = \bar{\tau}_1 \) during one period of vibration in the bar 2 is

\[
M_2' = \frac{1}{(\beta_0 + \beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7)(\beta_0 + \beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7)} (17) \beta_{11}\theta_{11}')
\]

where \( \bar{\eta}_1 \) through \( \bar{\eta}_4 \) are given by Eq. (29). In Eq. (25), taking \( \bar{e}_2 = 0, \bar{\eta}_2 = 1 \), we get the bending moment concentration factors at the joints \( B_2 \) and \( C_2 \).

5. Numerical examples

In this paper numerical examples are obtained only for the cases in which all constituent bars are of the same material and have the same cross section. It is assumed that the cross section is a circle and the material is mild steel with the following physical properties:

\[
\begin{align*}
E &= E_1 = E_2 = 21 \times 10^6 \text{ kg/m}^2 \\
G &= G_1 = G_2 = 8 \times 10^5 \text{ kg/m}^2 \\
\rho_2 &= \rho_3 = \rho_4 = 7.85 \times 10^{-3} \text{ kg/cm}^3 \\
\epsilon &= \epsilon_1 = \epsilon_3 = 0.866
\end{align*}
\]

Accordingly, we have

\[
\begin{align*}
\bar{e}_1 &= \frac{1}{l}, \quad \bar{f}_1 = \frac{1}{l}, \quad \bar{g} = \bar{g}_1 = \bar{g}_2 = \bar{g}_3 \\
\bar{h}_1 &= \frac{E_1 l}{E_1 l} = \frac{1}{3}, \quad \bar{\eta}_1 = \frac{E_1 l}{E_1 l} = \frac{1}{3}, \quad \bar{\gamma}_1 = \frac{E_1 l}{E_1 l} = \frac{1}{3}
\end{align*}
\]

Thus the bending moment concentration factors in a circular bend bar, an L-type and a Z-type bar are obtained from Eqs. (17), (21) and (25), respectively, and the above expressions.

First, we present the numerical results for circular bend bars and those for L-type bars for comparing the moment concentration factor of the bar with an angular rigid joint with that of the one with a circular arc bend, and then study the features of bending moment concentration under various states of joints. Next, the numerical results for Z-type bars are shown to clarify the property of the moment concentration factor depending upon the
frequency, slenderness ratio and joint angle. In the case of Z-type bars the results based on the elementary theory are also illustrated to be compared with those based on our improved theory.

Figures 4 and 5 show the curves of the moment concentration factors $M_C^e$ and $M_C^s$ versus frequency ratio $\alpha$ ($= R \omega \sqrt{\rho/\varepsilon}$) for the circular bend bar and the L-type bars with $\zeta$ ($= R / \sqrt{I/A}$) = 50 and 100, and the joint angle $\theta = 90^\circ$. The figures show the solid line shows the values at the middle point $D_C$ of the circular arc bend, the broken line at the joint $B_C$ of the circular arc bar, and the chain line at the joint $B_L$ of the L-type bar. In Fig. 6 are shown the curves of $M_C^e$ and $M_C^s$ versus $\alpha$ when $\zeta = 100$, and $\theta = 45^\circ$, $180^\circ$ respectively; the solid line shows the values at the point $D_C$ for $\theta = 45^\circ$, the broken line those at the point $D_C$ for $\theta = 180^\circ$, and the chain line at the point $B_L$ for $\theta = 45^\circ$ ($\theta = 135^\circ$). From Figs. 4 through 6 it is seen that the dependence of the moment concentration factor of a circular arc bend bar upon the frequency ratio is strong for small frequencies but very slight for the frequency ratio larger than unity, while, in the case of an L-type bar, the concentration factor changes gradually with the frequency. In Fig. 7, the relations of $M_C^e$ and $M_C^s$ to the joint angle $\theta$ are shown, and the solid line represents the values at the middle

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Bending moment concentration factor versus frequency ratio in circular bend bar and in L-type bar ($\zeta=50$, $\theta=90^\circ$)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Bending moment concentration factor versus frequency ratio in circular bend bar and in L-type bar ($\zeta=100$, $\theta=90^\circ$)}
\end{figure}
point $B_z$ of the bend and the broken line at the joint $B_z$ of the L-type bar. It is observed from this figure that the change of the concentration factor with the joint angle is large in the L-type bar but small in the circular bend one.

It is seen from these figures that, when the frequency becomes large, the moment concentration factor for a circular bend bar with a round bend is small independently of the change of joint angle, and is influenced by its bend less than is the same factor for an L-type bar with an angular sharp bend.

Figures 8, 9 and 10 show the relations between the moment concentration factors and frequency ratios for various slenderness ratios $\zeta$ ($=l/\sqrt{I/A}$) at the joints $B_z$ and $C_z$ in the Z-type bar with $\theta_1 = \theta_2 = 90^\circ$, in which the solid and the chain line show the values by means of our improved theory and the broken and the two-dot chain line those by means of the elementary theory. In this case, for the convenience of comparison with the results given by our improved theory, the moment concentration factor from the elementary theory is defined as the maximum nondimensional moment in a Z-type bar which is given by the elementary theory and is expressed in dimensionless form by using the maximum moment $M_0$ ($=F_1 \omega_1 \beta_3;1\beta_1;1 / l$) according to our improved theory due to an incident wave propagation in the Timoshenko beam.

![Fig. 6 Bending moment concentration factor versus frequency ratio in circular bend bar and in L-type bar ($\zeta=100$)](image)

![Fig. 7 Bending moment concentration factor versus joint angle in circular bend bar and in L-type bar ($\zeta=100$)](image)

![Fig. 8 Bending moment concentration factor versus frequency ratio at $B_z$ of Z-type bar ($\zeta=50, \theta_1=\theta_2=90^\circ$)](image)

![Fig. 9 Bending moment concentration factor versus frequency ratio at $B_z$ of Z-type bar ($\zeta=100, \theta_1=\theta_2=90^\circ$)](image)

![Fig. 10 Bending moment concentration factor versus frequency ratio at $C_z$ of Z-type bar ($\zeta=100, 300, \theta_1=\theta_2=90^\circ$)](image)

In Figs. 11 and 12 are shown the results obtained from our improved theory for two Z-type bars of $\zeta = 100$ with $\theta_1 = \theta_2 = 120^\circ$ and $\theta_1 = 90^\circ, \theta_2 = 270^\circ$ respectively, where the solid line shows the values at the joint $B_z$, and the broken line at the joint $C_z$.

From these figures it is observed that as the frequency becomes smaller and the slenderness ratio larger, the calculated results from these two theories agree well, and the moment concentration factor becomes larger on the incident side (at the joint $B_z$) than on the downstream side (at the point $C_z$).
an infinite straight bar with no bend, while that of $\theta_1=90^\circ$, $\theta_2=180^\circ$ does to an infinite L-type bar, accordingly their results agree with the result $M^* = 1$ of an infinite straight bar and that of an L-type bar (in Fig. 5), respectively.

When comparing these three types of bars, namely the circular bend bar, the L-type bar and the Z-type bar, as regards the maximum value of moment concentration factor given for the variation of frequency, we can see that the Z-type bar has a maximum value larger than those for two others. This is due to the fact that in the Z-type bar the moment concentration is significantly affected by multiple scattered waves propagating in the bar 2.

In the case of an infinite bar with bends, there are three kinds of waves propagating in the bar 2 as shown by the expression of the displacement $u_1$ in Eqs. (7), namely, an incident wave, a nondamping scattered wave (which is not damped even at infinity of the bar 1), and a damping scattered wave (which is damped as it proceeds through the bar 1 far from the joint), therefore at infinity of the bar 1 an incident wave and a nondamping wave remain, accordingly the bending moment due to these two kinds of waves at infinity can be found. When the frequency ratio $\omega$ approaches zero, the bending moment becomes equivalent to the static bending moment in the limit, and moreover the bending moment due to two waves at infinity becomes the moment acting statically to the bar at infinity. In this report, since the bending moment concentration factor is defined as a ratio of a maximum moment (during one period) in the bend to the maximum moment $M_2$ due to only the incident wave $\omega$. This is different from the usual definition of the bending moment concentration factor in the static case in which we shall use a ratio of a moment to the fixed moment acting statically at infinity which corresponds to the limit of the sum of $M_2$ and the moment due to the nondamping scattered wave. When reading the result in the limit, we must consider this discrepancy.

Further, we must have the root $\lambda_1$ of the cubic equation (11) to calculate the moment concentration factor in a circular bend bar. This task is performed by the Newton-Raphson method.

The numerical results were obtained by using NEAC - 2200 Model 700 Computer of Tohoku University Computing Center.

6. Conclusions

In this paper the problems of flexural wave propagation in a bar of infinite length with bends are investigated and features of dynamical bending moment concentration produced in the bar due to an incident wave are clarified by examining the numerical results. Summarizing our obtained results we can conclude as follows.

(1) As for an infinite bar with a circular arc bend and an infinite L-type bar, the equations for the calculation of the
dynamical bending moment concentration factors are given by considering the effects of the extension, rotatory inertia and shearing deformation; and the dynamical behaviors depending upon the state of joint are studied.

(2) As for an infinite Z-type bar, the moment concentration factors are obtained by means of the elementary theory and our improved theory, and two results are compared with each other to clarify the difference between them.

(3) Several examples are numerically calculated in which all constituent bars are assumed to be of the same material and of the same cross section. From these results it is shown that the moment concentration factor of an infinite bar with a round bend (a circular bend bar) approaches unity as the frequency increases, and the effect of a round bend on the concentration factor is small as compared with that of an infinite bar with a sharp bend (an L-type bar). In addition, it becomes clear that the maximum value of the moment concentration factor of an infinite Z-type bar is larger than those of an L-type and a circular bend bar.

References


