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Continuum Theory of Cumulative Damage

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A continuum theory of cumulative damage of materials which contain various types of distributed defects is proposed. At first, a study is made of general kinetic models representing three types of defect growth in the elastic material. Here it is assumed that the mechanism of such defect growth is governed by a kind of pure birth process. When the nucleation process of defects is known, the damage accumulation functional at any time after the application of loads is given by the convolution integral of the probability density of defects and the nucleation rate of the defect. The internal structural change of the material brought about by the cumulative damage is ascribable to the nucleation and propagation of the defect, and can be described by introducing internal state variables.

This theory is applied to the description of the mechanical and thermodynamical stability condition of equilibrium and steady state in cumulative damage.

1. Introduction

The application of constant or cyclic loads of long duration to polymeric solids or FRP frequently results in a type of local fracture called "cracking in irreversible manner". The character of cracking differs from that of fracture commonly treated in the linear fracture mechanics which is based upon the classical concept of the reversible thermodynamics. Cracking damage consists of a distribution of small independent cracks in the plane surface of or inside a body and results in complete separation of the body only if the number and size of these cracks are such as to cover the cross-sectional area of the sample on some plane.

Similarly, the application of impulsive loads of high intensity and short duration to material bodies frequently causes a type of fracture called spall. Spall is defined as the local fracture of materials resulting when interactions of plane wave of uniaxial strain produce tensile stress in excess of the strength of the material, and its character differs from that of fractures produced by quasistatic loads. The duration of impulsive load application is too short to permit the extensive crack propagation.

Most theoretical research on fracture is directed towards the establishment of criteria for propagation of an isolated crack inside a closed system under the condition of quasistatic loading. These criteria can be sufficiently described by the energy conditions. But little has

been done on the problem of fracture process of cracked or spalled materials with internal dissipation. There are several ways of accounting for the dissipative effects of damage. One approach is to postulate the existence of internal equations of the pure birth processes.

Our objective in this paper is to construct kinetic models representing four types of defect growth in the elastic material and to formulate a theory of continuous damage of cracking with application to the description of the stability conditions in cumulative damage of creep and fatigue testing.

2. Model of Defect Growth

A model of defect growth kinetics in an elastic material is investigated in which the mechanism is presumed to be an irreversible chemical reaction. The kinetics of irreversible defect-growth turn out to be intimately related to a problem known in statistics as the pure birth processes which predict particle size distributions both in the presence and in the absence of a continuous initiation mechanism of defects.

Suppose that a growing defect at any instant is accumulating void at a rate which can be written analogically as a sequence of bimolecular reactions with rate constants;

$$\dot{\tilde{P}} = \tilde{A} \tilde{P} \quad (1)$$

where $\tilde{P} = [\dot{P}_0, \dot{P}_1, \dots, \dot{P}_g]^t$,

$\tilde{P} = [P_0, P_1, \dots, P_g]^t$, t: transpose

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$$A = \begin{bmatrix} -\beta_0 M & 0 & \dots & 0 \\ \beta_0 M & -\beta_1 M & 0 & \dots & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \beta_{g-1} M & -\beta_g M \end{bmatrix} \quad (2)$$

Here $P_g(t)$ is the existence-probability of defects each of which has accrued g fracture units from the start of reaction, M is the concentration of vacancies induced by external forces in metals. In the case of polymeric solids, however, M is the concentration of ruptured molecular segments surrounding the crack or the void.

The equations (1) are nonlinear, but they may be linearized by alteration of the independent variable t to a synthetic time u defined by

$$du = M dt, \quad u = 0 \text{ at } t = 0 \quad (3)$$

There follow immediately the equations of the pure birth process

$$\frac{dP}{du} = \tilde{B} P \quad (4)$$

where the matrix $\tilde{B}$ is time-independent and given by

$$\tilde{B} = M^{-1} A \quad (5)$$

By differentiating both sides of Eq.(4) with respect to u , and then substituting the relation

$$f = \frac{dP}{du} \quad (6)$$

into (4) we obtain a set of $(g+1)$ linear first order differential equations: (20)

$$\frac{d f}{du} = \tilde{B} f \quad (7)$$

When once equations (7) have been solved in terms of u , the solution can be related to clock time t by inversion of integral

$$t = \int_0^u du/M \quad (8)$$

If by some conditions the concentration M is maintained constant throughout the course of the defect extension, then Eq. (7) can be easily transformed by the relation $u = Mt$ to the clock-time space. Accordingly Eq.(7) reduce to those of the pure birth processes and a solution to them under the initial conditions,

$$P_0(0) = 1, P_g(0) = 0 \text{ for } g > 0 \quad (9)$$

is known as (2)(3)

$$f_g(t) dg = (2\pi\sigma_g^2)^{-1/2} \exp \left[-(u - \tau_g)/2\sigma_g^2 \right] d\tau \quad (10)$$

In the above expression the first and the second temporal moments are given by

$$\tau_g = \int_0^g \beta_g^{-1} dg, \quad \sigma_g^2 = \int_0^g \beta_g^{-2} dg \quad (11)$$

3. General Growth Law of Line Crack

We assume that the rate of condensation of vacancies or the rate of rupture of molecular segments at the tip of the crack are proportional to the s -th power of the half length of the line crack. The model of the line crack is illustrated in Fig.1.

Fig.1 Growth model of line crack

The functional dependence of β_g on g may be written as (4)

$$\beta_g = \beta \{ \ell (g_0 + g) \}^s \quad 0 \leq s \leq 1 \quad (12)$$

in which s is the influential index which depends on stress levels at the vicinity of line crack and on the properties of materials, β is the reaction rate of crack growth, ℓ is the average length of a fracture unit which is formed by the condensation of vacancies in metals and by the rupture of molecular segments in polymeric solids in the rate process of the defect growth, g_0 is the number of fracture units originally presented in the crack, i.e. the half length of the original crack is given by $g_0 \ell$, and at time t it becomes $(g_0 + g) \ell$.

We may lay down for selected value of s the following growth law of line crack and its dispersion in Table 1. with the aid of Eqs.(11) and (12). In these calculations the variation of vacancy concentration with time is ignored, i.e. $M = \text{constant}$. A noteworthy fact is that the dispersion of crack length increases with the lapse of time, and for $s > 1/2$ it reaches a constant value; the inhomogeneity of the assembly of line cracks generally increases at the slower rate for the larger value of s , namely for the larger value of stresses. It should be

also noted that, according to the principles of extreme-value statistics, an increase in the dispersion is expected to result in a shift of the entire life time distribution towards shorter times. In other words, with time it makes materials unstable.

When transparent thermoplastics, such as Polymethylmethacrylate (PMMA) or Polycarbonate (PC), are exposed to a sufficiently high tensile stress, deterioration of their optical clarity which results from the formation of line cracks occurs. These quasi cracks are known as "craze".⁽⁵⁾

Sato has shown the distribution of crazes in plane surface of PC under various levels of nominal stresses.⁽⁶⁾ His experiment indicates that an increased stress level decreases the dispersion of craze length at the constant temperature (Fig.2).

Fig.2 Distribution of crazes in the plane surface of PC for various stress levels⁽¹¹⁾

This fact agrees with the dispersion law of the line crack, as shown in Table 1.

Table 1. Growth law of line crack and its dispersion with rate constant

$$\beta g = \beta [l(g_0 + g)]^s$$

s	Growth law	Dispersion
0	$\langle r \rangle = r_0 + \beta l M t$	$\text{var}(r) = (r - r_0)/l \sim t$
$\frac{1}{3}$	$\langle r^{1/3} \rangle = r_0^{1/3} + \frac{2}{3} \beta l M t$	$\text{var}(r^{1/3}) = 3 \left[\left(\frac{r}{l} \right)^{1/3} - \left(\frac{r_0}{l} \right)^{1/3} \right] \sim t^{1/3}$
$\frac{1}{2}$	$\langle r^{1/2} \rangle = r_0^{1/2} + \frac{1}{2} \beta l M t$	$\text{var}(r^{1/2}) = \log \left \frac{r}{r_0} \right \sim \log t^2$
$\frac{2}{3}$	$\langle r^{1/3} \rangle = r_0^{1/3} + \frac{1}{3} \beta l M t$	$\text{var}(r^{1/3}) = 3 \left[\left(\frac{r}{l} \right)^{-1/3} - \left(\frac{r_0}{l} \right)^{-1/3} \right] \rightarrow \text{const}$
1	$\langle \log r \rangle = \log r_0 + \beta l M t$	$\text{var}(\log r) = \left(\frac{r_0}{l} \right)^{-1} - \left(\frac{r}{l} \right)^{-1} \rightarrow \text{const}$

Our experiment shows that the dispersion of craze length in the surface of PC specimen increases with the lapse of time and seems to reach a constant value. (Fig.3)

Fig.3 Change of frequency of crazes at constant stress

So far we presumed that the concentration does not vary with time. In other cases the vacancies are consumed only through the growth reactions of cracks and they are not reproduced anyway in the system by means of stress on the boundary under the action of the applied stress, but they grow by the addition of preexisting vacancies.

In the case of polymeric solids, the activated molecular segments surrounding the crack are consumed through the growth reaction of cracks, so that the concentration M decreases in course of time.

The rate of change of the concentration these phenomena is expressed as follows,⁽²⁾

$$\frac{dM}{dt} = -M \sum_0^{\infty} \beta g f g \quad (13)$$

with the initial conditions

$$f_0(0) = N, f_g(0) = 0, M(0) = M_0 \quad (14)$$

The right hand side of Eq.(10) is sharply peaked at $u = \tau g$

Approximation of Eq.(10) is therefore

$$f_g \beta g = N \delta(u - \tau g) \quad (15)$$

where $\delta(x)$ is the Dirac delta function. By substituting (15) into (13) and assuming the influential index $s = 1$, there results

$$\frac{dM}{du} = -N \beta l g_0 e^{\beta l u} \quad (16)$$

Replacing the sum over g by an integral and using Eqs.(3) and (16), the relation between t and u is given by

$$t = \int_0^u \frac{du}{M} = [\beta l (M_0 + N r_0 / l)]^{-1} \log \left| \frac{M_0 e^{\beta l u}}{M_0 + N r_0 (1 - e^{\beta l u}) / l} \right| \quad (17)$$

In a special case that the dispersion is negligibly small compared with the average length of cracks at all time, the crack length can be written as a function of u

$$\frac{r}{r_0} = e^{\beta \lambda u} \quad (18)$$

Combining Eqs.(17) and (18) we have the following relation

$$\frac{r}{r_0} = \frac{(M_0 + Nr_0/\ell)e^{\beta \lambda (M_0 + Nr_0/\ell)t}}{M_0 + (Nr_0/\ell)e^{\beta \lambda (M_0 + Nr_0/\ell)t}} \quad (19)$$

In a limiting case, the initial crack length and the equilibrium crack length are calculated respectively:

$$\lim r/r_0 = 1 \quad (20)$$

$$\lim r/r_0 = 1 + M\ell/r_0 N \quad (21)$$

A system which includes distributed cracks reaches a state in which the transport of vacancies from boundaries ceases, and the crack length tends asymptotically to a time independent value, if the concentration M satisfies the differential equations (13). Then an equilibrium state is reached concerning the crack propagating phenomenon.

On the contrary in a stationary state the boundary conditions imposed on the system prevent it from reaching equilibrium; when a stationary state is given by a system which receives vacancies continuously from the boundary, the concentration remains non-vanishing and no longer varies with time ($M = \text{constant}$).

Fig. 4 Growth law of the steady line crack ($M = 0$)

Fig. 4 shows that an initially monodisperse set of nucleation centers of line cracks reacting at the tip of the crack via a bimolecular reaction with constant supply of vacancies M will grow from r_0 to r according to the growth law for $s = 1$, i.e.

$$\langle \log r \rangle = \log r_0 + \beta \lambda M t, \quad (22)$$

$$M = \text{const.}$$

$$d\langle \log r \rangle / dt = \text{const.}$$

Fig. 5 Growth law of the equilibrium line crack ($M \neq 0$)

Fig. 5 shows a schematic diagram of growth law for the equilibrium line crack following Eq.(19). A system which contains distributed line cracks will be at equilibrium if the length of the line crack reaches $r_0 + M_0/N$. This state corresponds to an endurance limit in fatigue and creep-testing. The craze length on the surface plane of PC specimen loaded with tension of constant stress reaches a time-independent value which differs from craze to craze in an equilibrium state (Fig. 6).

Fig. 6 Growth of crazes in the plane surface of PC(6)

This experimental result agrees with the abovementioned calculation for the equilibrium line crack.

4. Crack Growth with Continuous Initiation

With the aid of the superpose principle, a solution of the pure birth process for the number of cracks $c(t)$ other than a constant value is given by

$$F_g(r, t) = \int_0^t \frac{\partial f_g(r, t-\eta)}{\partial t} c(\eta) d\eta, \quad (23)$$

$$c(t) = \int_0^t \dot{c}(\eta) d\eta$$

Taking into consideration the following conditions,

$$\frac{\partial f_g(t-\eta)}{\partial t} = - \frac{\partial f_g(t-\eta)}{\partial \eta}, \quad c(0) = 0 \quad (24)$$

it follows from Eq.(23) that the damage accumulative functional F_g at any time t after the application of a constant stress is given by

$$F_g(r, t) = \int_0^t f_g(r, t-\eta) \frac{\partial c}{\partial \eta} d\eta \quad (25)$$

For cyclic loading at any number of cycles n it becomes

$$F_g(r, t) = \int_0^n f_g(r, n-\eta) \frac{\partial c}{\partial \eta} d\eta \quad (26)$$

The first and second convolution terms of the right hand members of Eqs.(25) and (26) are identified with the damage due to crack growth and the rate of damage nucleation, respectively.

Following the fracture theory of Kawabata⁽⁷⁾ we estimate the probabilistic nucleation rate of defects $\dot{c}(t)$. Let us first consider a solid in which at all points there is a uniform stress, with the reduced value σ . The expected functional relation between the probability of defect nucleation in a certain volume of a body B , designated by V , and the magnitude of V is easily obtained from the assumption that these inhomogeneities are uniformly distributed over the volume of the considered body. Now let us partition the volume V into M pieces of cells, the volume of the i -th cell being ΔV_i , which are sufficiently so small that the stress condition might be considered almost uniform in each, but sufficiently so large that there is an expected value of defect nucleation m_{ci} per unit volume and unit time. Then the probability of nonoccurrence of defect nucleation during a small time interval between t and $t+\Delta t$ is given by

$$q_0 = e^{-\sum_{i=1}^M m_{ci} \Delta V_i \Delta t} \quad (27)$$

under the assumption that the probabilities of nonoccurrence in each volume V_i are independent. Accordingly the probability of occurrence of defect nucleation during the time-interval Δt can be expressed as

$$q_1 = 1 - e^{-\sum_{i=1}^M m_{ci} \Delta V_i \Delta t} \quad (28)$$

The probability of nonoccurrence in interval between $t=0$ and t is

$$q_2 = e^{-\sum_{n=1}^{\ell} \sum_{i=1}^M m_{ci} \Delta t \Delta V_i} \quad (29)$$

where $\ell = t/\Delta t$.

The probability of occurrence at the time $t + \Delta t$ is

$$q_3 = q_1 \cdot q_2 \quad (30)$$

If now t is of extremely short duration, the average rate of defect nucleation over this interval is very nearly equal to the instantaneous rate at the beginning of the interval. By putting $q_3 = \dot{c}(t) dt$, we shall define the instantaneous rate of defect nucleation at the time t as Δt approaches zero,

$$\begin{aligned} \dot{c}(t) dt &= e^{-\int_0^t \left(\sum_{i=1}^M m_{ci} \Delta V_i \right) dt} \\ &\quad \left(1 - e^{-\sum_{i=1}^M m_{ci} \Delta V_i \Delta t} \right) \\ &\approx \sum_{i=1}^M m_{ci} \Delta V_i \exp \left[-\int_0^t \left(\sum_{i=1}^M m_{ci} \Delta V_i \right) dt \right] dt \quad (31) \end{aligned}$$

As the stress might be considered almost uniform in each cell, it is reasonable to consider that the defect nucleation in each cell is constant, i.e. $m_{ci} = \text{const.} = m_c$. In general, m_{ci} is a function of stress and temperature in cell, i.e. $m_{ci} = m_{ci}(\sigma, T)$. From Eq.(31) and the abovementioned fact, we obtain the probabilistic nucleation rate of defects as follows,

$$\begin{aligned} \dot{c}(t) &= m_c V \exp \left[-V \int_0^t m_c dt \right] \\ &= m_c V e^{-m_c V t} \quad (32) \end{aligned}$$

where $m_c V = (\bar{t})^{-1}$ is the number of defects nucleated in the body B per unit time and has dimension of t^{-1} . $\bar{t}$ denotes the mean time per defect-nucleation.

In the case of the retarded nucleation of defects after the time lapse of t_0 , the nucleation rate is written as

$$\dot{c}(t) = m_c V e^{-m_c V(t-t_0)} \quad t \geq t_0 \quad (33)$$

$$\dot{c}(t) = 0 \quad t < t_0 \quad (34)$$

In general, there exists a retardation time for the defect nucleation in polymeric solids, such as craze in PC, so that the defect nucleation process is a kind of rate process. According to the experiment of Satoh, there exists a relation between the retardation time t_0 and the temperature T as follows⁽⁶⁾

$$\log t_0 = A_0(\sigma) + \frac{B_0(\sigma)}{RT} \quad (35)$$

where $A_0(\sigma)$ and $B_0(\sigma)$ stress-dependent constant, R is gas constant.

Integrating Eq.(33), the number of defects can be expressed as an exponential function of the volume and time.

$$C(t) = 1 - e^{-\frac{m}{c} V(t-t_0)} \quad (36)$$

If the occurrence in the volume V of a single inhomogeneity of critical severity is assumed to produce fracture of this volume, then Eq.(36) reduces to the probability of fracture, as was pointed out by Freudenthal. For the same concentration of inhomogeneities, the probability of fracture increases rapidly with an increasing volume and time.(19)

For the stable defects such as craze in polymeric solid, Eq.(36) does not correspond to the probability of fracture. In this case the number of defects reaches a limited value for a sufficiently large value of time t under the condition of a constant stress (Fig. 7).

Fig.7 Stress dependence of craze growth rate in the surface of PC

By substituting Eq.(32) into Eq.(25), we obtain

$$F_g(r,t) = \frac{m_c V}{\sqrt{2\pi} \sigma_\infty} \int_0^t \exp \left\{ -\frac{[M(t-\eta) - \tau_g]^2}{2\sigma_\infty^2} \right\} e^{-\frac{m}{c} V \eta} d\eta \quad (37)$$

$$u = Mt$$

Eq.(37) indicates that the current value of the damage depends more strongly upon the recent history than upon the distant history of the damage nucleation. This character of the damage accumulation is consistent with the fading memory hypothesis in viscoelasticity(8). It is a sufficient condition of the fading memory that the magnitude of the slope of the growth function f_g be a continuously decreasing function of time, indeed

$$\left| \frac{df_g(t)}{dt} \right|_{t=t_1} < \left| \frac{df_g(t)}{dt} \right|_{t=t_2}$$

$$\text{for sufficiently large } t_1 > t_2 > 0 \quad (38)$$

This fading memory hypothesis is reasonable in the sense that it would be physically unrealistic to expect materials to have a growing memory of damage for a more distant event.

The rate of damage accumulation is given by differentiation of Eq.(25) with respect to t .

$$\begin{aligned} \dot{F}_g(r,t) &= f_g(r,0) \dot{c}(t) + \\ &\quad \cdot \int_0^t \dot{f}_g(r,t-\eta) \frac{dc}{d\eta} d\eta \\ &= \dot{c}(t) f_g(r) \quad \text{for } \dot{f}_g = 0 \\ &= c_0 \dot{f}_g(r,t) \quad \text{for } c = c_0 \quad (39) \end{aligned}$$

5. General Growth Law of Penny Shaped Crack (I)

We assume that the rate of condensation of vacancies or the rate of rupture of molecular segments at the tip of the crack is proportional to the s -th power of the radius of the crack, in the same way as the growth of line crack. The growth model of a penny shaped crack(I) is illustrated in Fig. 8.

Fig.8 Growth model of penny shaped crack (I)

The functional dependence of β_g on g may be written as

$$\beta_g = 2\pi\beta\{\ell(g_0+g)\}^s \quad 0 \leq s \leq 1 \quad (40)$$

in which s is the influential index, β is the reaction rate of crack growth, ℓ is the average length between coarsely dispersed fracture units forming a small penny shaped crack around the initial crack, g_0 is the number of fracture units originally present in the main crack as counted in the radial direction. In this case the radius of the original crack is

given by $r_0 = g_0 \ell$, and at time t it becomes $r = (g_0 + g) \ell$.

We may lay down for the selected value of s the following growth law of a penny shaped crack and its dispersion in Table 2.

Table 2. Growth law of penny shaped crack (I) and its dispersion with rate constant

$$\beta g = \beta L^s = \beta [2\pi \ell (g_0 + g)]^s$$

s	Growth law	Dispersion
0	$\langle r \rangle = r_0 + 2\pi \beta \ell M t$	$\text{var}(r/L) = 2\pi(r - r_0)/L \sim t$
$\frac{1}{3}$	$\langle r^{1/3} \rangle = r_0^{1/3} + \frac{2}{3} \pi \beta \ell M t$	$\text{var}\left(\frac{r}{L}\right)^{1/3} = \frac{4}{3} \left[\left(\frac{r}{L}\right)^{1/3} - \left(\frac{r_0}{L}\right)^{1/3} \right] \sim t^2$
$\frac{1}{2}$	$\langle r^{1/2} \rangle = r_0^{1/2} + \pi \beta \ell M t$	$\text{var}\left(\frac{r}{L}\right)^{1/2} = \frac{1}{4} \log\left(\frac{r}{r_0}\right) \sim \log t^2$
$\frac{2}{3}$	$\langle r^{2/3} \rangle = r_0^{2/3} + \frac{2}{3} \pi \beta \ell M t$	$\text{var}\left(\frac{r}{L}\right)^{2/3} = \frac{1}{3} \left[\left(\frac{r}{L}\right)^{-1/3} - \left(\frac{r_0}{L}\right)^{-1/3} \right] \rightarrow \text{const}$
1	$\langle \log r \rangle = \log r_0 + 2\pi \beta \ell M t$	$\text{var}(\log r) = \left(\frac{r_0}{L}\right)^{-1} - \left(\frac{r}{L}\right)^{-1} \rightarrow \text{const}$

Thereby the variation of vacancy-concentration with time is ignored, i.e. $M = \text{constant}$. Similarly the dispersion of crack length increases with the lapse of time, and for $s > 1/2$ it reaches a constant value.

Let us apply this growth law of the penny shaped crack to the problem of creep rupture by using the concept of the fracture mechanics. First of all we calculate the rate of growth of the penny shaped crack under the condition of constant stress. From Table 2 the growth law of the penny shaped crack for $s = 1$ having radius r can be written as

$$\langle \log r \rangle = \log r_0 + 2\pi \beta \ell M t \quad (41)$$

The reaction rate of crack growth and its simplified form are given by Yokobori⁽⁹⁾,

$$\begin{aligned} \beta &= A \frac{KT}{h} \exp\left[-\frac{\Delta f^* + \Delta F^*}{KT} + \frac{v_0^2 S^2}{2KET}\right] \\ &= A \frac{KT}{h} \left(\frac{qS}{S_0}\right)^{1/mKT} \exp\left[-\frac{\Delta f^*}{KT}\right] \end{aligned} \quad (42)$$

where A : material constant, h : Planck's constant, K : Boltzmann's constant, T : absolute temperature, Δf^* : free energy with the formation of a crack of radius r , ΔF^* : activation energy for the separation of crack surface, S : stress acting on the specimen, ΔV : effective volume, q : stress concentration, E : elastic modulus, S_0 : constant depending upon the shape of crack.

Assuming that the dispersion is negligibly small, and substituting Eq.(42) into Eq.(41) the growth law of the penny shaped crack is

$$\begin{aligned} \langle \log r \rangle &= \log r_0 + 2\pi \ell M \frac{AKT}{h} \left(\frac{qS}{S_0}\right)^{1/mKT} \\ &\quad \exp\left[-\frac{\Delta f^*}{KT}\right] t \end{aligned} \quad (43)$$

Instead of the local stress concentration qS we use the average stress σ over the ϵ region at the tip of crack such that

$$qS = 2\sqrt{\frac{r}{2\epsilon}} \sigma \quad (44)$$

Taking into account the stress intensity factor $K_I = \sqrt{r} \sigma$ of the fracture mode I, the rate of crack growth becomes

$$\frac{dr}{dt} = C_1 (K_I)^{1/mKT} r$$

where

$$C_1 = 2\pi \ell M \frac{AKT}{h} \exp\left[-\frac{\Delta f^*}{KT}\right] \left[\frac{\sqrt{2}}{S_0 \sqrt{\epsilon \pi}}\right]^{1/mKT} \quad (45)$$

The above expression is valid for static stress loading, for example crack growth in creep testing.

In a fatigue testing the applied stress is a periodic function of time and can be approximated by

$$qS = 2\sqrt{\frac{r}{2\epsilon}} (\sigma_m + \sigma_a \sin 2\pi \omega t) \quad (46)$$

where σ_a is the cyclic stress amplitude, ω is the cyclic frequency of loading and σ_m is the mean stress of loading.

The net rate of the crack growth per cycle is then obtained by combining Eq.(43) and Eq.(46),

$$\begin{aligned} \frac{dr}{dn} &= 2\pi \ell M \frac{AKT}{h} (S_0)^{-1/mKT} \\ &\quad \exp\left[-\frac{\Delta f^*}{KT}\right] r \int_0^{1/\omega} (qS)^{1/mKT} dt \end{aligned} \quad (47)$$

Neglecting the effect of the mean stress σ_m in comparison with that of the stress amplitude σ_a we obtain the following expression for the growth rate of a fatigue crack having the stress intensity factor range $K = \sqrt{\pi r} \sigma_a$,

$$\begin{aligned} \frac{dr}{dn} &= \frac{H}{\omega} \frac{1}{\sqrt{\pi r} \sigma_a}^{1/mKT} r \\ &= \frac{H}{\omega} \frac{1}{(\Delta K/2)^{1/mKT}} r \end{aligned} \quad (48)$$

where

$$\begin{aligned} H_1 &= (\pi \ell M A K T / h) \exp \\ &\quad \left[-\frac{\Delta f^*}{KT}\right] \left(\frac{4}{\pi} \frac{1}{\sqrt{2\pi \epsilon} S_0}\right)^{1/mKT} \end{aligned}$$

On the other hand, taking into account the effect of the mean stress the growth rate of crack per cycle is written as

$$\begin{aligned} \frac{dr}{dn} &= \frac{H}{\omega} \frac{1}{[\sqrt{\pi r} (\sigma_m + \frac{2}{\pi} \sigma_a)]^{1/mKT}} r \\ &= \frac{H}{\omega} \frac{1}{(K_m)^{1/mKT}} r, \quad K_m = \sqrt{\pi r} (\sigma_m + \frac{2}{\pi} \sigma_a) \end{aligned} \quad (49)$$

The form of Eq.(48) is closely related to Frost's equation of fatigue crack propagation (12), Prevorsek's equation (10) and the author's (11) equation of the rate of crack growth in fatigue testing for polymeric solids take the same form as Eq.(48).

6. General Growth Law of Penny Shaped Crack (II)

This type of penny shaped crack grows in the accumulating process of penny shaped micro-cracks around the initial crack which consists of densely packed small penny shaped cracks of g_0 . The radius of the initial crack is given by

$$r_0 = (g_0 a / \pi)^{1/2} \quad (50)$$

and at time t it becomes

$$r = \{(g_0 + g) a / \pi\}^{1/2}$$

where a is the area of a penny shaped micro crack.

Fig.9 Growth model of penny shaped crack (II)

Fig.9 shows the growth model of a penny shaped crack II, and in Table 3 the growth law of the penny shaped crack II and its dispersion are shown for the selected value of s , in which the functional dependence of βg on g is given by the m -th power of the perimeter of the penny shaped crack II.

$$\beta g = \beta (2\pi r)^m = \beta (2\pi)^m [(g_0 + g) a / \pi]^{m/2} \quad (51)$$

Table 3. Growth law of penny shaped crack (II) and its dispersion with rate constant

s	Growth law	Dispersion
0	$\langle r^2 \rangle = r_0^2 + \frac{a}{\pi} \beta M t$	$var(r^2) = \frac{a}{\pi} (r^2 - r_0^2) \sim t$
$\frac{1}{3}$	$\langle r^{4/3} \rangle = r_0^{4/3} + \frac{2}{3} \left(\frac{4}{\pi}\right)^{1/3} a \beta M t$	$var(r^{4/3}) = \frac{4a}{3\pi} (r^{2/3} - r_0^{2/3}) \sim t^{1/2}$
$\frac{1}{2}$	$\langle r \rangle = r_0 + a \beta M t$	$var(r) = \frac{a}{4\pi} (\log r^2 - \log r_0^2) \sim \log t^2$
$\frac{2}{3}$	$\langle r^{2/3} \rangle = r_0^{2/3} + \frac{2}{3} \sqrt[3]{\frac{16}{\pi}} a \beta M t$	$var(r^{2/3}) = \frac{3a}{\pi} (r_0^{-2/3} - r^{-2/3}) \rightarrow \text{const}$
1	$\langle \log r^2 \rangle = \log r_0^2 + 4\pi a \beta M t$	$var(\log r^2) = \frac{a}{\pi} (r_0^{-2} - r^{-2}) \rightarrow \text{const}$

From Table 3 the growth law of the penny shaped crack II for $m = 1$, $s = 1/2$ can be written as

$$\langle r \rangle = r_0 + \frac{AKTMa}{h} \left(\frac{gS}{S_0}\right)^{1/mKT} \exp\left[-\frac{\Delta f^*}{KT}\right] t \quad (52)$$

Assuming that the dispersion of r is negligibly small, the growth rate of the penny shaped crack II for constant stress loading is written as

$$\frac{dr}{dt} = C_2 (K_I)^{1/mKT} \quad (53)$$

where

$$C_2 = \frac{AKTMa}{h} \exp\left[-\frac{\Delta f^*}{KT}\right] \left(\frac{\sqrt{2}}{S_0 \sqrt{\pi \epsilon}}\right)^{1/mKT}$$

Sakata's equation of the growth rate of crack in creep testing takes the same form as Eq.(52).

In fatigue testing it can be written as

$$\frac{dr}{dn} = \frac{\textcircled{H}}{\omega} (K_m)^{1/mKT} \quad (54)$$

Neglecting the effect of the mean stress, Eq.(54) reduces to

$$\frac{dr}{dn} = \frac{\textcircled{H}}{\omega} (\Delta K/2)^{1/mKT} \quad (55)$$

where

$$\textcircled{H}_2 = \frac{AKTMa}{2h} \exp\left(-\frac{\Delta f^*}{KT}\right) \left(\frac{4}{\pi} \frac{1}{\sqrt{\pi \epsilon} S_0}\right)^{1/mKT}$$

The form of Eq.(54) is closely related to Paris' (13) and Yokobori's equations (9) of fatigue crack propagation for metals.

7. General Growth Law of Void

The original size of the nucleation center is measured by g_0 , which, in the case that the nucleation center is composed of the same vacancies as those in the continuum matrix, is the number of rupture units, voids, originally present in the initial void. The functional dependence of βg on g will be defined by the assumption that the rupture process leading to the accretion of vacancies on the growing void is surface catalyzed and each βg may be written

$$\beta g = \beta s(g) = \beta (4\pi r^2)^m = \beta (36\pi v^2)^{s/2} (g_0 + g)^s \quad (56)$$

where $s(g)$ is the surface area of a void, assumed spherical, which has accrued g rupture units from the start of reaction, v is the volume of the rupture unit, small void, in the condensed phase (Fig.10).

Fig.10 Growth model of void

From Eq.(56) and Eq.(11) we may lay down for the selected value of s the growth law of void and its dispersion in Table 4.

Table 4. Growth law of void and its dispersion with rate constant

$$\beta g = \beta (36\pi v^2)^{s/2} (g_0 + g)^s$$

s	Growth law	Dispersion
0	$\langle r^3 \rangle = r_0^3 + \frac{3v}{4\pi} \beta M t$	$\text{var}(r^3) = \frac{3v}{4\pi} (r^3 - r_0^3) \sim t$
$\frac{1}{3}$	$\langle r^2 \rangle = r_0^2 + 6v \beta M t$	$\text{var}(r^2) = \frac{v}{\pi} (r - r_0) \sim t^{1/2}$
$\frac{1}{2}$	$\langle r^{3/2} \rangle = r_0^{3/2} + \frac{\sqrt{2}}{\pi} v \beta M t$	$\text{var}(r^{3/2}) = \frac{3v}{16\pi} \log \left(\frac{r^3}{r_0^3} \right) \sim \log t$
$\frac{2}{3}$	$\langle r \rangle = r_0 + \beta v M t$	$\text{var}(r) = \frac{v}{4\pi} (r_0^{-1} - r^{-1}) \rightarrow \text{const}$
1	$\langle \log r^3 \rangle = \log r_0^3 + 6\pi^{1/2} v \beta M t$	$\text{var}(\log r^3) = \frac{3v}{4\pi} (r_0^{-3} - r^{-3}) \rightarrow \text{const}$

These four models can be applied to various materials other than polymeric solids. For sufficiently brittle materials spall damage takes the form of a distribution of small plane cracks, so that the growth models of a penny shaped crack are convenient for the description of mechanical behavior of such materials. Examples of materials in this class are iron and beryllium. In more ductile materials such as copper, aluminum and tantalum spall damage takes the form of small open voids. For such materials we can use the growth model of void.⁽¹⁾

8. Thermodynamic Process of Damage Accumulation

In the present continuum theory of cumulative damage, we consider a body B with material points X and assume there is no diffusion of mass in B , but B may deform and conduct heat. The thermodynamic process of a body B is described by nine functions⁽¹⁴⁾

$$\{ \chi, T, b, \psi, q, r, \eta, \theta, \alpha \}$$

of the particle X and time t .

These functions have the following physical interpretations.

The function of motion $x = \chi(X, t)$ determines the spatial position occupied by the material point X at time t . The function $T(X, t)$ is a symmetric Cauchy stress tensor, $b(X, t)$ is the body force per unit mass, $\psi(X, t)$ denotes the specific free energy per unit mass, $q(X, t)$ the heat flux vector, $r(X, t)$ the heat supply per unit mass and unit time, $\eta(X, t)$ is the specific entropy, $\theta(X, t)$ the local absolute temperature, and the damage accumulation as the internal state variable $\alpha(X, t)$.

The deformation gradient F is determined by

$$F = \text{Grad} \chi(X, t) \quad (57)$$

where Grad is computed with respect to the material coordinate X , and it is assumed that $\det F > 0$.

The system of nine functions $\{ \chi, T, b, \psi, q, r, \eta, \theta, \alpha \}$ defined for every particle X in B and for any time t is called the thermodynamic process in B , if and only if it is compatible with the condition for the balance of linear momentum

$$\rho x - \text{div } T = \rho b \quad (58)$$

and with the balance of energy

$$\rho(\dot{\psi} + \dot{\theta}\eta + \dot{\theta}\eta) - \text{tr} \{ F^{-1} T \dot{F} \} + \text{div } q = \rho r \quad (59)$$

Here ρ is the mass density of bone.

Let us call the tensor S , defined by

$$S = \rho^{-1} T (F^T)^{-1} \quad (60)$$

the Piola-Kirchhoff stress tensor. Using this tensor we can write (59) in the form

$$\rho(\dot{\psi} + \dot{\theta}\eta + \dot{\theta}\eta) - \text{ptr}(S \dot{F}) + \text{div } q = \rho r \quad (61)$$

In the present theory, the damaged material at the point X is characterized by five response functions as follows

$$\begin{aligned} \psi &= \hat{\psi}(e_{PR}, e'_{PR}, \theta, g, fg, c) \\ \eta &= \hat{\eta}(e_{PR}, e'_{PR}, \theta, g, fg, c) \\ S &= \hat{S}(e_{PR}, e'_{PR}, \theta, g, fg, c) \\ q &= \hat{q}(e_{PR}, e'_{PR}, \theta, g, fg, c) \\ \dot{\alpha}_i &= f_i(e_{PR}, e'_{PR}, \theta, g, fg, c) \end{aligned} \quad (62)$$

where $g = \text{grad } \theta$ and $\dot{\alpha}_i = f_i$ ($i = 1, 2$) are given by Eqs.(7) and (32), the total strain tensor e_{PR} being defined by

$$2e_{PR} = F^T F - I \quad (63)$$

In the above equations (62) we introduce the difference

$$e'_{PR} = e_{PR} - e''_{PR} \quad (64)$$

which we call an elastic strain tensor and we assume an inelastic strain tensor $e_{PR}^{\prime\prime}$ (15).

It is required that for any time t , for a thermodynamic process in B the thermodynamic postulate of positive production of entropy be satisfied. This is equivalent to the following inequality.

$$\rho\theta\dot{\eta} - \rho r + \operatorname{div} q - qg/\theta \geq 0 \quad (65)$$

From Eqs. (61, (62) and (65) we obtain the following expression for the entropy inequality in an admissible process.

$$\begin{aligned} (S - \partial_{e_{KL}} \hat{\psi}) \dot{e}_{KL} + \partial_{e_{KL}}'' \hat{\psi} \dot{e}_{KL}'' \\ - (\eta + \partial_{\theta} \hat{\psi}) \dot{\theta} - \partial_g \hat{\psi} \dot{g} - \partial_{\alpha} \hat{\psi} \dot{\alpha} \\ - qg/\rho\theta \geq 0 \end{aligned} \quad (66)$$

Choosing arbitrary values of $\dot{e}_{KL}$, $\dot{\theta}$ and $\dot{g}$ it is possible to determine an admissible process in B . Hence, to satisfy the inequality (66) we must assume

$$S = \partial_{e_{KL}} \hat{\psi}, \quad \eta = -\partial_{\theta} \hat{\psi}, \quad \partial_g \hat{\psi} = 0 \quad (67)$$

and

$$qg/\rho\theta \leq \theta\sigma + \partial_{e_{KL}}'' \hat{\psi} \dot{e}_{KL}'' \quad (68)$$

where $\sigma = \eta_{\dot{g}}$ is called the internal dissipation or the entropy production due to damage accumulation.

From Eq. (68) the internal dissipation inequality and the heat conduction inequality take the following form

$$\theta\sigma + \partial_{e_{KL}}'' \hat{\psi} \dot{e}_{KL}'' \geq 0 \quad (69)$$

$$qg \leq 0 \quad (70)$$

The internal dissipation due to the cumulative damage is then given by

$$\begin{aligned} \sigma = -\frac{1}{\theta} \partial_{\alpha} \hat{\psi} \dot{\alpha} = -\frac{1}{\theta} [\partial_{f_g} \hat{\psi} \dot{f}_g + \partial_c \hat{\psi} \dot{c}] \\ \geq 0 \end{aligned} \quad (71)$$

since the inelastic work $S_{KL} e_{KL}^{\prime\prime}$ is non-negative.

Intrinsic thermodynamic instability cannot occur, if the inequality (71) is consistent for some $F_g[f_g, c]$. Conversely if the inequality (71) is inconsistent for some $F_g[f_g, c]$, the structure cannot be strictly stable. A damaged material whose thermodynamic state is described by internal variables f_g and c which are governed by the equations of evolution of the form of Eqs. (7) and (32) possesses the property of fading memory.

More precisely, Helmholtz free energy ψ may be written in the form of a functional (16)

$$\psi = \Gamma[u, F_g(\tau)]_{-\infty}^t \quad (72)$$

where u represents the external variables, namely deformation and temperature. Let $h(s)$ be an influence function which is positive and decreases monotonically to zero as $t \rightarrow \infty$ and is integrable and normalized in the sense that

$$\int_0^{\infty} h(s) ds = 1 \quad (73)$$

In our case the defect nucleation function $\dot{c} = a e^{-at}$ is such a function. Then the damage accumulation functional F_g can be written as

$$\begin{aligned} F_g(\tau) &= \int_{-\infty}^{\tau} f_g(\tau-s) h(s) ds \\ &= \int_{-\infty}^{\tau} f_g(s) h(\tau-s) ds, \\ h(s) &= a e^{-as} \end{aligned} \quad (74)$$

The function $f_g(\tau)$ is always limited in absolute value ($<M$) in the interval $(-\infty, t_1)$ and can be converged to f_g^c by taking an interval (t_1, t) sufficiently large. Then for fixed u^*

$$\psi^c = \Gamma[f_g^c(\tau)]_{-\infty}^t = \phi[f_g(t)] \quad (75)$$

ψ^c is a function only of the current value of f_g .

From Eq. (74) we may write

$$\begin{aligned} F_g(\tau) &= -\int_{-\infty}^{\tau} h(\tau-\sigma) (f_g(t) - f_g(\sigma)) d\sigma \\ &\quad + \int_{-\infty}^{\tau} h(\tau-\sigma) f_g(t) d\sigma \\ &= f_g(t) - \int_{-\infty}^{\tau} h(\tau-\sigma) \delta f_g(\sigma) d\sigma, \end{aligned} \quad (76)$$

$$\delta f_g(\sigma) = f_g(t) - f_g(\sigma)$$

and putting

$$F_g^* = \int_{-\infty}^t h(t-\sigma) \delta f_g(\sigma) d\sigma \quad (77)$$

it then follows that

$$\psi = \Gamma[f_g^c(\tau)_{-\infty}^t - F_g^*(\tau)] \quad (78)$$

Evidently if $|\delta f_g(\tau)| = 0$ in $(-\infty, t)$ then $F_g^*(\tau) = 0$

ψ may be expanded in a functional power series and all terms of $O(\epsilon^n)$, $n > 1$ may be omitted. Then

$$\begin{aligned} \psi &= \Gamma[f_g^c(\tau)_{-\infty}^t - F_g^*(\tau)] \\ &= \psi^c - \int_{-\infty}^t \Gamma'[f_g^c(\tau); t-\zeta] F_g^*(\zeta) d\zeta \\ &= \phi(f_g) - \int_{-\infty}^t \phi'(f_g; t-\zeta) \\ &\quad \int_{-\infty}^{\zeta} h(\zeta-s) \delta f_g(s) ds d\zeta \end{aligned} \quad (79)$$

where ϕ and ϕ' are functions.

Putting

$$\int_0^t h(\sigma) \phi'(f_g; t-\sigma) d\sigma = \Psi'(t) \quad (80)$$

then Eq.(79) becomes

$$\psi = \phi(f_g) - \int_{-\infty}^t \{f_g(t) - f_g(s)\} \Psi'(t-s) ds \quad (81)$$

After some manipulations Eq.(81) becomes

$$\psi = \phi(f_g(t)) + \int_{-\infty}^t \Psi(f_g(t); t-\tau) \frac{df_g}{d\tau} d\tau \quad (82)$$

where

$$\Psi(f_g; t) = \int_t^{\infty} \Psi'(\sigma) d\sigma \quad (83)$$

Evidently $\Psi(f_g, t) \rightarrow 0$ as $t \rightarrow \infty$

From Eqs.(82) and (83) the mathematical definition of fading memory of damage accumulation appears to be

$$\Psi = \Gamma[F_g(\tau) + \delta F_g(\tau)]_{-\infty}^t - \Gamma[F_g(\tau)]_{-\infty}^t \rightarrow 0 \quad (84)$$

as the interval (t_1, t) becomes large.

9. Stability of Equilibrium and Steady States

If the function of damage is characterized by the average size of defects $\langle r \rangle$, the variation of defect-size σ^2 and the number of defects c ,

$$\dot{F}_g^* = \dot{F}_g(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c^*) = 0 \quad (85)$$

as $t \rightarrow \infty$

is called an equilibrium state for the damaged materials.⁽⁸⁾⁽¹⁷⁾

An equilibrium state $(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c^*)$ is said to be asymptotically stable at constant strain e_{KL}^* and constant temperature θ^* , if there exists such δ that every initial value $F_g(e_{KL}^*, \theta^*, \langle r \rangle_0, \sigma_0^2, c_0) = F_0$ with $|F_0 - F_g^*| < \delta$ is in the domain $(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c^*)$. In some neighborhoods of c^* , provided that the following form holds

$$\psi(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c) > \psi(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c^*) \quad (86)$$

then the set $(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c^*)$ is Lyapunov stable at constant strain, constant temperature and constant distribution of defects.

In this case the equilibrium state is reached with the condition

$$\{\partial \psi(e_{KL}^*, \theta^*, \langle r \rangle^*, \sigma^{2*}, c) / \partial c\} \dot{c} = 0 \quad (87)$$

$c=c^*$

From Eq.(32) the defect nucleation rate $\dot{c}$ is given by $\dot{c} = a e^{-at}$, so that the state reaches equilibrium with respect to c as $t \rightarrow \infty$.

Both the classical equilibrium conditions and the corresponding stability conditions are derived directly from the entropy production per unit time $P = \eta_d$ ⁽¹⁸⁾ The steady state which occurs sufficiently near the equilibrium may be characterized by the minimum of the entropy production P . It is convenient to split the time change of the entropy production into two parts,

$$\frac{dP}{dt} = \frac{d_X P}{dt} + \frac{d_J P}{dt} \quad (88)$$

where the subscript X denotes "generalized" thermodynamic force and J the flow. In the linear range, using the Onsager reciprocity relations we have the following equation,⁽¹⁸⁾

$$\frac{d_X P}{dt} = \frac{d_J P}{dt} \quad (89)$$

with the definition

$$\frac{d_X P}{dt} = \int_{\alpha} J_{\alpha} d_t X_{\alpha}, \quad \frac{d_J P}{dt} = \int_{\alpha} X_{\alpha} d_t J_{\alpha} \quad (90)$$

In our case

$$X_1 = -\frac{1}{\theta} \frac{\partial \psi}{\partial \langle r \rangle}, \quad X_2 = -\frac{1}{\theta} \frac{\partial \psi}{\partial \sigma^2}, \quad X_3 = -\frac{1}{\theta} \frac{\partial \psi}{\partial c} \quad (91)$$

$J_1 = \dot{\langle r \rangle}, \quad J_2 = \dot{\sigma}^2, \quad J_3 = \dot{c}$

For simplicity we introduce the three vectors:

$$\dot{\tilde{a}} = (\dot{\langle r \rangle}, \dot{\sigma}^2, \dot{c}), \quad \ddot{\tilde{a}} = (\ddot{\langle r \rangle}, \ddot{\sigma}^2, \ddot{c}) \quad (92)$$

$$\frac{\partial \psi}{\partial \tilde{a}} = \left(\frac{\partial \psi}{\partial \langle r \rangle}, \frac{\partial \psi}{\partial \sigma^2}, \frac{\partial \psi}{\partial c} \right)$$

Then the intrinsic stability criterion of equilibrium is written as

$$\theta \rho = -\frac{\partial \psi}{\partial \tilde{a}} \cdot \dot{\tilde{a}} \geq 0 \quad (93)$$

or

$$\frac{\partial A}{\partial \tilde{a}} \cdot \dot{\tilde{a}} \leq 0 \quad (94)$$

The system will be at the equilibrium state if the equality sign is fulfilled in Eq.(94). The evolution proceeds in the direction of decreasing ψ , until for

stable equilibrium, the minimum of free energy is reached.

Let us now proceed similarly with $d_J P/dt$ for a steady state. The evolution criterion to the steady state for a dissipative system of the damaged body is given by

$$\theta \frac{d_J P}{dt} = - \frac{\partial \psi}{\partial \tilde{a}} \cdot \ddot{\tilde{a}} \leq 0 \quad (95)$$

$$\frac{\partial \psi}{\partial \tilde{a}} \cdot \ddot{\tilde{a}} \geq 0 \quad (96)$$

From Eq.(96) a sufficient condition for the stability of the steady state is given by inequality

$$\frac{\partial \psi}{\partial \tilde{a}} \cdot \delta^2 \tilde{a} \leq 0 \quad (97)$$

where δ represents an increment from the steady state.

A system will be at the steady state if the equality sign is fulfilled in (98). The change of the flow proceeds always in such a way as to lower the value of the entropy production until stable steady state. The geometrical interpretation of (94) and (96) is given on Fig.(11).

Fig.11 Geometrical interpretation of Eqs.(94) and (96)

According to Eq.(94) the scalar product of the "vector" $\partial \psi / \partial \tilde{a}$ with "vector" $\dot{\tilde{a}}$ is a negative quantity, while according to Eq. (96), the scalar product of "vector" $\partial \psi / \partial \tilde{a}$ with "vector" $\ddot{\tilde{a}}$ is a positive quantity. These facts imply that the sufficient condition for equilibrium is given by

$$\dot{\tilde{a}} = (0, 0, 0) \quad (98)$$

and for the steady state

$$\ddot{\tilde{a}} = (0, 0, 0) \quad (99)$$

Conclusions

Mechanical and thermodynamical stability theories of a damaged material were proposed. In these theories the change of internal structure of material due to damage accumulation are described by introducing internal variables, such as defect nucleation and defect-growth. These results were derived under the assumption of uniform deformation, i.e. the spatial variance of the defect-growth and defect-nucleation are neglected. If the deformation were nonuniform the system of equations would have been coupled with the equation of the balance of linear momentum.

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