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The Subsequent Yield Surfaces after Proportional Preloading of the Tresca-type Material*

By Eiryo SHIRATORI**, Kozo IKEGAMI***, and Fusahito YOSHIDA****

The subsequent yield surfaces after proportional preloading are experimentally investigated for the materials whose stress-strain curves in the proportional loading are represented by the effective stress-strain relation of Tresca-type. The yield surfaces are obtained by subjecting thin-walled tubular specimens of aluminium alloy to combined axial load and internal pressure. These results are compared with those of combined axial load and torsion. For the Tresca-type materials, the shapes of the yield surfaces represented in the π -plane and in the deviatoric stress plane are found to depend on the direction of the prestress. A new stress plane modified from the deviatoric stress plane is proposed, in which the yield surfaces are given uniquely as a function of the length of the prestrain path.

1. Introduction

Many attempts have been made to describe the stress-strain relation of metals in the plastic region by regarding the subsequent yield surface after preloading as a plastic potential and using the associated flow rule, and various yield criteria considering the loading history have been proposed. (1), (2), (3) However, it is a difficult problem to establish the yield criterion for an arbitrary combined loading of a material having a complex preloading history, and the yield criteria ever proposed are hardly successful in the exact description of the flow characteristics observed in the experiment. For the Mises-type material whose initial yielding and the plastic flow under proportional loading are given by the Mises yield condition and its associated flow rule, the authors show that the shapes of the subsequent yield surfaces after proportional preloading depend only on the amount of prestrain in the deviatoric stress space as well as in the π plane, (4) and derive an equation of the subsequent yield condition after complex preloading from the fact that the strain history can be represented by the intrinsic

geometry of the hodograph of the plastic strain vector in the deviatoric strain space. For a material in which plastic behaviour is not exactly defined by the Mises condition and the shape of the subsequent yield surface depends on the direction of the preloading as well as the amount of prestrain and varies with the rotation of the principal stress axes, the authors give an equation of the subsequent yield surface only for the case of preloading under combined axial load and torsion. (6) In this paper the subsequent yield surfaces after proportional preloading for a material of the latter type are investigated in a more general combination of preloadings by assuming the material as the Tresca-type whose initial yielding and plastic flow under proportional loading can be defined by Tresca's condition.

2. Experimental procedure

The tests are made by subjecting thin-walled tubular specimens to combined axial

Al	Cu	Si	Fe	Mn	Mg	Zn	Cr
93.76	4.26	0.50	0.47	0.63	0.36	0.01	0.01 %

Table 1. Chemical composition of the material

Fig.1 Specimen

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** Professor, The Research Laboratory of Precision Machinery and Electronics, Tokyo Institute of Technology, Yokohama.

*** Associate Professor, The Research Laboratory of Precision Machinery and Electronics, Tokyo Institute of Technology.

**** Associate, The Research Laboratory of Precision Machinery and Electronics, Tokyo Institute of Technology.

load and internal pressure in a combined stress testing machine.(4) The material of the specimen is aluminium alloy(17S), annealed at 410°C for two hours. Figure 1 and Table 1 show the shape of the specimen and the chemical composition of the material, respectively. The method of the measurement of stress and strain is previously described.(1) The yield point is defined as 0.02% off-set point in Tresca-type effective plastic strain. Figure 2 shows the initial yield surface of the material in the plane (σ_x , σ_y) determined by combined axial load and internal pressure, where σ_x and σ_y are axial stress and circumferential stress of the

Fig. 2 Initial yield surface

Fig. 3 Stress-strain curves of the material

Fig. 4 Subsequent yield surfaces after tensile preloading

specimen, respectively. Figure 3 shows the effective stress-strain relations of the Tresca-type under proportional loading of various combinations of axial load

Fig. 5 Subsequent yield surface after preloading in the direction of $\sigma_y/\sigma_x = 1$

Fig. 6 Subsequent yield surface after preloading in the direction of $\sigma_y/\sigma_x = 2$

Fig. 7 Subsequent yield surface after preloading in the direction of $\sigma_y/\sigma_x = -1$

and internal pressure, and torsion, where the effective stress $\sigma_F(T)$ and the effective strain $\rho(T)$ are given by the following equation;

$$\sigma_F(T) = |\sigma_{\max} - \sigma_{\min}| \quad (1)$$

$$\rho(T) = \int |d\epsilon_{\max}| \quad (2)$$

where $\sigma_{\max}$, $\sigma_{\min}$, and $\epsilon_{\max}$ are the maximum principal stress, the minimum principal stress, and the maximum principal strain, respectively. The results of Fig. 2 and Fig. 3 show that the initial yielding and the plastic flow under proportional loading of this material are uniquely represented by the equations of Tresca-type.

3. Experimental results and discussion

3.1 Experimental results

Figures 4 to 7 show the subsequent yield surfaces in the plane (σ_x, σ_y) after proportional preloadings under combined axial load and internal pressure. The arrows in these figures indicate the directions of the plastic strain increment vectors on the yield surfaces. The normality of the plastic strain increment vector to the yield surface is observed.

3.2 Equi-strain surfaces of the Tresca-type material in the π plane and the deviatoric stress space.

For the Mises-type material, the initial yield surface and equi-strain surfaces under proportional loadings describe circles in the π plane and hyperboloids in the deviatoric stress space. The subsequent yield surfaces after proportional preloadings are determined uniquely as a function of the amount of prestrain. The subsequent yield surfaces after complex

Fig. 8 Equi-strain surfaces in the π plane

preloadings such as those including the rotation of the principal stress axes have the same shape in the deviatoric space proposed by Ilyushin(8) which is simply termed "deviatoric space", provided that the intrinsic geometry of the loading trajectories are mutually congruent, namely one is the reflection or the rotation of the other in the space.(5) From these results, the deviatoric space by Ilyushin is considered as the "isotropic space" for the Mises-type material. For the Tresca-type material, on the contrary, the shapes of the initial yield surface and equi-strain surfaces in the π plane are represented as hexagons as shown in Fig. 8. In the deviatoric stress space, the shapes for biaxial loading without rotation of the principal stress axes are represented as hexagons in the $(\bar{\sigma}_1, \bar{\sigma}_2)$ plane as shown in Fig. 9(a), and those for combined axial load and torsion with

Fig. 9(a) Equi-strain surfaces in the deviatoric stress plane $(\bar{\sigma}_1, \bar{\sigma}_2)$

Fig. 9(b) Equi-strain surfaces in the deviatoric stress plane $(\bar{\sigma}_1, \bar{\sigma}_3)$

rotation of the principal axes are expressed as ellipses in the $(\bar{\sigma}_1, \bar{\sigma}_3)$ plane as shown in Fig.9(b). Consequently, for the Tresca-type material, the shapes of the subsequent yield surfaces after proportional preloading depend not only on the amount of prestrain but also on the direction of the preloading and the rotation of the principal stress axes, even if they are represented in the π plane and in the deviatoric stress space. Here the Cartesian deviatoric stress space for the plane stress condition is represented as $(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ where

$$\bar{\sigma}_1 = \sigma_x - (1/2)\sigma_y, \quad \bar{\sigma}_2 = \sqrt{3}/2\sigma_y, \quad \bar{\sigma}_3 = \sqrt{3}\tau_{xy} \quad (3)$$

and the corresponding deviatoric strain space $(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3)$ is expressed as

$$\bar{\epsilon}_1 = \epsilon_x^{(P)}, \quad \bar{\epsilon}_2 = \frac{1}{\sqrt{3}} (\epsilon_x^{(P)} + 2\epsilon_y^{(P)}), \quad (4)$$

$$\bar{\epsilon}_3 = 2/\sqrt{3}\gamma_{xy}^{(P)}$$

where $\epsilon_x^{(P)}$ is the axial plastic strain, $\epsilon_y^{(P)}$ the circumferential plastic strain, and $\gamma_{xy}^{(P)}$ is the shear plastic strain in tensor definition. In Fig.8 and Fig.9, experimental data are also included.

3.3 The subsequent yield surfaces for the loading without rotation of the principal stress axes.

Let us investigate the subsequent yield surfaces determined by the combined axial load and internal pressure after preloading by this combination. In this case the principal stress axes do not rotate. For the Mises-type material a stress vector σ in the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ is given as

$$\sigma = \bar{\sigma}_1 e_1 + \bar{\sigma}_2 e_2$$

where e_1 and e_2 indicate the unit vectors in the directions of $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ respectively, and its value $|\sigma|$ is equal to the effective stress value of Mises-type

$$\sigma_F(M) = \sqrt{\sigma_x^2 + \sigma_x\sigma_y + \sigma_y^2}.$$

Hence,

$$\bar{\sigma}_1^2 + \bar{\sigma}_2^2 = \sigma_F(M)^2 \quad (5)$$

For the Tresca-type material the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ is not isotropic as mentioned in section 3.2. Hence a quasi-vector plane* $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ which is regarded as approximately isotropic plane for the Tresca-type material is derived from the modification of the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$. We assume that, in the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$, a stress vector σ' is represented as

$$\sigma' = \bar{\sigma}'_1 e_1 + \bar{\sigma}'_2 e_2$$

and the value $|\sigma'|$ is equal to the value of effective stress of Tresca-type

$$\sigma_F(T) = |\sigma_{\max} - \sigma_{\min}|. \quad \text{Then,}$$

$$\bar{\sigma}'_1{}^2 + \bar{\sigma}'_2{}^2 = \sigma_F(T)^2 \quad (6)$$

The plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ must be derived from linear transformation of the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ to let the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ has the characteristic of a "perfect" vector plane. But Eq.(6), is not satisfied with the value of $(\bar{\sigma}_1, \bar{\sigma}_2)$ obtained by the linear transformation. Among the transformations of the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ to the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ which satisfy Eq.(6) the following is the simplest one in which only the value of the stress vector changes like Fig.10 while the direction of the stress vector is held constant. Then

$$\begin{pmatrix} \bar{\sigma}'_1 \\ \bar{\sigma}'_2 \end{pmatrix} = \alpha(\theta_\sigma) \begin{pmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{pmatrix} \quad (7-a)$$

$$\theta_\sigma = \tan^{-1}(\bar{\sigma}_2/\bar{\sigma}_1)$$

where $\alpha(\theta_\sigma) = |\sigma'|/|\sigma|$ is the ratio of the value of the vector σ' in the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ to the value of the corresponding vector σ in the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ as shown in Fig.10 and given by

$$\alpha(\theta_\sigma) = \sqrt{3}/\{2\cos(\frac{\pi}{6} - \xi)\} \quad (0 \leq \xi \leq \frac{\pi}{3})$$

$$\alpha(\theta_\sigma) = \begin{cases} \sqrt{3}/\{2\cos(\frac{\pi}{6} - \xi)\} & (0 \leq \xi \leq \frac{\pi}{3}) \\ \sqrt{3}/(2\sin\xi) & (\frac{\pi}{3} \leq \xi \leq \frac{\pi}{2}) \end{cases} \quad (7-b)$$

$$\xi = \tan^{-1}|\bar{\sigma}_2/\bar{\sigma}_1| = \begin{cases} |\theta_\sigma| & (0 \leq |\theta_\sigma| \leq \frac{2}{3}\pi) \\ \pi - |\theta_\sigma| & (\frac{2}{3}\pi \leq |\theta_\sigma| \leq \pi) \end{cases} \quad (7-c)$$

Figure 11 gives the experimental results of the equi-surfaces shown in the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ introduced here. Marks $\bullet$ and $\circ$ show the initial yield point and equi-strain point, in the same way as in Fig.9 (a), respectively. Figure 12 shows the subsequent yield surfaces after various

Fig.10 Transformation of the plane $(\bar{\sigma}_1, \bar{\sigma}_2)$ into the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$

* It is impossible to derive the "perfect" isotropic vector plane from the assumption that the plastic behaviour of the material is expressed by Tresca's yield criterion and its associated flow rule.

proportional preloadings in the plane (σ'_I, σ'_II) where the preloading direction is taken as σ'_I axis, the yield surfaces of which correspond to those shown in Figs. 4 to 7. The relation between $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ and (σ'_I, σ'_II) is given by

$$\begin{pmatrix} \sigma'_I \\ \sigma'_II \end{pmatrix} = \begin{pmatrix} \cos \bar{\theta}'_\sigma & \sin \bar{\theta}'_\sigma \\ -\sin \bar{\theta}'_\sigma & \cos \bar{\theta}'_\sigma \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_1 \\ \bar{\sigma}'_2 \end{pmatrix} \quad (8)$$

where $\bar{\theta}'_\sigma = \tan^{-1}(\bar{\sigma}'_2/\bar{\sigma}'_1)$ denotes the preloading direction, shown in Fig. 13. It is found from Fig. 12 that the shapes in the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ of the subsequent yield surfaces after proportional preloading direction are independent of the preloading direction and represented as a function of the amount of prestrain. Hence the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ is regarded as an approximately isotropic plane for the Tresca-type material.

3.4 The subsequent yield surfaces for the loading with rotation of the principal stress axes

Let us investigate the subsequent yield surfaces determined by the combined axial load and torsion after preloading by this combination. The principal stress axes rotate in this case. The isotropic plane $(\bar{\sigma}''_1, \bar{\sigma}''_3)^*$ for the Tresca-type material is derived by transforming the plane $(\bar{\sigma}'_1, \bar{\sigma}'_3)$, i.e., the isotropic plane for the Mises-type material, as follows:

Fig. 11 Equi-strain surfaces in the modified stress planes $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ and $(\bar{\sigma}'_1, \bar{\sigma}'_3)$

* The planes $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ and $(\bar{\sigma}''_1, \bar{\sigma}''_3)$ are given, respectively, by modifying the planes (σ_1, σ_2) and (σ_1, σ_3) namely, the subspaces of the space $(\sigma_1, \sigma_2, \sigma_3)$. But the planes $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ and $(\bar{\sigma}''_1, \bar{\sigma}''_3)$ do not belong to the same modified space of $(\sigma_1, \sigma_2, \sigma_3)$. Hence, the modified plane of $(\bar{\sigma}'_1, \bar{\sigma}'_3)$ is written as $(\bar{\sigma}''_1, \bar{\sigma}''_3)$, not as $(\bar{\sigma}'_1, \bar{\sigma}'_3)$.

$$\begin{pmatrix} \bar{\sigma}''_1 \\ \bar{\sigma}''_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2/\sqrt{3} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_1 \\ \bar{\sigma}'_3 \end{pmatrix} \quad (9)$$

Initial yield points and equi-strain points in the plane $(\bar{\sigma}''_1, \bar{\sigma}''_3)$ are shown by the marks $\blacktriangle$ and $\triangle$ in Fig. 11, respectively. The results previously obtained (6) for the subsequent yield surfaces after preloading under combined axial load and torsion are shown in a simpler shape like Fig. 14 in the plane (σ'_I, σ'_II) , where the preloading direction is taken as the axis σ'_I , and (σ'_I, σ'_II) is related to $(\bar{\sigma}'_1, \bar{\sigma}'_3)$ by the following equation

$$\begin{pmatrix} \sigma'_I \\ \sigma'_II \end{pmatrix} = \begin{pmatrix} \cos \bar{\theta}''_\sigma & \sin \bar{\theta}''_\sigma \\ -\sin \bar{\theta}''_\sigma & \cos \bar{\theta}''_\sigma \end{pmatrix} \begin{pmatrix} \bar{\sigma}''_1 \\ \bar{\sigma}''_3 \end{pmatrix} \quad (10)$$

with $\bar{\theta}''_\sigma = \tan^{-1}(\bar{\sigma}''_3/\bar{\sigma}''_1)$. Similarly to the plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$ the plane $(\bar{\sigma}''_1, \bar{\sigma}''_3)$ is regarded as an isotropic plane for the Tresca-type material.

Fig. 12 Subsequent yield surfaces in the stress plane $(\bar{\sigma}'_1, \bar{\sigma}'_2)$

Fig. 13 Stress plane $(\bar{\sigma}'_1, \bar{\sigma}'_{II'})$

3.5 The relation between the subsequent yield surfaces in the two planes $(\bar{\sigma}_1', \bar{\sigma}_2'')$ and $(\bar{\sigma}_1'', \bar{\sigma}_3'')$

The subsequent yield surfaces shown in Fig.12 and Fig.14 are put one upon another in the plane (σ_I, σ_{II}) of Fig.15 by taking σ_I' or σ_I'' as σ_I and σ_{II}' or σ_{II}'' as σ_{II} . The shapes of the subsequent yield surfaces in both $(\bar{\sigma}_1', \bar{\sigma}_2'')$ plane and $(\bar{\sigma}_1'', \bar{\sigma}_3'')$ plane are found mutually congruent, provided that the amount of prestrain is equal to one another. Hence the subsequent yield surfaces after proportional preloading in various loading directions can be uniquely formulated in these stress planes, regardless of the rotation of the principal stress axes.

3.6 Equation of the subsequent yield surface after proportional preloading.

Let us express the experimental results by the following equation of the subsequent yield surface previously reported(6):

$$\left(\frac{\sigma_I - \sigma_*}{Y - \zeta \sigma_*}\right)^2 + \left(\frac{\sigma_{II}}{Y - \sigma_C}\right)^2 = 1 \quad (11-a)$$

$$\zeta = \{(\sigma_B / \sigma_* - 2) / (2Y - \sigma_B)\} (Y - \sigma_I) + 1$$

where Y , σ_B , σ_C and σ_* are following characteristic values defined by the difference between the actual subsequent yield surface (solid line) and the ideal surface of isotropic hardening type (broken line) as shown in Fig.16, as follows: Y is the yield stress value in the preloading direction, σ_B is the drop of the yield stress in the opposite direction to the preloading direction, σ_C is the drop of the yield stress in the perpendicular direction to the preloading direction, and σ_* is the amount of the translation of the center of the subsequent yield surface. They are dependent on the amount of the prestrain $\rho(T)$ and expressed as follows:

Fig. 14 Subsequent yield surfaces in the stress plane $(\bar{\sigma}_1'', \bar{\sigma}_3'')$

$$\left. \begin{aligned} Y &\approx \sigma_F(T) \\ \sigma_B &= k\rho(T)^m \\ \sigma_C &= C_1\sigma_B \\ \sigma_* &= C_2\sigma_B \end{aligned} \right\} \quad (11-b)$$

where k , m , C_1 and C_2 are material constants. The solid lines in Figs.12,14 and 15 show the calculated results of the yield surface from Eq.(11) with the material constants of $k=78.8\text{kg/mm}^2$, $m=0.383$ and $C_1=C_2=0.31$. The solid lines in Figs. 4 to 7 show the calculated yield surfaces obtained by the transformation of the plane (σ_I, σ_{II}) into the plane (σ_x, σ_y) . The calculation shows a fairly good agreement with the experimental results.

4. Conclusions

The subsequent yield surfaces after proportional preloadings of the Tresca-type material are experimentally investigated.

Fig. 15 Subsequent yield surfaces in $(\bar{\sigma}_1', \bar{\sigma}_2'')$ and $(\bar{\sigma}_1'', \bar{\sigma}_3'')$

Fig. 16 Definition of the characteristic values of the actual subsequent yield surface

The shape of the subsequent yield surface of the Tresca-type depends on the direction of the preloading and varies with the rotation of the principal stress axes even in the π plane and in the deviatoric stress space of Ilyushin. To obtain an isotropic stress space, the deviatoric stress space is modified so that the subsequent yield surfaces after various preloadings under combined axial load and internal pressure, and under axial load and torsion can be given uniquely as a function of the amount of prestrain.

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