621-522

The Transient Characteristics of Fluid Pipes including Mechanical Load

(2nd Report, Theoretical Analysis on Effects of Valve Closure)*

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This paper presents a theoretical study of the transient characteristics for a system consisting of hydraulic actuator, fluid pipe and operation valve. In this analysis fluid motion in pipe is treated as a two-dimensional unsteady laminar flow, and non-linear characteristics of the flow through a valve is considered exactly. The solution for pressure variations at arbitrary pipe section and actuator displacement are obtained in terms of convolution integral and their numerical results are presented. The effects of system parameters and operating parameters, especially, valve orifice shape, valve movement (e.g., instantaneous closure, partial closure, gradual closure, exponential gradual closure, etc.) on the transient response of the system are discussed. Two approximate solutions, one obtained by treating a fluid in pipe as a one-dimensional steady laminar flow, and the other obtained by considering only the compressibility of the fluid in pipe are also presented and compared with an exact solution.

1. Introduction

When a shock load is applied to a hydraulic actuator externally, or the moving actuator driving large inertia load is stopped rapidly by sudden valve closure, especially high surge pressures will arise in pipe. These surge pressures become important if the effects of possible disturbances exceed the design criteria of any component parts.

Water-hammer phenomenon in a pipe with one end connected to the pressure vessel due to rapid valve closure has been studied by numerous investigators. However in regard to water-hammer in which a motion of actuator (mechanical factor) takes place as well as a motion of fluid (fluid factor) detailed studies have not been done so much.

This author investigated pressure surge in a pipe and motion of actuator for a system in which a shock load is applied from outside to a hydraulic actuator connecting to the pipe.⁽¹⁾ Although the effects of mechanical factor on oil-hammer phenomena clarified to some extent by this study, there is not enough of necessary data for a valve-controlled hydraulic driving system. For such systems, effects of port shape, closure time and closure process of operation valve on transient phenomenon should be clarified besides. In the case of gradual valve closure,

analytical treatment becomes too complicated on account of non-linearity of flow characteristics through valve. Several analytical solutions for these problems have been published, but they are not wholly satisfactory. The analyses made by McCloy⁽²⁾ and Montgomery⁽³⁾ are good only for a relatively short pipe because the inertia effect of fluid in pipe is neglected. Moreover their analyses concern only the frequency response. An improved analysis, considering the effect of connecting pipe, was made by Anno⁽⁴⁾ and Akiyama⁽⁵⁾. However, their analysis, in which a pipe is divided into a finite number of smaller sections and each section is treated as a lumped constant system to use an analogue computer, is not wholly sufficient for the effect of connecting pipe.

The primary purpose of this paper is to present engineering basic data for transient problem as an aid to the designer of a valve-controlled hydraulic driving system.

2. Nomenclature

- $a(t)$: orifice area
- A : cross-sectional area of pipe
- A_1, A_2 : effective cross-sectional areas of cylinder of inlet and outlet sides
- B : equivalent bulk modulus of liquid
- c : speed of sound in pipe
- $C_d(t)$: discharge coefficient of orifice
- c_{fm}, c_{fs} : coefficients of friction drag (proportional to pressure difference) of motor and cylinder
- c_{vm}, c_{vs} : coefficients of viscous drag of motor and cylinder
- D : swept volume per revolution of motor

* Received 7th January, 1974.

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- F_c : Coulomb friction force
 F_f : initial friction drag ($=c_{f1}(p_{12,0}-p_{21,0})$)
 J_c : equivalent moment of inertia of motor
 K_{1m} K_{1c} : coefficients of leakage flow of motor and cylinder
 l_{p2} : length of outlet pipe
 m_c : equivalent mass of cylinder
 p : pressure deviation in pipe
 p_0 : initial (before valve closure) pressure in pipe
 $P(s)$: Laplace transform of p
 $\Delta p_{v,0}$: initial pressure drop through valve
 r_0 : pipe inner radius
 R_f : ($=8\nu/r_0^2$)
 s : Laplace variable
 t : time
 t_c : valve closure time
 T : wave travel time ($=2l_{p2}/c$)
 T_c : constant friction torque
 T_f : initial friction torque ($=c_{f1}(p_{12,0}-p_{21,0})$)
 u : velocity of fluid in pipe, deviation
 u_0 : initial velocity of fluid in pipe
 $U(s)$: Laplace transform of u
 V_c : cylinder volume of outlet side
 x : distance along pipe from actuator end
 y : velocity of cylinder, deviation
 y_0 : initial velocity of cylinder
 y^* : displacement of cylinder from the time $t=0$

Greek symbols

- $\dot{\theta}$: rotational velocity of motor, deviation
 $\dot{\theta}_0$: initial rotational velocity of motor
 θ^* : rotational displacement of motor from the time $t=0$
 ν : fluid kinematic viscosity
 ρ : fluid density
 $\phi(t, x)$: wave function (progressive wave)
 $\varphi(t, x)$: wave function (regressive wave)

Dimensionless quantities

([] refers to cylinder)

- $\bar{a}_c(t)$: $C_d(t)a(t)/C_d(0)a(0)$
 $\bar{c}_f$: $2\pi c_{f1}/D$ c_{f1}/A_{12}
 $\bar{c}_v$: $c_{v1}T/J_c$ [$c_{v1}T/m_c$]
 $\bar{K}_1$: $\rho c K_{1m}/A_p$ [$\rho c K_{1c}/A_p$]
 $\bar{R}_f$: $R_f T$
 $\bar{s}$: Ts
 $\bar{x}$: x/l_{p2}
 α : $(\rho l_{p2}/A_p J_c)(D/2\pi)^2[(\rho A_p l_{p2}/m_c)(A_{12}/A_p)^2]$
 β : 0 [$V_c/A_p l_{p2}$]
 ζ_c : $(1/\rho c u_0)(2\pi/D)T_c$ [$(1/\rho c u_0)(1/A_{12})F_c$]
 ζ_f : $(1/\rho c u_0)(2\pi/D)T_f$ [$(1/\rho c u_0)(1/A_{12})F_f$]
 τ : t/T
 τ_c : t_c/T
 $\bar{\sigma}_a$: real part of root of Eq.(14)
 $\bar{\omega}_a$: imaginary part of root of Eq.(14)

Subscripts (additional)

- 1 : upstream section of outlet pipe
 2 : downstream section of outlet pipe

3. The basic differential equation

The hydraulic system to be analyzed is composed of accumulator ①, hydraulic actuator ②, inertia load ③, fluid pipe ④ and operation valve ⑤ as shown in Fig. 1. In this paper, pressure at inlet of actuator is assumed to be maintained at constant level even after valve closure. The equation governing a transient flow in pipe can be expressed by the following two types of equations. First, using wall-shear stress in an unsteady laminar flow as fluid resistance, transforms of the pressure and velocity deviation at a section a distance x from the upstream end are given by the following equations⁽⁶⁾

$$P_2(s, x) = -U_{21}(s)j\rho c \zeta(s) \sin\left\{\frac{\zeta(s)s}{jc}x\right\} + P_{21}(s) \cos\left\{\frac{\zeta(s)s}{jc}x\right\} \quad (1)$$

$$U_2(s, x) = U_{21}(s) \cos\left\{\frac{\zeta(s)s}{jc}x\right\} - P_{21}(s) \frac{j}{\rho c \zeta(s)} \sin\left\{\frac{\zeta(s)s}{jc}x\right\} \quad (2)$$

$$\zeta(s) = [1 - 2J_1(jr_0(s/\nu)^{1/2})/jr_0(s/\nu)^{1/2} \times J_0(jr_0(s/\nu)^{1/2})]^{-1/2} \quad (3)$$

Next, using wall-shear stress in a steady laminar flow approximately, the pressure and velocity deviation can be expressed in time domain with wave functions $\phi(t, x)$ and $\varphi(t, x)$ as follows⁽⁷⁾

$$p_2(t, x) = e^{-(R_f/2c)x} \phi\left(t - \frac{x}{c}\right) + e^{(R_f/2c)x} \varphi\left(t + \frac{x}{c}\right) \quad (4)$$

$$u_2(t, x) = \frac{1}{\rho c} \left\{ e^{-(R_f/2c)x} \phi\left(t - \frac{x}{c}\right) - e^{(R_f/2c)x} \varphi\left(t + \frac{x}{c}\right) \right\} \quad (5)$$

Assuming that effect of stiffness of shaft (or rod) between actuator and inertia load can be neglected, the equation of motion of actuator is given as follows

$$J_c \frac{d\dot{\theta}}{dt} = \frac{D}{2\pi} (p_{12,0} + p_{12} - p_{21,0} - p_{21}) - c_{v1}(\dot{\theta}_0 + \dot{\theta})$$

Fig.1 Physical model

$$\begin{aligned}
& -c_{f2} \frac{\dot{\theta}_0 + \dot{\theta}}{|\dot{\theta}_0 + \dot{\theta}|} |p_{12,0} + p_{12} - p_{21,0} - p_{21}| \\
& - \frac{\dot{\theta}_0 + \dot{\theta}}{|\dot{\theta}_0 + \dot{\theta}|} T_c \dots \dots \dots (6) \\
m_s \frac{dy}{dt} &= A_{s1}(p_{12,0} + p_{12}) - A_{s2}(p_{21,0} + p_{21}) \\
& - c_{v1}(\dot{y}_0 + \dot{y}) - c_{f1} \frac{\dot{y}_0 + \dot{y}}{|\dot{y}_0 + \dot{y}|} \\
& \times |p_{12,0} + p_{12} - p_{21,0} - p_{21}| - \frac{\dot{y}_0 + \dot{y}}{|\dot{y}_0 + \dot{y}|} F_c \dots (6)'
\end{aligned}$$

where Eq.(6) and (6)' are for hydraulic motor and cylinder, respectively.

Putting the time when pressure rises to $p_{21}=p_{12,0}-p_{21,0}$ after valve closure as t_1 , the time at $\dot{\theta}_0 + \dot{\theta}$ (or $\dot{y}_0 + \dot{y}$) = 0 as t_2 , and the time when the pressure falls to $p_{21}=p_{12,0}-p_{21,0}$ again as t_3 , and rewriting Eqs.(6) and (6)' in terms of deviation, the following equations are obtained

$$\begin{aligned}
J_s \frac{d\dot{\theta}}{dt} &= -\varepsilon \frac{D}{2\pi} p_{21} - c_{em} \dot{\theta} + 2T_f u_s(t-t_1) \\
& - 2T_f u_s(t-t_2) + 2T_c u_s(t-t_2) + 2T_f u_s(t-t_3) \\
& \dots \dots \dots (7) \\
m_s \frac{dy}{dt} &= -\varepsilon A_{s2} p_{21} - c_{v1} \dot{y} + 2F_f u_s(t-t_1) \\
& - 2F_f u_s(t-t_2) + 2F_c u_s(t-t_2) + 2F_f u_s(t-t_3) \\
& \dots \dots \dots (7)'
\end{aligned}$$

where $u_s(t)$ is unit step function, $\varepsilon=1-\bar{\varepsilon}_f$ (for $0 < t \leq t_1$, $t_2 < t \leq t_3$), $1+\bar{\varepsilon}_f$ (for $t_1 < t \leq t_2$, $t_3 < t$).

Assuming that leakage flow in actuator is in proportion to pressure difference, flow equation for actuator end of outlet pipe can be written as follows

$$\begin{aligned}
u_{21} &= \frac{D}{2\pi A_p} \dot{\theta} - \frac{K_{l1}}{A_p} p_{21} \dots \dots \dots (8) \\
u_{21} &= \frac{A_{s2}}{A_p} \dot{y} - \frac{K_{l1}}{A_p} p_{21} - \frac{V_s}{A_p B} \frac{dp_{21}}{dt} \dots \dots \dots (8)'
\end{aligned}$$

Assuming that the pressure at outlet of valve is maintained at constant level, flow equation for valve end of outlet pipe can be written as follows

$$u_{22} = -u_0 \left\{ 1 - \bar{\alpha}_s(t) \sqrt{1 + \frac{p_{22}}{\Delta p_{v,0}}} \right\} \dots \dots \dots (9)$$

Equations (7) and (8) are boundary conditions of pipe at $\bar{x}=0$, and Eq.(9) is that at $\bar{x}=1$

Further, the rotational displacement of hydraulic motor, θ^* , and the displacement of cylinder, y^* , from the beginning of valve closure ($t=0$) can be written as follows

$$\begin{aligned}
\theta^* &= (\dot{\theta}_0 + \dot{\theta})t \dots \dots \dots (10) \\
y^* &= (\dot{y}_0 + \dot{y})t \dots \dots \dots (10)'
\end{aligned}$$

Equations (1)~(3) (or Eqs.(4) and (5)) and Eqs.(7)~(10) are a set of basic equations for the problem.

In this paper, for convenience, we express the solution based on Eqs.(1)~(3) for an unsteady pipe flow in terms of harmonic solution, and the solution based on Eqs.(4) and (5) in terms of wave solution.

4. Transient characteristics

4.1 The case of instantaneous valve closure (harmonic solution) First, a relatively simple system subject to instantaneous valve closure is taken up. The boundary condition at valve end of pipe (at $\bar{x}=1$) is given as follows

$$U_{22}(s) = -\frac{u_0}{s} \dots \dots \dots (11)$$

Taking the Laplace transform of Eqs. (7), (8) and (10), substituting Eqs.(1) and (2) in these equations and expressing in terms of non-dimensional parameters, we obtain the solutions of pressure deviation and actuator displacement

$$\frac{P_2(\bar{s}, \bar{x})}{\rho c u_0} = \frac{f_{p1}(\bar{s}, \bar{x})}{\bar{s} g(\bar{s})} + (\zeta_f e^{-\tau_1 \bar{s}} - \zeta_f e^{-\tau_2 \bar{s}} + \zeta_c e^{-\tau_2 \bar{s}} + \zeta_f e^{-\tau_3 \bar{s}}) \frac{f_{p2}(\bar{s}, \bar{x})}{\bar{s} g(\bar{s})} \dots \dots \dots (12)$$

$$\frac{\theta^*(\bar{s})}{\dot{\theta}_0 T} \left(\frac{Y^*(\bar{s})}{\dot{y}_0 T} \right) = \frac{1}{\bar{s}^2} + \frac{f_{a1}(\bar{s})}{\bar{s}^2 g(\bar{s})} + (\zeta_f e^{-\tau_1 \bar{s}} - \zeta_f e^{-\tau_2 \bar{s}} + \zeta_c e^{-\tau_2 \bar{s}} + \zeta_f e^{-\tau_3 \bar{s}}) \frac{f_{a2}(\bar{s})}{\bar{s}^2 g(\bar{s})} \dots \dots \dots (13)$$

where

$$\begin{aligned}
g(\bar{s}) &= \xi(\bar{s}) \left\{ \frac{\beta}{2} \bar{s}^2 + \left(\frac{\beta \bar{c}_v}{2} + K_t \right) \bar{s} + 2\varepsilon \alpha + \bar{c}_v K_t \right\} \cos \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} + j(\bar{s} + \bar{c}_v) \sin \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} \dots \dots \dots (14) \\
f_{p1}(\bar{s}, \bar{x}) &= \xi(\bar{s}) \left[(\bar{s} + \bar{c}_v) \cos \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \bar{x} \right\} + j \xi(\bar{s}) \left\{ \frac{\beta}{2} \bar{s}^2 + \left(\frac{\beta \bar{c}_v}{2} + K_t \right) \bar{s} + 2\varepsilon \alpha + \bar{c}_v K_t \right\} \sin \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \bar{x} \right\} \right] \\
f_{p2}(\bar{s}, \bar{x}) &= 4\alpha \xi(\bar{s}) \left[\sin \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} \sin \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \bar{x} \right\} + \cos \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} \cos \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \bar{x} \right\} \right] \\
f_{a1}(\bar{s}) &= -2\varepsilon \alpha \xi(\bar{s}), \quad f_{a2}(\bar{s}) = 4\alpha \left[\xi(\bar{s}) \left(\frac{\beta}{2} \bar{s} + K_t \right) \cos \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} + j \sin \left\{ \frac{\xi(\bar{s}) \bar{s}}{2j} \right\} \right] \dots \dots \dots (15)
\end{aligned}$$

To find p_2 and θ^* (or y^*) as a function of $\bar{x}$ and τ we must take inverse transform of

Eqs.(12) and (13). One pole of Eqs.(12) and (13) is evidently $\bar{s}=0$ (the first rank pole for Eq.(12) and the second rank pole for Eq.(13)). The other poles are the value of satisfying $g(\bar{s})=0$. These values, being impossible to obtain analytically in general, may be obtained by numerical analysis. An inverse transform of Eqs.(12) and (13) by using these values of pole ($\bar{s}=\bar{\sigma}_n \pm j\bar{\omega}_n$, $\bar{\omega}_n > 0$, $n=1, 2, 3 \dots$) gives the following results:

$$\begin{aligned} \frac{p_2(\tau, \bar{x})}{\rho c u_0} = & 2 \sum_{n=1}^{\infty} \left[G_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) e^{s_n \tau} \cos \left\{ \bar{\omega}_n \tau + \tan^{-1} \left(\frac{D_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})}{C_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})} \right) - \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \right. \\ & + \zeta_f u_s(\tau - \tau_1) G_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) e^{s_n(\tau - \tau_1)} \cos \left\{ \bar{\omega}_n(\tau - \tau_1) + \tan^{-1} \left(\frac{D_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})}{C_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})} \right) - \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \\ & + (\zeta_c - \zeta_f) u_s(\tau - \tau_2) G_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) e^{s_n(\tau - \tau_2)} \cos \left\{ \bar{\omega}_n(\tau - \tau_2) + \tan^{-1} \left(\frac{D_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})}{C_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})} \right) \right. \\ & \left. \left. - \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} + \zeta_f u_s(\tau - \tau_3) G_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) e^{s_n(\tau - \tau_3)} \cos \left\{ \bar{\omega}_n(\tau - \tau_3) \right. \right. \\ & \left. \left. + \tan^{-1} \left(\frac{D_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})}{C_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})} \right) - \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \right] + \frac{1}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} [\bar{c}_v + 4\alpha(\zeta_f u_s(\tau - \tau_1) \\ & + (\zeta_c - \zeta_f) u_s(\tau - \tau_2) + \zeta_f u_s(\tau - \tau_3))] \dots \dots \dots (16) \end{aligned}$$

$$\begin{aligned} \frac{\theta^*(\tau)}{\theta_0 T} \left(\frac{y^*(\tau)}{y_0 T} \right) = & 2 \sum_{n=1}^{\infty} \left[G_{a1}(\bar{\sigma}_n, \bar{\omega}_n) e^{s_n \tau} \cos \left\{ \bar{\omega}_n \tau + \tan^{-1} \left(\frac{D_{a1}(\bar{\sigma}_n, \bar{\omega}_n)}{C_{a1}(\bar{\sigma}_n, \bar{\omega}_n)} \right) - \tan^{-1} \left(\frac{B_a(\bar{\sigma}_n, \bar{\omega}_n)}{A_a(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \right. \\ & + \zeta_f u_s(\tau - \tau_1) G_{a2}(\bar{\sigma}_n, \bar{\omega}_n) e^{s_n(\tau - \tau_1)} \cos \left\{ \bar{\omega}_n(\tau - \tau_1) + \tan^{-1} \left(\frac{D_{a2}(\bar{\sigma}_n, \bar{\omega}_n)}{C_{a2}(\bar{\sigma}_n, \bar{\omega}_n)} \right) - \tan^{-1} \left(\frac{B_a(\bar{\sigma}_n, \bar{\omega}_n)}{A_a(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \\ & + (\zeta_c - \zeta_f) u_s(\tau - \tau_2) G_{a2}(\bar{\sigma}_n, \bar{\omega}_n) e^{s_n(\tau - \tau_2)} \cos \left\{ \bar{\omega}_n(\tau - \tau_2) + \tan^{-1} \left(\frac{D_{a2}(\bar{\sigma}_n, \bar{\omega}_n)}{C_{a2}(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right. \\ & \left. \left. - \tan^{-1} \left(\frac{B_a(\bar{\sigma}_n, \bar{\omega}_n)}{A_a(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} + \zeta_f u_s(\tau - \tau_3) G_{a2}(\bar{\sigma}_n, \bar{\omega}_n) e^{s_n(\tau - \tau_3)} \cos \left\{ \bar{\omega}_n(\tau - \tau_3) + \tan^{-1} \left(\frac{D_{a2}(\bar{\sigma}_n, \bar{\omega}_n)}{C_{a2}(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right. \right. \\ & \left. \left. + \tan^{-1} \left(\frac{B_a(\bar{\sigma}_n, \bar{\omega}_n)}{A_a(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right\} \right] + \frac{\bar{c}_v \bar{K}_l}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} \tau + \frac{\varepsilon\alpha(\bar{c}_v + 2\bar{K}_l)}{(2\varepsilon\alpha + \bar{c}_v \bar{K}_l)^2} + \zeta_f u_s(\tau - \tau_1) \left\{ 2\alpha \frac{1 + 2\bar{K}_l(\tau - \tau_1)}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} \right. \\ & \left. - \frac{2\alpha \bar{K}_l(\bar{c}_v + 2\bar{K}_l)}{(2\varepsilon\alpha + \bar{c}_v \bar{K}_l)^2} \right\} + (\zeta_c - \zeta_f) u_s(\tau - \tau_2) \left\{ 2\alpha \frac{1 + 2\bar{K}_l(\tau - \tau_2)}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} - \frac{2\alpha \bar{K}_l(\bar{c}_v + 2\bar{K}_l)}{(2\varepsilon\alpha + \bar{c}_v \bar{K}_l)^2} \right\} \\ & + \zeta_f u_s(\tau - \tau_3) \left\{ 2\alpha \frac{1 + 2\bar{K}_l(\tau - \tau_3)}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} - \frac{2\alpha \bar{K}_l(\bar{c}_v + 2\bar{K}_l)}{(2\varepsilon\alpha + \bar{c}_v \bar{K}_l)^2} \right\} \dots \dots \dots (17) \end{aligned}$$

where

$$\begin{aligned} A_p(\bar{\sigma}_n, \bar{\omega}_n) &= R_e \{ \bar{s} dg(\bar{s}) / d\bar{s} \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & B_p(\bar{\sigma}_n, \bar{\omega}_n) &= I_m \{ \bar{s} dg(\bar{s}) / d\bar{s} \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ C_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= R_e \{ f_{p1}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & D_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= I_m \{ f_{p1}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ C_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= R_e \{ f_{p2}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & D_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= I_m \{ f_{p2}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ A_a(\bar{\sigma}_n, \bar{\omega}_n) &= R_e \{ \bar{s}^2 dg(\bar{s}) / d\bar{s} \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & B_a(\bar{\sigma}_n, \bar{\omega}_n) &= I_m \{ \bar{s}^2 dg(\bar{s}) / d\bar{s} \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ C_{a1}(\bar{\sigma}_n, \bar{\omega}_n) &= R_e \{ f_{a1}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & D_{a1}(\bar{\sigma}_n, \bar{\omega}_n) &= I_m \{ f_{a1}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ C_{a2}(\bar{\sigma}_n, \bar{\omega}_n) &= R_e \{ f_{a2}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n}, & D_{a2}(\bar{\sigma}_n, \bar{\omega}_n) &= I_m \{ f_{a2}(\bar{s}) \} |_{\bar{s}=\bar{\sigma}_n + j\bar{\omega}_n} \\ G_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= [C_{p1}^2(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) + D_{p1}^2(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) / (A_p^2(\bar{\sigma}_n, \bar{\omega}_n) + B_p^2(\bar{\sigma}_n, \bar{\omega}_n))]^{1/2} \\ G_{p2}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) &= [C_{p2}^2(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) + D_{p2}^2(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) / (A_p^2(\bar{\sigma}_n, \bar{\omega}_n) + B_p^2(\bar{\sigma}_n, \bar{\omega}_n))]^{1/2} \\ G_{a1}(\bar{\sigma}_n, \bar{\omega}_n) &= [C_{a1}^2(\bar{\sigma}_n, \bar{\omega}_n) + D_{a1}^2(\bar{\sigma}_n, \bar{\omega}_n) / (A_a^2(\bar{\sigma}_n, \bar{\omega}_n) + B_a^2(\bar{\sigma}_n, \bar{\omega}_n))]^{1/2} \\ G_{a2}(\bar{\sigma}_n, \bar{\omega}_n) &= [C_{a2}^2(\bar{\sigma}_n, \bar{\omega}_n) + D_{a2}^2(\bar{\sigma}_n, \bar{\omega}_n) / (A_a^2(\bar{\sigma}_n, \bar{\omega}_n) + B_a^2(\bar{\sigma}_n, \bar{\omega}_n))]^{1/2} \end{aligned} \dots \dots \dots (18)$$

Equations(16) and (17) are the solutions for p_2 and θ^* (or y^*) as a function of $\bar{x}$ and τ in case of instantaneous valve closure.

In such a special case that inertia of fluid in pipe can be neglected, pressure deviation by the following equation

$$U_{21}(s) = -\frac{u_0}{s} + \frac{I_{p2}}{B} s P_2(s) \dots \dots \dots (19)$$

Using Eqs.(7), (8), (10), (10)' and (19), the following results are obtained for this case:

$$\begin{aligned} \frac{p_2(\tau)}{\rho c u_0} = & \frac{2\sqrt{2\varepsilon\alpha}}{(1+\beta)\bar{\omega}^* \sqrt{2\varepsilon\alpha + \bar{c}_v \bar{K}_l}} e^{s^* \tau} \sin(\bar{\omega}^* \tau - \bar{\delta}_1) - \frac{8\alpha}{(1+\beta)\bar{\omega}^* \sqrt{2\varepsilon\alpha + \bar{c}_v \bar{K}_l}} \\ & \times [\zeta_f u_s(\tau - \tau_1) e^{s^*(\tau - \tau_1)} \sin(\bar{\omega}^*(\tau - \tau_1) + \bar{\delta}_2) - \zeta_f u_s(\tau - \tau_2) e^{s^*(\tau - \tau_2)} \sin(\bar{\omega}^*(\tau - \tau_2) + \bar{\delta}_2) \\ & + \zeta_c u_s(\tau - \tau_2) e^{s^*(\tau - \tau_2)} \sin(\bar{\omega}^*(\tau - \tau_2) + \bar{\delta}_2) + \zeta_f u_s(\tau - \tau_3) e^{s^*(\tau - \tau_3)} \sin(\bar{\omega}^*(\tau - \tau_3) + \bar{\delta}_2)] \\ & + \frac{1}{2\varepsilon\alpha + \bar{c}_v \bar{K}_l} [\bar{c}_v + 4\alpha(\zeta_f u_s(\tau - \tau_1) - \zeta_f u_s(\tau - \tau_2) + \zeta_c u_s(\tau - \tau_2) + \zeta_f u_s(\tau - \tau_3))] \dots \dots \dots (20) \end{aligned}$$

$$\begin{aligned} \frac{\theta^*(\tau)}{\theta_0 T} \left(\frac{y^*(\tau)}{y_0 T} \right) = & -\frac{2\varepsilon\alpha}{\bar{\omega}^*(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)} e^{i\bar{\omega}^* \tau} \sin(\bar{\omega}^* \tau + 2\delta_2) + \frac{4\alpha\sqrt{\varepsilon\alpha(1+\beta)}}{\bar{\omega}^*(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)} \\ & \times [\zeta_f u_s(\tau - \tau_1) e^{i\bar{\omega}^*(\tau - \tau_1)} \sin(\bar{\omega}^*(\tau - \tau_1) + 2\delta_2 + \delta_3) - \zeta_f u_s(\tau - \tau_2) e^{i\bar{\omega}^*(\tau - \tau_2)} \sin(\bar{\omega}^*(\tau - \tau_2) + 2\delta_2 + \delta_3) \\ & + \zeta_c u_s(\tau - \tau_2) e^{i\bar{\omega}^*(\tau - \tau_2)} \sin(\bar{\omega}^*(\tau - \tau_2) + 2\delta_2 + \delta_3) + \zeta_f u_s(\tau - \tau_3) e^{i\bar{\omega}^*(\tau - \tau_3)} \sin(\bar{\omega}^*(\tau - \tau_3) + 2\delta_2 + \delta_3) \\ & + \frac{\bar{c}_s \bar{K}_1}{2\varepsilon\alpha + \bar{c}_s \bar{K}_1} \tau + \frac{\varepsilon\alpha(\bar{c}_s + 2\bar{K}_1)}{(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)^2} + \zeta_f u_s(\tau - \tau_1) \left\{ 2\alpha \frac{1 + 2\bar{K}_1(\tau - \tau_1)}{2\varepsilon\alpha + \bar{c}_s \bar{K}_1} - \frac{2\alpha \bar{K}_1(\bar{c}_s + 2\bar{K}_1)}{(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)^2} \right\} \\ & + (\zeta_c + \zeta_f) u_s(\tau - \tau_2) \left\{ 2\alpha \frac{1 + 2\bar{K}_1(\tau - \tau_2)}{2\varepsilon\alpha + \bar{c}_s \bar{K}_1} - \frac{2\alpha \bar{K}_1(\bar{c}_s + 2\bar{K}_1)}{(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)^2} \right\} + \zeta_f u_s(\tau - \tau_3) \\ & \times \left\{ 2\alpha \frac{1 + 2\bar{K}_1(\tau - \tau_3)}{2\varepsilon\alpha + \bar{c}_s \bar{K}_1} - \frac{2\alpha \bar{K}_1(\bar{c}_s + 2\bar{K}_1)}{(2\varepsilon\alpha + \bar{c}_s \bar{K}_1)^2} \right\} \dots\dots\dots (21) \end{aligned}$$

where

$$\begin{aligned} \bar{\omega}^* &= \frac{1}{2(1+\beta)} \{16\varepsilon\alpha(1+\beta) + 4\bar{c}_s \bar{K}_1(1+\beta) - \bar{c}_s^2(1+\beta)^2 - 4\bar{K}_1^2\}^{1/2} \\ \bar{\sigma}^* &= -\frac{1}{2} \left(\bar{c}_s + \frac{2\bar{K}_1}{1+\beta} \right) \\ \delta_1 &= \tan^{-1} \left\{ \frac{2\bar{c}_s^*(1+\beta)\bar{\omega}^*}{8\varepsilon\alpha + 2\bar{c}_s \bar{K}_1 - \bar{c}_s^2(1+\beta)} \right\} \\ \delta_2 &= \tan^{-1} \left\{ \frac{2(1+\beta)\bar{\omega}^*}{\bar{c}_s(1+\beta) + 2\bar{K}_1} \right\} \\ \delta_3 &= \tan^{-1} \left\{ \frac{2(1+\beta)\bar{\omega}^*}{2\bar{K}_1 - \bar{c}_s(1+\beta)} \right\} \end{aligned}$$

4.2 The case of gradual valve closure (harmonic solution) As mentioned in foregoing section, in the case of instantaneous valve closure, the valve end of pipe may be regarded as closed end after valve closure. However, in the case of gradual valve closure or partial valve closure, we must take the flow characteristics of valve (nonlinear) into consideration during valve closure.

First, for the sake of simplicity in investigating fundamental characteristics, a hydraulic actuator on which mechanical friction force does not act will be taken up here. From Eqs.(1)~(3), (7)~(8) and (10), putting $\zeta_f=0$ and $\zeta_c=0$, and expressing $P_2(\bar{s}, \bar{x})$ and $\theta^*(\bar{s})$ (or $Y^*(\bar{s})$) in terms of $U_{22}(\bar{s})$, we get

$$\frac{P_2(\bar{s}, \bar{x})}{\rho c u_0} = -\frac{f_p(\bar{s}, \bar{x})}{u_0 g(\bar{s})} U_{22}(\bar{s}) \dots\dots\dots (22)$$

$$\frac{\theta^*(\bar{s})}{\theta_0 T} \left(\frac{Y^*(\bar{s})}{y_0 T} \right) = \frac{1}{\bar{s}^2} - \frac{f_{s1}(\bar{s})}{u_0 \bar{s} g(\bar{s})} U_{22}(\bar{s}) \dots\dots\dots (23)$$

Using Eq.(9) and taking convolution integral of the foregoing equations we get

$$\begin{aligned} p_2(\tau, \bar{x})/\rho c u_0 &= \int_0^\tau [h_p(\zeta, \bar{x}) F(\bar{a}_s(\tau - \zeta), p_{22}(\tau - \zeta))] d\zeta \\ &\dots\dots\dots (24) \end{aligned}$$

$$\begin{aligned} \frac{\theta^*(\tau)}{\theta_0 T} \left(\frac{y^*(\tau)}{y_0 T} \right) &= \tau + \int_0^\tau [h_s(\tau) F(\bar{a}_s(\tau - \zeta), p_{22}(\tau - \zeta))] d\zeta \\ &\dots\dots\dots (25) \end{aligned}$$

where

$$\begin{aligned} \Delta \bar{p}_{s,0} &= \Delta p_{s,0}/\rho c u_0 \\ F(\bar{a}_s(\tau), p_{22}(\tau)) &= 1 - \bar{a}_s(\tau) \sqrt{1 + p_{22}(\tau)/\rho c u_0 \Delta \bar{p}_{s,0}} \end{aligned}$$

and $h_p(\tau, \bar{x})$ and $h_s(\tau)$ denote inverse transforms of $f_p(\bar{s}, \bar{x})/g(\bar{s})$ and $f_{s1}(\bar{s})/\bar{s}g(\bar{s})$, respectively, which are given by the following equations

$$h_p(\tau, \bar{x}) = 2 \sum_{n=1}^{\infty} \left[\frac{f_p(\bar{s}, \bar{x})}{dg(\bar{s})/d\bar{s}} e^{i\bar{s}\tau} \right]_{\bar{s}=\bar{s}_n+j\omega_n} \dots\dots\dots (26)$$

$$h_s(\tau) = 2 \sum_{n=1}^{\infty} \left[\frac{f_{s1}(\bar{s})}{\bar{s}dg(\bar{s})/d\bar{s}} e^{i\bar{s}\tau} \right]_{\bar{s}=\bar{s}_n+j\omega_n} + \frac{f_{s1}(0)}{g(0)} \dots\dots\dots (27)$$

Setting $\bar{x}=1$ in Eq.(24), then the following Volterra type integral equation for $p_{22}(\tau)$ is given

$$\frac{p_{22}(\tau)}{\rho c u_0} = \int_0^\tau [h_p(\zeta, 1) F(\bar{a}_s(\tau - \zeta), p_{22}(\tau - \zeta))] d\zeta \dots\dots\dots (28)$$

The kernel of this equation, $h_p(\zeta, 1)$, is a singular one, which takes infinite value for $\zeta=0$ (for $\zeta=1, 2, 3, \dots$ in case of $\bar{K}_1=0$ besides). But integration of $h_p(\zeta, 1)$ gives a finite value for all values of ζ . Hence, assuming that the value $F(\quad)$ in integrand of Eq.(28) can be substituted for average of values at both ends of domain in taking integration of small time domain $(m-1)\Delta\tau, m\Delta\tau$, and numerical integration of Eq.(28) gives a solution for $p_{22}(\tau)$ value at $\tau=i\Delta\tau$, $p_{22}(i\Delta\tau)$, as follows

$$\begin{aligned} \frac{p_{22}(i\Delta\tau)}{\rho c u_0} &= \sum_{n=1}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau) \right. \right. \\ &\times \sqrt{1 + \frac{p_{22}(i-m\Delta\tau)}{\rho c u_0 \Delta \bar{p}_{s,0}}} - \bar{a}_s(i-m+1\Delta\tau) \\ &\times \sqrt{1 + \frac{p_{22}(i-m+1\Delta\tau)}{\rho c u_0 \Delta \bar{p}_{s,0}}} \left. \right\} (H_p(m\Delta\tau, 1) \\ &- H_p(m-1\Delta\tau, 1)) \left. \right] \dots\dots\dots (29) \end{aligned}$$

where

$$\begin{aligned} H_p(\tau, 1) &\equiv \int_0^\tau h_p(\zeta, 1) d\zeta \\ &= 2 \sum_{n=1}^{\infty} \left[G_p(\bar{\sigma}_n, \bar{\omega}_n, 1) e^{i\bar{\sigma}_n \tau} \cos \left\{ \bar{\omega}_n \tau \right. \right. \\ &+ \tan^{-1} \left(\frac{D_p(\bar{\sigma}_n, \bar{\omega}_n, 1)}{C_p(\bar{\sigma}_n, \bar{\omega}_n, 1)} \right) \\ &- \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \left. \right] + \frac{\bar{c}_s}{2\alpha + \bar{c}_s \bar{K}_1} \dots\dots\dots (30) \end{aligned}$$

Setting $\zeta_f=0$ and $\zeta_c=0$, then Eq.(16) coincides with Eq.(30). That is, $H_p(\tau, 1)$ represents pressure variation at the valve

end in case of instantaneous valve closure.

By considering $H_p(0,1)=0$, $H_p(\Delta\tau,1)=1$ and $p_{22}(0)=0$, from Eq.(30) the following recurrence relation for $p_{22}(i\Delta\tau)$ can be obtained:

$$\frac{p_{22}(i\Delta\tau)}{\rho c u_0} = \frac{1}{2} + \frac{\bar{a}_s^2(i\Delta\tau)}{8\Delta\bar{p}_{s,0}} + W(i-1\Delta\tau) - \frac{1}{2}\bar{a}_s(i\Delta\tau)\sqrt{1 + \frac{1}{2\Delta\bar{p}_{s,0}} + \frac{\bar{a}_s^2(i\Delta\tau)}{16\Delta\bar{p}_{s,0}^2} + \frac{W(i-1\Delta\tau)}{\Delta\bar{p}_{s,0}}} \quad (31)$$

where

$$W(i-1\Delta\tau) = \frac{1}{2} \left\{ 1 - \bar{a}_s(i-1\Delta\tau)\sqrt{1 + \frac{p_{22}(i-1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} + \sum_{m=2}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right. \right. \\ \left. \left. - \bar{a}_s(i-m+1\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m+1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} (H_p(m\Delta\tau,1) - H_p(m-1\Delta\tau,1)) \right] \quad (32)$$

The second term on the right hand side of Eq.(32) is to be zero whenever $i=1$.

With determined $p_{22}(i\Delta\tau)$, it is now relatively easy to find the pressure variation at arbitrary section of pipe, $p_2(i\Delta\tau, \bar{x})$, and actuator displacement, $\theta^*(i\Delta\tau)$ (or $y^*(i\Delta\tau)$), from Eq.(24) and Eq.(25). The results may be expressed as follows:

$$\frac{p_2(i\Delta\tau, \bar{x})}{\rho c u_0} = \sum_{m=1}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} - \bar{a}_s(i-m+1\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m+1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} \right. \\ \left. \times (H_p(m\Delta\tau, \bar{x}) - H_p(m-1\Delta\tau, \bar{x})) \right] \quad (33)$$

$$\frac{\theta^*(i\Delta\tau)}{\theta_0 T} \left(\frac{y^*(i\Delta\tau)}{y_0 T} \right) = \tau + \sum_{m=1}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right. \right. \\ \left. \left. - \bar{a}_s(i-m+1\Delta\tau)\sqrt{1 + \frac{p_{22}(i-m+1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} (H_s(m\Delta\tau) - H_s(m-1\Delta\tau)) \right] \quad (34)$$

where

$$H_p(\tau, \bar{x}) \equiv \int_0^\tau h_p(\zeta, \bar{x}) d\zeta = 2 \sum_{n=1}^\infty \left[G_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x}) e^{*n\tau} \cos\{\bar{\omega}_n \tau\} \right. \\ \left. + \tan^{-1} \left(\frac{D_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})}{C_{p1}(\bar{\sigma}_n, \bar{\omega}_n, \bar{x})} \right) - \tan^{-1} \left(\frac{B_p(\bar{\sigma}_n, \bar{\omega}_n)}{A_p(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right] + \frac{\bar{c}_s}{2\alpha + \bar{c}_s R_l} \quad (35)$$

$$H_s(\tau) \equiv \int_0^\tau h_s(\zeta) d\zeta = 2 \sum_{n=1}^\infty \left[G_{s1}(\bar{\sigma}_n, \bar{\omega}_n) e^{*n\tau} \cos\{\bar{\omega}_n \tau + \tan^{-1} \left(\frac{D_{s1}(\bar{\sigma}_n, \bar{\omega}_n)}{C_{s1}(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right. \right. \\ \left. \left. - \tan^{-1} \left(\frac{B_s(\bar{\sigma}_n, \bar{\omega}_n)}{A_s(\bar{\sigma}_n, \bar{\omega}_n)} \right) \right] - \frac{2\alpha}{2\alpha + \bar{c}_s R_l} \tau + \frac{\alpha(\bar{c}_s + 2R_l)}{(2\alpha + \bar{c}_s R_l)^2} \quad (36)$$

In the same manner, for the case of fluid inertia effect being negligible the following results are obtained:

$$\frac{p_2(i\Delta\tau)}{\rho c u_0} = \frac{1}{2} + \frac{\bar{a}_s^2(i\Delta\tau)}{8\Delta\bar{p}_{s,0}} + W(i-1\Delta\tau) - \frac{1}{2}\bar{a}_s(i\Delta\tau)\sqrt{1 + \frac{1}{2\Delta\bar{p}_{s,0}} + \frac{\bar{a}_s^2(i\Delta\tau)}{16\Delta\bar{p}_{s,0}^2} + \frac{W(i-1\Delta\tau)}{\Delta\bar{p}_{s,0}}} \quad (37)$$

$$\frac{\theta^*(i\Delta\tau)}{\theta_0 T} \left(\frac{y^*(\tau)}{y_0 T} \right) = \tau + \sum_{m=1}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau)\sqrt{1 + \frac{p_2(i-m\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} - \bar{a}_s(i-m+1\Delta\tau) \right. \right. \\ \left. \left. \times \sqrt{1 + \frac{p_2(i-m+1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} (H_{s,1}(m\Delta\tau) - H_{s,1}(m-1\Delta\tau)) \right] \quad (38)$$

where

$$H_{p,1}(\tau) = \frac{2\sqrt{2\alpha}}{(1+\beta)\bar{\omega}^* \sqrt{2\alpha + \bar{c}_s R_l}} e^{*n\tau} \sin(\bar{\omega}^* \tau - \delta_1) + \frac{\bar{c}_s}{2\alpha + \bar{c}_s R_l} \quad (39)$$

$$H_{s,1}(\tau) = -\frac{2\alpha}{\bar{\omega}^*(2\alpha + \bar{c}_s R_l)} e^{*n\tau} \sin(\bar{\omega}^* \tau + 2\delta_2) - \frac{2\alpha}{2\alpha + \bar{c}_s R_l} \tau + \frac{\alpha(\bar{c}_s + 2R_l)}{(2\alpha + \bar{c}_s R_l)^2} \quad (40)$$

$$W(i-1\Delta\tau) = \frac{1}{2} \left\{ 1 - \bar{a}_s(i-1\Delta\tau)\sqrt{1 + \frac{p_2(i-1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} + \sum_{m=2}^i \left[\frac{1}{2} \left\{ 2 - \bar{a}_s(i-m\Delta\tau) \right. \right. \\ \left. \left. \times \sqrt{1 + \frac{p_2(i-m\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} - \bar{a}_s(i-m+1\Delta\tau)\sqrt{1 + \frac{p_2(i-m+1\Delta\tau)}{\rho c u_0 \Delta\bar{p}_{s,0}}} \right\} \right. \\ \left. \times (H_{p,1}(m\Delta\tau) - H_{p,1}(m-1\Delta\tau)) \right] \quad (41)$$

As is obvious from present analysis, the results obtained above are also applicable to the hydraulic system in which mechanical friction is applied on the actuator by replacing $H_p(\tau, \bar{x})$, $H_{p,i}(\tau)$, $H_s(\tau)$ and $H_{s,i}(\tau)$ by Eq.(16), Eq.(20), Eq.(17)- τ , and Eq.(21)- τ , respectively.

4.3 The case of gradual valve closure (wave solution)

The actuator on which mechanical friction does not act is taken up also here. Now let $\phi_n(\tau)$ and $\varphi_n(\tau)$ denote the progressive wave and regressive wave at the actuator end of pipe ($\bar{x}=0$) in the time interval of $n+1/2 \leq \tau < n+3/2$ ($n=0, 1, 2, 3, \dots$). Substituting Eq.(4) and Eq.(5) in Eq.(9) gives

$$\frac{\varphi_n(\tau)}{\rho c u_0} = \frac{\phi_{n-1}(\tau-1)}{\rho c u_0} e^{-R_f/2} - \left\{ 1 + \frac{\bar{a}_s^2(\tau-1/2)}{2\Delta\bar{p}_{s,0}} - \bar{a}_s\left(\tau-\frac{1}{2}\right) \sqrt{1 + \frac{1}{\Delta\bar{p}_{s,0}} + \frac{\bar{a}_s^2(\tau-1/2)}{4\Delta\bar{p}_{s,0}^2} + \frac{2}{\Delta\bar{p}_{s,0}} \frac{\phi_{n-1}(\tau-1)}{\rho c u_0} e^{-R_f/4}} \right\} e^{-R_f/4} \dots (42)$$

Next, substituting Eq.(4) and Eq.(5) in Eq.(7) and Eq.(8) (or Eq.(7) and Eq.(8)) gives

$$\frac{\beta}{2} \frac{d^2\phi_n(\tau)}{d\tau^2} + \left(1 + R_l + \frac{\beta}{2} \bar{c}_s\right) \frac{d\phi_n(\tau)}{d\tau} + (2\alpha + \bar{c}_s + \bar{c}_s R_l) \phi_n(\tau) = - \left\{ \frac{\beta}{2} \frac{d^2\varphi_n(\tau)}{d\tau^2} + \left(-1 + R_l + \frac{\beta}{2} \bar{c}_s\right) \frac{d\varphi_n(\tau)}{d\tau} + (2\alpha - \bar{c}_s + \bar{c}_s R_l) \varphi_n(\tau) \right\} \dots (43)$$

It is assumed that the wave generated at the valve end ($\bar{x}=1$) at time $\tau=0$ is reflected at a dead end under the condition $\beta=0$ (at an open end $\beta \neq 0$) at time $\tau=1/2, 3/2, 5/2, \dots$, and at the valve end at time $\tau=1, 2, 3, \dots$. Then, initial condition decided from wave reflection at pipe end is given as follows

$$\left. \begin{aligned} \{\phi_n(\tau)\}_{\tau=n+1/2} &= \kappa e^{-n(R_f/2)} \{\varphi_0(\tau)\}_{\tau=1/2} + \{\phi_{n-1}(\tau)\}_{\tau=n+1/2} \\ \left\{ \frac{d\phi_n(\tau)}{d\tau} \right\}_{\tau=n+1/2} &= \kappa e^{-n(R_f/2)} \left\{ \frac{d\varphi_0(\tau)}{d\tau} \right\}_{\tau=1/2} + \left\{ \frac{d\phi_{n-1}(\tau)}{d\tau} \right\}_{\tau=n+1/2} \end{aligned} \right\} \dots (44)$$

where $\kappa=1$ for $\beta=0$ and $\kappa=(-1)^{n+1}$ for $\beta \neq 0$.

By solving Eq.(43) under initial condition given by Eq.(44), it is found that

$$\begin{aligned} \phi_n(\tau) = & -\varphi_n(\tau) + \frac{4(\lambda_1 + \lambda_2 - \bar{c}_s)}{\lambda_1 \beta} e^{-(\lambda_1 + \lambda_2)\tau} \int_{n+1/2}^{\tau} e^{(\lambda_1 + \lambda_2)\tau} \varphi_n(\tau) d\tau + \frac{4(\bar{c}_s - \lambda_2)}{\lambda_1 \beta} \\ & \times e^{-\lambda_2\tau} \int_{n+1/2}^{\tau} e^{\lambda_2\tau} \varphi_n(\tau) d\tau + \left\{ 1 + \frac{\lambda_2}{\lambda_1} - \frac{\lambda_2}{\lambda_1} e^{-\lambda_1(\tau - n + 1/2)} \right\} [\kappa e^{-n(R_f/2)} \{\varphi_0(\tau)\}_{\tau=1/2} \\ & + \{\phi_{n-1}(\tau)\}_{\tau=n+1/2}] e^{-\lambda_2(\tau - n + 1/2)} + \frac{1}{\lambda_1} \{1 - e^{-\lambda_1(\tau - n + 1/2)}\} \left[\kappa e^{-n(R_f/2)} \left\{ \frac{d\varphi_0(\tau)}{d\tau} \right\}_{\tau=1/2} \right. \\ & + \left. \left\{ \frac{d\phi_{n-1}(\tau)}{d\tau} \right\}_{\tau=n+1/2} \right] + \left\{ 1 + \frac{\lambda_2}{\lambda_1} - \frac{4}{\lambda_1 \beta} + \left(\frac{4}{\lambda_1 \beta} - \frac{\lambda_2}{\lambda_1} \right) e^{-\lambda_1(\tau - n + 1/2)} \right\} \{\varphi_n(\tau)\}_{\tau=n+1/2} e^{-\lambda_2(\tau - n + 1/2)} \\ & + \frac{1}{\lambda_1} \{1 - e^{-\lambda_1(\tau - n + 1/2)}\} \left\{ \frac{d\varphi_n(\tau)}{d\tau} \right\}_{\tau=n+1/2} e^{-\lambda_2(\tau - n + 1/2)} \dots (45) \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= -\frac{2}{\beta} \sqrt{\left(1 + R_l - \frac{\beta}{2} \bar{c}_s\right)^2 - 4\alpha\beta} \\ \lambda_2 &= \frac{1}{\beta} \left(1 + R_l + \frac{\beta}{2} \bar{c}_s\right) + \frac{1}{\beta} \sqrt{\left(1 + R_l - \frac{\beta}{2} \bar{c}_s\right)^2 - 4\alpha\beta} \end{aligned}$$

For the system of $\beta=0$, from limitation of Eq.(45), $\lim_{\beta \rightarrow 0} \phi_n(\tau)$, it follows that

$$\begin{aligned} \phi_n(\tau) = & \frac{1-R_l}{1+R_l} \varphi_n(\tau) - \frac{4\alpha}{(1+R_l)^2} \exp\left\{-\left(\bar{c}_s + \frac{2\alpha}{1+R_l}\right)\tau\right\} \int_{n+1/2}^{\tau} \exp\left\{\left(\bar{c}_s + \frac{2\alpha}{1+R_l}\right)\tau\right\} \varphi_n(\tau) d\tau \\ & + \exp\left\{-\left(\bar{c}_s + \frac{2\alpha}{1+R_l}\right)(\tau - n + 1/2)\right\} [\kappa e^{-n(R_f/2)} \{\varphi_0(\tau)\}_{\tau=1/2} + \{\phi_{n-1}(\tau)\}_{\tau=n+1/2}] \\ & - \frac{1-R_l}{1+R_l} \exp\left\{-\left(\bar{c}_s + \frac{2\alpha}{1+R_l}\right)(\tau - n + 1/2)\right\} \{\varphi_n(\tau)\}_{\tau=n+1/2} \dots (46) \end{aligned}$$

Equation(45) (or Eq.(46)) and Eq.(42) completely determine wave motion. The method of

calculation is as follows. As $\phi_{-1}(\tau)$ is zero evidently, $\phi_0(\tau)$ during first interval $1/2 \leq \tau < 3/2$ can be obtained from Eq.(42) at first. Substituting this $\phi_0(\tau)$ for $\phi_{-1}(\tau)$ in Eq.(45) or Eq.(46) and making numerical integration, value of $\phi_0(i\Delta\tau)$ at time $\tau=i\Delta\tau$ during first time interval can be obtained. Proceeding further in the same manner, $\phi_1(i\Delta\tau)$, $\phi_2(i\Delta\tau)$, $\phi_3(i\Delta\tau)$... can be obtained successively from Eq.(42) and Eq.(45) (or Eq.(46)). By using these results in Eq.(4) we can find pressure variation at arbitrary time τ and pipe section $\bar{x}$.

Moreover, in case of instantaneous valve closure, Eq.(42) becomes the following linear equation

$$\frac{\phi_n(\tau)}{\rho c u_0} = \frac{\phi_{n-1}(\tau-1)}{\rho c u_0} e^{-R_f/2} - e^{-R_f/4} \dots (47)$$

Then the analytical integration of Eq.(45) and Eq.(46) becomes possible. Hence, in the same manner as mentioned in author's preceding report,⁽¹⁾ wave functions during every time interval can be obtained analytically. But further descriptions are omitted here.

5. Results and discussions

Fig.2 Fig.5 show pressure variation at actuator end ($\bar{x}=0$) calculated from Eq.(16) due to instantaneous valve closure, they indicate the effects of load inertia parameter α , leakage flow parameter K_l , pipe friction parameter $\bar{R}_f$, and mechanical friction parameter of actuator ζ_f , respectively.

As a matter of course, pressure variation and surge period increase with an increase in the value α . Besides, as the value α decreases, shapes of variation deviate from a sinusoidal curve and properties of wave appear clearly. Even if the inertia of load, length and cross-sectional area of pipe, etc. are constant, the value of α decreases in inverse pro-

Fig.2 Pressure variation at $\bar{x}=0$ due to instantaneous valve closure (effect of α)

portion to 2nd power of swept volume D (or effective cross-sectional area of cylinder A_{s2}). Therefore value of D (or A_{s2}) has remarkable influence upon the shape of wave.

As the leakage flow parameter K_l increases, the amount of energy dissipated (the product of pressure and leakage flow) increases, and consequently pressure variation decreases. However, the value of K_l is ordinarily small (value of the order of 0.01), so it can be assumed that there is no effect of leakage flow on pressure variation unless a special leakage pass is provided for shock absorption (Fig.3).

Pressure wave is affected considerably by pipe friction parameter $\bar{R}_f$ in the neighbourhood of each maximum value, which includes higher harmonic components, but not affected very much by other values. It follows that the effective wall-shear resistance to fluid flow in a pipe increases monotonically with the frequency and then the higher harmonic waves attenu-

Fig.3 Pressure variation at $\bar{x}=0$ due to instantaneous valve closure (effect of K_l)

Fig.4 Pressure variation at $\bar{x}=0$ due to instantaneous valve closure (effect of $\bar{R}_f$)

ate rapidly (Fig.4).

Figs.6 and 7 show pressure variations and actuator displacement respectively, for the case that the value of non-dimensional effective orifice area $\bar{a}_e(\tau)$ changes linearly from unity to zero in time τ_c . As non-dimensional valve closure time τ_c increases, pressure wave curve becomes gentle and value of maximum pressure decreases too. But maximum peak pressure does not so much decrease as actuator displacement increases with τ_c .

Fig.8 shows pressure variation calculated from Eq.(4) in the forms of wave function, for a system with parameters equal to those used in Fig.6. Comparing Fig.8 with Fig.6, it is found that the difference between the two appears distinctly in the neighbourhood of each maximum pressure especially in case of short valve closure time. This difference is not due to calculation method itself, but chiefly due to the difference in treating an unsteady pipe flow, that is, either treating it as a two-dimensional model or as a one-dimensional model. This fact can be understood in terms of velocity profile. That is, as well known for laminar flow at high frequency pulsation velocity profile approaches a slug-flow distribution with a thin boundary-layer, then wall-shear resistance increases with the frequency.

As the valve closure time becomes longer and then wave curve becomes gentle, however, the difference between the two becomes smaller.

Fig.9 shows pressure variation in case that the valve of non-dimensional orifice area $\bar{a}_e(\tau)$ changes from unity to zero in the same τ_c by the following movement. (1) linearly, (2) exponentially ($\bar{a}_e(\tau) = (e^{-\tau} - e^{-\tau_c}) / (1 - e^{-\tau_c})$ for $0 \leq \tau \leq \tau_c$), (3) parabolically ($\bar{a}_e(\tau) = 1 - \tau^2/\tau_c^2$ for $0 \leq \tau \leq \tau_c$), (4) sinusoidally ($\bar{a}_e(\tau) = 0.5 + 0.5 \cos(\pi\tau/\tau_c)$ for $0 \leq \tau \leq \tau_c$), respectively.

Fig.5 Pressure variation due to instantaneous valve closure calculated by Eq.(20), (effect of ζ_f)

The pressure variations and actuator displacement due to instantaneous partial valve closure, that is, $\bar{a}_e(\tau)$ change instantaneously from unity to the value $\bar{a}_e(0_+)$, as shown in Fig.10 and Fig.11 respectively. For $\bar{a}_e(0_+)$ above a certain value, energy of load can not wholly be absorbed by valve orifice, then the actuator does not stop. For $\bar{a}_e(0_+)$ less than a certain value, a rapid back motion of actuator is brought about and pressure variation becomes higher. In this instance, for about $\bar{a}_e(0_+) = 0.1$, the actuator stops

Fig.6 Pressure variation at $\bar{z}=0$ due to gradual valve closure (effect of τ_c)

Fig.7 Actuator displacement due to gradual valve closure (effect of τ_c)

Fig.8 Pressure variation at $\bar{z}=0$ due to gradual valve closure calculated by Eq.(42) and (45) (effect of τ_c)

smoothly and pressure variation becomes relatively low. These results are useful as basic data for the research on effects of lap of four-way spool valve in a common valve-controlled hydraulic driving system too.

6. Conclusions

The transient responses due to sudden valve closure were investigated for a system composed of hydraulic cylinder, fluid pipe and operation valve. The effect of the fluid pipe, the actuator (including mechanical load), valve movement, etc. upon the transient responses were clarified theoretically. In carrying out the analysis, the following three equations were used independently to express an unsteady pipe flow. (a) water hammer equation of two-dimensional laminar flow, (b) water hammer equation of one-dimensional laminar flow, and (c) equation obtained by considering only the compressibility of fluid. The transient responses were derived in form of harmonic solution for cases (a) and (c), and in form of wave solution for case (b). These three calculated results were compared and discussed.

The appropriateness of theoretical solutions obtained in the present analysis and applicable limitation of approximate solutions (based on (b) and (c)) will be discussed in detail in the following report.

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Fig.9 Pressure variation at $\bar{x}=1$ due to gradual valve closure (effect of valve closure process)

Fig.10 Pressure variation at $\bar{x}=1$ due to partial valve closure (effect of $\bar{a}_e(0_+)$)

Fig.11 Actuator displacement due to partial valve closure (effect of $\bar{a}_e(0_+)$)