Analysis of Spherical Symmetric Problems
in Thermoelastically Coupled Field*

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The present paper deals with quasi-static coupled thermoelastic problems for an infinite medium with a spherical cavity and for a solid sphere, and the effect of thermoelastic coupling on variations of temperature and thermal stresses is examined in detail.

From results obtained by numerical calculation the difference between the uncoupled and coupled solutions for a solid sphere is recognized to be larger than that for an infinite medium with a spherical cavity. Particularly, with regard to a solid sphere the following fact is obtained. Even for usual industrial materials, e.g. aluminum alloy, the difference between the values of the uncoupled and coupled stresses amounts to about 10 per cent in some portions of the sphere at a certain time. Maximum values of the coupled thermal stresses become larger than those of the uncoupled ones and the former increases with a larger coupling coefficient and smaller Poisson's ratio. Moreover, under consideration of the thermoelastic coupling effect, it can be explained that an adiabatic change of volume and a variation of temperature occur even in the region where heat flow from the boundary has not reached.

1. Introduction

In a field of thermoelasticity, coupled problems have attracted interest recently, owing to a variety of industrial materials and the importance of high rate deformation5. In Fourier's heat-conduction equation the temperature field is assumed to be independent of the displacement field, but a heat energy supplied to an elastic body should give rise to variations of the displacement field in it as well as changes of the temperature field. A modified Fourier's heat-conduction equation, in which the above mentioned coupling of the temperature and strain fields is taken into account, was originally established by Duhamel and Neumann. Further contributions, concerned primarily with the structure of the coupled field equations, are attributed to M.A. Biot12, M. Lessed13, B.A. Boley and J. Weiner14, and Y. Takeuti and N. Noda15. Considering the thermoelastic coupling effect, the equations of motion and the modified Fourier's heat-conduction equation should be solved as simultaneous equations such that the mechanical and temperature boundary conditions are simultaneously satisfied. Since the effect of thermoelastic coupling is related to a high rate deformation, numerous investigations of the coupled problem are usually treated as a dynamic one, however, most of them are concerned with problems of infinite or semi-infinite bodies on the thermal shock or on the wave propagation. The dynamic treatment of finite body problems will receive an increased practical interest, due to the combinations between interferences of reflected wave and a thermoelastic damping, but the analysis becomes extremely difficult. With respect to the analysis of thermoelastic coupled problems on a finite medium, we can find only a few references, for example, a study of the quasi-static thermal stresses in a solid cylinder by T. Koizumi and I. Nakahara16 and a report on the dynamic short-time behaviors at the center and on the surface of a solid sphere by C. Jen-yi17.

Even in a quasi-static problem, for materials of a large value of the coupling coefficient we should take account of the thermoelastic coupling effect. From this point of view, in the present paper we have examined by the quasi-static treatment how a great influence of thermoelastic coupling on the temperature, displacement, and stresses fields of a finite medium appears. Thus, the quasi-static coupled thermoelastic problems of spherical symmetry, namely, a solid sphere as well as an infinite medium with a spherical cavity, are solved under the assumption of temperature independent properties of a material. By means of the Laplace transform exact solutions are obtained. Numerical calculations are compared with those in the absence of the thermoelastic coupling, and the effect is discussed in detail.

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2. Fundamental equations

We assume that the elastic medium under consideration is undisturbed and at reference temperature uniformly. Let \((r, \theta, \phi)\) respectively denote the radial coordinate, the co-latitude, and the longitude of a spherical coordinate system. Since we are only dealing with problems characterized by spherical symmetry about the origin, we may assume that all quantities depend on the radial coordinate \(r\) and the time \(t\) only.

\[
T=T(r,t), \quad u_r=u(r,t), \quad w_\theta=w_\phi=0 \quad \cdots \cdots (1)
\]

In which \(T\) is the temperature and \((u_r, u_\theta, u_\phi)\) are the spherical components of the displacement vector. In view of Eqs. (1), the spherical components of normal stress denoted by \(\sigma_r, \sigma_\theta, \sigma_\phi\) are independent of \(\theta\) and \(\phi\), while the corresponding components of shearing stress vanish identically. Therefore, the stress-displacement relations become

\[
\sigma_r = \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + 2 \frac{u_r}{r} \right) - \frac{\lambda + 2\mu}{2} \frac{\partial T}{\partial r},
\]

\[
\sigma_\theta = \sigma_\phi = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} + 2 \frac{u_\theta}{r} \right) + 2\mu \left( \frac{\partial u_r}{\partial r} - \frac{\lambda + 2\mu}{2} \frac{\partial T}{\partial r} \right) \quad \cdots \cdots (2)
\]

where \(\lambda\) and \(\mu\) are Lamé's constants, and \(\alpha_t\) is the coefficient of linear thermal expansion. On the other hand, the coupled thermoelastic equations for this case, in the absence of inertia term, reduce to

\[
\frac{\partial T}{\partial r} + \frac{2}{r} \frac{\partial T}{\partial r} = \frac{(\lambda + 2\mu)\alpha_t T}{\alpha_t \rho \kappa} \quad \cdots \cdots (3)
\]

\[
\frac{\partial u_r}{\partial r} + 2 \frac{u_r}{r} = \frac{(\lambda + 2\mu)\alpha_t T}{\alpha_t \rho \kappa} \quad \cdots \cdots (4)
\]

where \(T^*\) is the referential absolute temperature at which the stress and strain components are zero, \(\kappa\) is the thermal diffusivity, \(c_s\) is the specific heat at constant volume, and \(\rho\) is the density of the material. If the coupling term in Eq. (3) is neglected, the governing equations become uncoupled ones, which can be easily solved separately, first for the temperature field and then for the displacement and stress fields. However, the coupled equations should be solved as simultaneous equations such that the mechanical and thermal boundary conditions are simultaneously satisfied. At this stage, it is expedient to introduce a non-dimensional radial coordinate, time, temperature, and displacement defined by the following:

\[
R = \frac{r}{a}, \quad t = \frac{t}{t_0}, \quad \Theta = \frac{T}{T_0}, \quad \Phi = \frac{u_r}{a} \quad \cdots \cdots (5)
\]

where \(T_0\) is the constant temperature, and \(a\) is the reference radius. By means of Eqs. (5), the components of the stresses in Eqs. (2) can be written as

\[
(1-2\nu)\sigma_r - (1-\nu)\frac{\partial U}{\partial R} + 2\nu U = \mu \frac{\partial U}{\partial R} \quad \cdots \cdots (6)
\]

\[
\sigma_\theta = \mu \frac{\partial U}{\partial R} + (1-\nu) \frac{\partial U}{\partial R} \quad \cdots \cdots (7)
\]

and

\[
(\nu, \mu, \sigma_r, \sigma_\theta) = \frac{1}{\mu} (\sigma_r, \sigma_\theta, \sigma_\phi) \quad \cdots \cdots (8)
\]

where \(\nu\) is Poisson's ratio and \(E\) is Young's modulus. Similarly, introducing the non-dimensional quantities into Eqs. (3) and (4), and eliminating \(U(R, \tau)\) or \(\Theta(R, \tau)\) from those equations, we obtain the following equations:

\[
\frac{\partial^2 \Theta}{\partial R^2} + 2 \frac{\partial^2 \Theta}{\partial R \partial \tau} = (1+\nu) \frac{\partial \Theta}{\partial \tau} \quad \cdots \cdots (9)
\]

in which \(\delta\) is the coefficient of thermoelastic coupling, expressed as

\[
\delta = \frac{(1+\nu)\kappa}{\alpha_t \rho \kappa} \quad \cdots \cdots (10)
\]

3. Coupled thermal stresses

3.1 General solution in the Laplace transform domain

In this paper the Laplace transform is used for solving problems. Assuming that a medium is initially free from thermal and mechanical disturbances, initial conditions in the non-dimensional form are given as follows:

\[
\Theta(t=0) = 0, \quad \frac{\partial \Theta}{\partial t}(t=0) = 0 \quad \cdots \cdots (11)
\]

Application of the Laplace transform to Eqs. (9) and (10) with the initial conditions Eqs. (11) yields the following Bessel's equations.

\[
\left( \frac{d^2 \Theta}{dR^2} + \frac{1}{R} \frac{d \Theta}{dR} - (\mu \sqrt{\tau}) \frac{\Theta}{R^2} \right) = 0 \quad \cdots \cdots (12)
\]

\[
\left( \frac{d^2 U}{dR^2} + \frac{1}{R} \frac{d U}{dR} - (\mu \sqrt{\tau}) \frac{U}{R^2} \right) = 0 \quad \cdots \cdots (13)
\]

where \(\mu^2 = 1+\delta\). \(\Theta\) and \(U\) are respectively the Laplace transform of \(\Theta\) and \(U\), defined by

\[
\Theta(R, \rho) = \int_0^\infty \Theta(R, \tau) e^{-\rho \tau} d\tau \quad \cdots \cdots (14)
\]

The general solutions of Eqs. (12) and (13) are given by

\[
\Theta(R, \rho) = \frac{C_1}{R^{1+\delta}} \mu \sqrt{\tau} \Gamma(1, \mu \sqrt{\tau} R) - C_3 \delta \quad \cdots \cdots (15)
\]
\[ U(R, p) = \frac{C_1}{R^{1/4}} I_4(\mu \sqrt{\frac{\mu}{\sigma}} R) + \frac{C_4}{R^{1/4}} K_4(\mu \sqrt{\frac{\mu}{\sigma}} R) + 2C_3 + \frac{C_2}{R^4} \]  

\[ \sigma_0^* (R, p) = -\frac{2C_1}{R^{1/4}} I_4(\mu \sqrt{\frac{\mu}{\sigma}} R) - \frac{2C_1}{R^{1/4}} K_4(\mu \sqrt{\frac{\mu}{\sigma}} R) + 2C_3 - \frac{C_2}{R^4} \]  

\[ \sigma_0^* (R, p) = \frac{C_1}{R^{1/4}} I_4(\mu \sqrt{\frac{\mu}{\sigma}} R) - \mu \sqrt{\frac{\mu}{\sigma}} I_4(\mu \sqrt{\frac{\mu}{\sigma}} R) + \frac{C_1}{R^{1/4}} \]  

Here \( \delta_0^* \) and \( \sigma_0^* \) are the Laplace transforms of \( \sigma_0^* \) and \( \sigma_0^* \), respectively. The integral constants in Eqs. (15)~(18) are decided by boundary conditions. In the following sections, let us consider two particular cases, namely, the case of an infinite space with a spherical cavity, and a solid sphere.

3.2 Infinite medium with a spherical cavity
Denote by a the radius of a spherical cavity, which is initially at uniform temperature \( T_a \) and by \( h \) the heat transfer coefficient. Assuming that there is a heat exchange between the spherical cavity and the surrounding infinite medium

\[ M_1(R, \tau) = \frac{H}{1 + H} \left[ \text{erfc} \left( \frac{R - 1}{2\sqrt{\tau}} \right) - \exp \left( \frac{1 + H}{\mu} \right) \right] \]  

\[ M_2(R, \tau) = \frac{H}{1 + H} \left[ \frac{R + 1}{\mu} + \frac{2}{1 + H} \right] \sqrt{\frac{\tau}{\pi}} \exp \left( \frac{-R^2 - 1}{4\tau} \right) + \left( \frac{\mu}{1 + H} \right)^{1/2} \right] \]  

\[ \left[ \frac{R^2 - 1}{2} + \frac{R}{\mu} + \left( \frac{R - 1}{1 + H} \right) \exp \left( \frac{1 + H}{\mu} \right) \right] \]  

where

\[ \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \]  

In the limiting case, as \( H \to \infty \) in Eqs. (23) and (24), which corresponds to the spherical surface being suddenly heated or cooled and held at the constant temperature, \( M_1 \) and \( M_2 \) reduce to
\[ \lim_{n \to \infty} M_n(R, r) = \text{erfc} \left( \frac{\mu (R - 1)}{2\sqrt{r}} \right) \]
\[ \lim_{n \to \infty} M_n(R, r) = \frac{1}{\mu (R + 1)} \frac{1}{\pi} \exp \left( -\frac{\mu^2 (R - 1)^2}{4r} \right) \left[ 1 - \text{erfc} \left( \frac{\mu (R - 1)}{2\sqrt{r}} \right) \right] \]

Further, neglecting the coupling effect in Eqs. (22) and (24), namely, setting \( \delta = 0 \) and \( \mu = 1 \), they agree exactly with the conventional solutions as obtained by E. Sternberg. Otherwise, as \( r \to \infty \) in Eqs. (23) and (24), the steady-state solutions can be obtained as:

\[ \Theta(R, \infty) = \frac{H}{1 + H} \frac{1}{R^2}, \quad U(R, \infty) = \frac{H}{1 + H} \frac{1}{2} \left( \frac{1}{R^4} \right), \]
\[ \sigma^*(R, \infty) = -\frac{H}{1 + H} \frac{1}{R^4} \]

3.3 Solid sphere

We next proceed to the second particular case, of a solid sphere of radius \( a \), which radiates into the surrounding medium at \( T_0 \) under Newton's cooling law. In this case, the boundary conditions are given as:

\[ \left( \frac{\partial \Theta}{\partial r} \right)_{r=a} = 0, \quad (\sigma^*)_{a} = 0 \]

Taking the Laplace transforms of Eqs. (27) and (28), and substituting Eqs. (15) \( \sim \) (18) into those conditions, the integral constants are obtained as follows:

\[ C_1 = C_2 = 0 \]
\[ C_3 = \frac{H}{\mu} \left( \pi \mu v \right)^{1/2} \left( (H-1)(\mu v \nu \mu v \nu) + ((\mu v \nu \mu v \nu) H) \mu v \nu \cos (\mu v \nu) \right) \]
\[ C_4 = \frac{1}{\mu} \frac{H}{\nu} \left( (H-1)(\mu v \nu \mu v \nu) + ((\mu v \nu \mu v \nu) H) \mu v \nu \cos (\mu v \nu) \right) \]

where

\[ \epsilon = \frac{3a}{\mu} \frac{6a(1-2\nu)}{1+\nu+3a(1-\nu)} \]

Consequently, the expressions for the transformed temperature, displacement, and stress components are now obtained as follows:

\[ \Theta(R, p) = \frac{1}{D(p)} \left( \frac{\mu^2}{R} \sinh (\mu v \nu R) + \mu \frac{N(\mu v \nu)}{R} \right) \]
\[ U(R, p) = \frac{1}{D(p)} \left( \frac{1}{R} \right) \left( \frac{N(\mu v \nu R)}{N(\mu v \nu)} \right) \]
\[ \sigma^*(R, p) = -\frac{1}{D(p)} \left( \frac{\mu^2}{R} \sinh (\mu v \nu R) - \frac{1}{R^2} \frac{N(\mu v \nu R)}{N(\mu v \nu)} \right) \]

where

\[ D(p) = \left( (1-1)(\mu v \nu \mu v \nu) + \epsilon \sinh (\mu v \nu \mu v \nu) + ((\mu v \nu \mu v \nu) H - 1) \mu v \nu \cos (\mu v \nu) \right) \]
\[ N(\mu v \nu R) = \sinh (\mu v \nu R) - \mu v \nu R \cos (\mu v \nu R), \quad N(\mu v \nu) = \left( \frac{N(\mu v \nu)}{\mu v \nu} \right) \]

At this stage, to perform the inverse Laplace transforms of Eqs. (30) we shall examine the multi-valued properties of the functions. Although Eqs. (30) contain apparently a double-valued function \( \mu \) of the complex variable \( p \), we can find that the double-valued property vanishes under consideration of the entire equation, namely, these equations have only single-valued functions. The inverse Laplace transforms of Eqs. (30) can be done with the contour integral by virtue of Jordan's lemma, and further can be reduced to the evaluation of roots of the function \( D(p) = 0 \); each equation of Eqs. (30) has a single pole at \( p = 0 \) and \( p = \sigma^* (m = 1, 2, 3, \ldots) \) in complex \( p \)-plane. Here \( \sigma^* \) is the \( n \)-th positive root of \( \sigma^* \) of the complex variable \( \mu \).

Analytical expressions are not available for \( \sigma^* \), and one must resort to numerical techniques for their determination. For the uncoupled case and the coupled case \( \nu = 1/3 \), \( \delta = 0.03 \), the first ten roots of Eqs. (31) for various values of the Biot number \( H \) are shown in Tables 1 and 2, respectively. Bearing in mind the above argument and performing the inverse Laplace transforms of Eqs. (30), we finally obtain the temperature, displacement, and stress components, expressed by the nondimensional quantities, as follows:

\[ \Theta(R, \tau) = 1 + 2 \sum_{n=1}^{\infty} \frac{e^{-\mu v \nu \tau}}{D(\mu v \nu)} \left( \frac{1}{R} \sin (\mu v \nu R) - R \frac{N(\mu v \nu)}{\mu v \nu} \right) \]

\[ \sigma^*(R, \tau) = -\frac{1}{D(\mu v \nu)} \left( \frac{\mu^2}{R} \sinh (\mu v \nu R) - \frac{1}{R^2} \frac{N(\mu v \nu R)}{N(\mu v \nu)} \right) \]
\[ U(R, \tau) = \frac{\eta_0}{\tau} + \frac{1}{a_0 + 1} \sum_{n=1}^{\infty} \frac{a_n}{D(\alpha_n)} \left( \frac{N(\alpha_n R)}{\alpha_n} \right)^{1/3} \]  

(32)

\[ \sigma_n(0, \tau) = \sigma_n(0, \tau) = \frac{1}{a_0 + 1} \sum_{n=1}^{\infty} \frac{a_n}{D(\alpha_n)} \left( \frac{N(\alpha_n R)}{\alpha_n} \right)^{1/3} \]  

(33)

The displacement at the center of the sphere \( R = 0 \) in Eqs. (32) satisfies the boundary condition of Eq. (28) and at this point the radial stress coincides with the tangential one.

By substituting \( \mu = 1 \) and \( \phi = 0 \) in the solutions (32), these solutions become the uncoupled differential equations. Moreover, as \( H = \infty \), we obtain the solutions for the case of the spherical surface being suddenly subjected to a constant temperature \( T_0 \), which are equivalent to the results given by G. Grünberg and H. Parkus on.

### Table 1: Roots of \( \mu \alpha_n(\alpha_n^2 + H_0) \cot(\alpha_n) = (1 - H_0)(\alpha_n^2 + H_0) \) (Uncoupled case \( \delta = 0 \))

<table>
<thead>
<tr>
<th>( H )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_n )</td>
<td>0.542 28</td>
<td>1.185 58</td>
<td>1.570 80</td>
<td>2.028 76</td>
<td>2.570 43</td>
<td>2.836 30</td>
<td>3.078 84</td>
</tr>
<tr>
<td>( \mu \alpha_n(\alpha_n^2 + H_0) \cot(\alpha_n) = (1 - H_0)(\alpha_n^2 + H_0) )</td>
<td>0.672 18</td>
<td>1.240 77</td>
<td>1.630 89</td>
<td>2.013 25</td>
<td>2.445 92</td>
<td>2.806 60</td>
<td>3.078 84</td>
</tr>
</tbody>
</table>

### Table 2: Roots of \( \mu \alpha_n(\alpha_n^2 + H_0) \cot(\alpha_n) = (1 - H_0)(\alpha_n^2 + H_0) \) (Coupled case \( \delta = 0.03, \nu = 1/3 \))

<table>
<thead>
<tr>
<th>( H )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_n )</td>
<td>0.538 98</td>
<td>1.157 20</td>
<td>1.559 64</td>
<td>2.014 82</td>
<td>2.554 40</td>
<td>2.820 71</td>
<td>3.064 50</td>
</tr>
<tr>
<td>( \mu \alpha_n(\alpha_n^2 + H_0) \cot(\alpha_n) = (1 - H_0)(\alpha_n^2 + H_0) )</td>
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</tr>
</tbody>
</table>

### 4. Numerical Calculation

Next we shall illustrate numerical results of solutions obtained in the preceding chapter and discuss the effect of thermoelastic coupling on thermal stresses and temperature distribution.

Figures 1 (a) ~ (d) show the nondimensional temperature, displacement, and stress components distributions, respectively, in an infinite medium with a spherical cavity for the limit case of \( H = \infty \), as a function of the nondimensional radius \( R \) for various values of the nondimensional time \( \tau \).

In these figures the solid lines denote the uncoupled case and the dashed lines denote the coupled case of the coupling coefficient \( \delta = 0.1 \), which is a somewhat large value as compared with that for most of common materials. The coupled and uncoupled solutions gradually approach each other as time progresses and finally they agree exactly with thier corresponding steady-state solution which depend merely on the radial position. It can be seen from the figures that the thermoelastic coupling effect acts on displacement and thermal stresses as well as on temperature as a damping effect. However, for the coupling coefficient of common industrial materials, such as aluminum alloy \( \delta = 0.03 \), steel \( \delta = 0.01 \) at reference temperature \( T_0 = 293^\circ \) K, the difference between the coupled and uncoupled solutions is not appreciable and under 1.0 per cent.

Figure 2 shows the time-dependence of the components of thermal stress on the
Fig. 1 Infinite medium with a spherical cavity (H = ∞)

Fig. 2 Time dependence of stress at the center of a sphere and tangential stress on the spherical surface.

Fig. 3 Influence of Poisson's ratio and the coefficient of thermoelastic coupling on maximum stresses at the center and on the surface of a sphere.
surface and at the center of a solid sphere for various values of the Biot number \( H \), the nondimensional time \( \tau \) ranging in the interval \( \tau = 10^{-4} \sim 10^0 \). As represented by Eq. (33), the radial and tangential stresses at \( R = 0 \) have the same value. In Fig. 2 the solid lines denote uncoupled solutions while the dashed lines denote coupled ones for the case of \( \nu = 1/3 \) and \( \delta = 0.1 \). It is clear from these results that for the same Biot number the coupled thermal stresses will be larger than the uncoupled ones before an appearance of the maximum stress in the absolute value; and in contrast to this, the former will be smaller than the latter after it. The reason is that as explained by Koizumi and Nakaahara, before the maximum value is reached a part of heat is consumed for the work done by thermal stresses and then the increasing rate of the stresses is suppressed but after the appearance of the maximum the work done by thermal stresses generate heat and then the decreasing rate of stresses slows down. We now notice that the radial stress becomes maximum at the center of a sphere while the tangential stress does on the spherical surface.

Figure 3 illustrates the fact that maximum values of the radial and tangential stresses in the coupled case increase with an increasing value of the coefficient of thermoelastic coupling. In Fig. 3 the maximum values of the radial stress at the center of a sphere and of the tangential stress on the surface for the case \( H = 10 \) are depicted as a function of the coupling coefficient for various values of Poisson's ratio. In the case of an infinite medium with a spherical cavity Poisson's ratio is included only in the coupling coefficient and related only to the magnitude of \( \delta \).

In contrast to this, in the case of a solid sphere, as obtained also from Eqs. (32) and (33) the values of stress components vary with the value \( \nu \) even though for the same value of \( \delta \), since there exists the term \( \epsilon \) including \( \nu \) besides \( \delta \). On the other hand, in the uncoupled case \( \delta \) and \( \epsilon \) are identically zero regardless of Poisson's ratio, then the unchangeable maximum values of stress components for \( H = 10 \) can be obtained as

\[
|\sigma^*(0, t)_{\text{max}}| = |\sigma^*(0, t)_{\text{max}}| = 0.31971 \\
|\sigma^*(1, t)_{\text{max}}| = 0.44995
\]

After all the values of stress components become large as either \( \delta \) increases under

Fig. 4 Solid sphere ( \( H = \infty \)).

Fig. 5 Time dependence of volume and temperature in the element of a sphere ( \( H = \infty \)).
a constant $\nu$ or $v$ decreases at a fixed $\delta$.

In Figs. 1 and 2, for the purpose of the magnification of the effect of thermoelastic coupling, a comparatively large value $\delta = 0.1$ is expediently used. Numerical calculations for $\delta = 0.03$ corresponding to aluminum alloy are displayed in Figs. 4(a) $\sim$ (c), which indicate the radial, tangential stresses, and temperature distributions in a solid sphere for various values of time. The solid and dashed lines denote respectively the uncoupled case and the coupled case with $\nu = 1/3$ and $\delta = 0.03$. Although these figures show the results for the limiting case of $K = \infty$, similar figures can be observed also in the case of finite Biot number except in the neighborhood of $\sigma^*(1, t)$. Moreover, it was found, from the examination of the time-dependence of stress components corresponding to Fig. 4(a) $\sim$ (c), that the noticeable difference between the coupled and uncoupled results appeared especially in the time region when the stress components began to decrease after its maximum; namely, the values of the coupled stresses at the center of sphere are $1 \sim 10$ per cent larger than those of the uncoupled stresses within $t = 0.06 \sim 0.2$. In the temperature distributions shown in Fig. 4(c) the coupled results for $\delta = 0.01$ and 0.025 have negative values in the region of the center of the sphere. This is due to an adiabatic change of volume.

The time-dependence of change of volume at $R = 0$ and 0.5, along with temperature changes, is illustrated in Fig. 5. In this figure e denotes the change of volume, whose expression is taken in the form

$$
e(R, t) = \frac{2U}{R} = \frac{2U}{R} + \sum \frac{e^{*\mu i} \left\{ \frac{1}{R} \sin(\mu \alpha, R) \right\}}{\frac{3}{\zeta} N(\mu \alpha)}$$

(34)

It can be seen from Fig. 5 that the change of volume should occur even at the position and time regions where a flow of heat through the spherical surface of a sphere has not yet reached. This is caused by the thermal stress in the region of the spherical surface at which the heat flow has already arrived. Hence, if the thermoelastic coupling effect is taken into consideration, it can be explained that in certain regions of position and time the temperature falls below the reference one due to an adiabatic expansion of the volume element of a sphere when $T > 0$, or the temperature rises due to an adiabatic compression of the volume element when $T < 0$.

5. Conclusions

In the present paper we have investigated the thermoelastic problems of the spherical system according to the coupled thermoelastic theory in which the coupling of the temperature and displacement fields is taken into account. The treatment adopted here is quasi-static in the sense that inertia effects are disregarded. Thus, the coupled quasi-static solutions to the thermoelastic problems of an infinite medium with a spherical cavity, and of a solid sphere, have been obtained. As a result of this investigation, the following conclusions have been reached.

(1) The effect of thermoelastic coupling on variations of temperature and thermal stresses acts as a damping effect.

(2) In regard to an infinite medium with a spherical cavity, the difference between the coupled and uncoupled quasi-static results is not appreciable and less than 1 per cent within the region of the coupling coefficient for most of industrial materials. On the other hand, with respect to a solid sphere, even for aluminum alloy the values of the coupled stress in the neighborhood of the its center are about 10 per cent larger than those of the uncoupled stress.

(3) Likewise, in connection with the sphere, for the finite Biot number the maximum of absolute values of component of the coupled stress is larger than that of the uncoupled one; it becomes larger either with an increasing value of the coupling coefficient for the constant Poisson's ratio or with a decreasing Poisson's ratio at the fixed value of the coupling coefficient.

(4) In the interior part of a solid sphere whose surface is subjected to a thermal loading, an adiabatic change of volume in it occurs giving rise to a small variation of the temperature, regardless of a heat flow from the spherical boundary. However, it is impossible to explain the above phenomenon by the uncoupled treatment. Only the coupled treatment allows us to understand it quantitatively.

References


