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Subharmonic Oscillation of Order $1/3$ in Roller Chain*1st Report, Theoretical Analysis of the System Having
Multiple Degrees of Freedom

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The subharmonic oscillations of order $1/3$ in a roller chain regarded as a system with multiple degrees of freedom are analysed under the boundary conditions that its one end is fixed and the other end is subject to lateral forced displacement. Such oscillations are excited by the fluctuations of tension, namely of the extension of the chain during vibration. The solutions for these subharmonic oscillations are obtained by applying the harmonic balance method. It is clear that only two kinds of these solutions can be realized with the limited regions of stability. Further, the modes of these vibrations and the fluctuation patterns of tension are determined and characters of their stable steady vibrations are discussed.

1. Introduction

There have been a number of investigations relating to the transverse vibration of roller chains. In these analyses that regard the chain as a system having multiple degrees of freedom, only the linear natural frequencies are presented⁽¹⁾⁽²⁾⁽³⁾. On the other hand, there have been few investigations that discuss the nonlinear forced vibrations of roller chains.

The assumption that the chain is a string with infinite degrees of freedom, has been often adopted in the analysis of the vibrating chain. Concerning lateral vibrations of stretched strings, Oplinger⁽⁴⁾, Murthy⁽⁵⁾, Miles⁽⁶⁾, and Anand⁽⁷⁾ have discussed the nonlinear forced oscillations taking into account the fluctuation of tension during vibration, neglecting the effect of gravity, and their studies have been limited to the harmonic vibrations with the same frequency as the forced one. In the preceding report⁽⁸⁾, the authors investigated the nonlinear forced harmonic vibration that considered the effects of the weight of the chain and the fluctuation of tension during vibration, regarding the chain as a system with multiple degrees of freedom, and they compared these results with the experimental results.

Later, from the experiments under the boundary condition that one end is fixed and the other end is subject to the lateral forced displacement, they have found the subharmonic oscillations of order $1/3$ with symmetric modes. (In this report, the symmetric modes only are

treated. Hence, these are simply called the subharmonic oscillations of order $1/3$ here.) The characteristics of such vibrations are as follows in brief.

- (1) The harmonic vibration occurs in the vicinity of the fundamental linear natural frequency, but the subharmonic oscillation takes place at a frequency more than three times the fundamental natural frequency.
- (2) There are only two kinds of steady oscillations not only with different amplitudes but also explicitly with distinguishable vibration modes at certain forced frequencies. They can be realized with the limited regions of stability.

As a roller chain possesses a high modulus of elasticity for elongation and its mean line density is comparatively large, such subharmonic oscillations may be excited by the nonlinear character due to the fluctuation of tension during vibration. In order to explain the above mentioned phenomena, in this report, the authors consider a vertically stretched roller chain containing link plates of an arbitrary shape in operation with a rather small tension, and analyze theoretically what effect the temporal fluctuation of tension during vibration and the spatial variation of tension due to gravity can have upon its transverse forced vibration, above all, upon the subharmonic oscillations of order $1/3$. And they obtain the first approximate solutions of the subharmonic oscillations of order $1/3$ among the periodic solutions of the nonlinear simultaneous ordinary differential equations by applying the method of harmonic balance, then determine the steady solution to be realized by the stability conditions.

The lateral excitations of the roller chain are divided into two kinds, namely the one caused by force and the other caused by displacement. The excitations to which the chain is subject are either those caused by the eccentricity of the sprocket

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shaft or those caused by the impacts of the encounter and detachment of the chain and the sprocket, both of which are considered nearly as excitations by forced displacement at the ends. Hence, the authors have set the boundary condition that the roller chain is subject to forced displacement at one end but is fixed at the other end.

As a result, the theoretical analysis has shown that the subharmonic oscillation of order 1/3, just as the harmonic vibration, is governed by the parameter λ containing the initial tension, the amplitude of the forced displacement and the modulus of elasticity of the chain, and by the parameter p representing the effect of gravity.

In this report, the subharmonic oscillations are mainly treated in the ranges of the forced frequencies between about three times and about ten times the fundamental linear natural frequency and of the parameter λ between 0.001 and 0.1.

2. Fundamental equation

Nomenclature (refer to Fig.1b)

- N: total number of links
P: pitch length of a chain cm
m: mass of one pitch of chain kg s²/cm
k: position number of roller and link,
k=0,1,2,...,N, for roller,
k=1,2,3,...,N, for link.
 $T_k(T_{k-1})$: tension of upper (lower) side of the link "k" kg
 T_m : arithmetical mean of initial tension at the upper and the lower ends
 $=(\bar{T}_0 + \bar{T}_N)/2$ kg
Notation ($\bar{}$) means the tension in no vibration.
t: time s
 y_k : displacement of the center of the roller "k" cm
g: acceleration of gravity cm/s²
 E_d : equivalent modulus of elasticity of a link for elongation kg/unit strain
 $S(t)$: length of chain during vibration cm
 F_k : transverse force acting between the adjoining links "k" and "k+1" kg
I: moment of inertia around the horizontal axis through the gravity point of a link (contained the roller) = $m\kappa^2$ kg s²cm
 κ : radius of gyration mentioned above
K: shape parameter of a link = $2\kappa/P$
 ℓ : span length = PN cm
W: total weight of a chain = mgN kg
 $p = W/2T_m = 0 \sim 1$
A: amplitude of the forced displacement at the lower end cm
 ω : circular frequency of the forced displacement rad/s
 $\tau = \omega t$
 $\phi = \tau/3$
 $\omega_0 = \frac{\pi}{N} \sqrt{\frac{T_m}{Pm}}$ rad/s
 Ω_0 : fundamental linear natural frequency ratio not considering the fluctuation of tension during vibration
 Ω : ratio of forced frequency ω to standard frequency $\omega_0 = \omega/\omega_0$

Fig.1 Chain system

$$\lambda = \frac{E_d (A/PN)^2}{T_m}$$

$\Delta T_{\max}$: maximum fluctuation of tension during vibration kg

$$\Delta T = \Delta T_{\max}/T_m$$

$Y = \{Y_1, Y_2, \dots, Y_{N-1}\}$: displacement ratio vector

$Y_{1/3} = \{Y_{1/3,1}, Y_{1/3,2}, \dots, Y_{1/3,N-1}\}$: displacement vector of the fundamental harmonic of subharmonic oscillation of order 1/3

$Y_{1/1} = \{Y_{1/1,1}, Y_{1/1,2}, \dots, Y_{1/1,N-1}\}$: displacement vector of the third higher harmonic of subharmonic oscillation of order 1/3

We consider a chain with a span of length ℓ stretched in the direction of gravity, as shown in Fig.1a. In order to simplify the analysis, the following assumptions are made.

- Motions of rollers and links occur only in the y direction (in the sprocket plane), and their displacements are small, i.e., $|y_k| < P$.
- Inertia forces in the x direction are neglected.
- Variation of tension due to gravity is expressed as a linear function of longitudinal coordinate (x), and is independent of time.
- Tension is not constant during vibration, but fluctuates with time.
- Travelling velocity of a chain is neglected.

The tension at a roller "k" during vibration, $T_k = T_k(k, t)$, may be approximated by the following relation;

$$T_k = T_m + mg(k - \frac{N}{2}) + E_d \frac{S(t) - PN}{PN} \quad (1)$$

Neglecting the friction between bushes and rollers and so assuming that the links are subject to only the tensile force, $S(t)$ may be expressed as follows:

$$S(t) = \sum_{j=1}^N \sqrt{P^2 + (y_j - y_{j-1})^2} \approx P \sum_{j=1}^N \left\{ 1 + \frac{(y_j - y_{j-1})^2}{2P^2} \right\}$$

Substituting the above equation into Eq. (1) yields

$$T_k = T_m \left[1 + p \left(\frac{2k}{N} - 1 \right) + \frac{E_d}{2P^2 N T_m} \sum_{j=1}^N (y_j - y_{j-1})^2 \right] \quad (2)$$

Referring to Fig.1b, if the frictional effect is neglected, the equations of motion of a link "k" for the displacement of the center of gravity in the y direction and for revolution about the center of gravity in the x-y plane are given respectively as follows:

$$\left. \begin{aligned} \frac{m}{2}(\ddot{y}_k + \ddot{y}_{k-1}) &= F_k - F_{k-1} \\ \frac{I}{P}(\ddot{y}_k - \ddot{y}_{k-1}) + (T_k + T_{k-1})\frac{y_k - y_{k-1}}{2} &= (F_k + F_{k-1})\frac{P}{2} \end{aligned} \right\} \quad (3.a)$$

Eliminating F_{k-1} from Eqs.(3-a) yields

$$\frac{m}{2}\left(1 + \frac{4I}{mP^2}\right)\ddot{y}_k + \frac{m}{2}\left(1 - \frac{4I}{mP^2}\right)\ddot{y}_{k-1} + (T_k + T_{k-1})\frac{y_k - y_{k-1}}{P} = 2F_k \quad (3.b)$$

For the link "k+1", the similar equations hold. Eliminating F_k between the equation yielded by eliminating F_{k+1} from equations corresponding to Eq.(3-a) and Eq.(3-b), then substituting Eq.(2) into the resultant equation and rearranging, we have the following fundamental equation neglecting the frictional terms:

$$\begin{aligned} \frac{1}{\omega_s^2} \left(\frac{\pi}{N} \right)^2 \left[\frac{1+K^2}{2} \ddot{y}_k + \frac{1-K^2}{4} (\ddot{y}_{k+1} + \ddot{y}_{k-1}) \right] - \left[\left\{ 1 + p \left(\frac{2k+1}{N} - 1 \right) \right\} \ddot{y}_{k+1} \right. \\ \left. - 2 \left\{ 1 + p \left(\frac{2k}{N} - 1 \right) \right\} \ddot{y}_k + \left\{ 1 + p \left(\frac{2k-1}{N} - 1 \right) \right\} \ddot{y}_{k-1} \right] \\ - \frac{E_d}{2P^2NT_m} \sum_{j=1}^N (y_j - y_{j-1})^2 (y_{k+1} - 2y_k + y_{k-1}) = 0 \quad (k=1, 2, \dots, N-1) \end{aligned} \quad (4)$$

3. Steady solutions of subharmonic oscillations of order 1/3

We apply the boundary conditions of the lower end ($k=0$) being subject to the lateral forced displacement, and the upper end ($k=N$) being fixed, namely, the following conditions are assumed:

$$y_0 = A \cos \omega t \quad \text{at } k=0, \quad y_N = 0 \quad \text{at } k=N \quad (5)$$

Replacing the independent variable t by $\tau = \omega t$ in Eq.(4), we can rewrite Eq.(4) in a matrix form as follows:

$$\begin{aligned} \Omega^2 \left(\frac{\pi}{N} \right)^2 M \mathbf{y}'' + K_1 \mathbf{y} + \frac{E_d}{2P^2NT_m} \sum_{j=1}^N (y_j - y_{j-1})^2 K_0 \mathbf{y} \\ = A \left[\Omega^2 \left(\frac{\pi}{N} \right)^2 \frac{1-K^2}{4} + 1 + p \left(\frac{1}{N} - 1 \right) + \frac{E_d}{2P^2NT_m} \sum_{j=1}^N (y_j - y_{j-1})^2 \right] \cos \tau \end{aligned} \quad (6)$$

(a)

where

$$\begin{aligned} \Omega = \omega/\omega_s, \quad \mathbf{0} = [0, 0, \dots, 0]; (N-2) \times 1, \quad \mathbf{y} = [y_1, y_2, \dots, y_{N-1}]; (N-1) \times 1 \\ M = \begin{bmatrix} Q_1 & Q_2 & 0 & \dots & 0 \\ & Q_1 & Q_2 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & Q_1 & Q_2 \\ (N-1) \times (N-1) & \text{Symmetric} & & & \end{bmatrix}, \quad K_1 = \begin{bmatrix} B_1 & C_1 & 0 & \dots & 0 \\ & B_2 & C_2 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & B_{N-3} & C_{N-3} \\ (N-1) \times (N-1) & \text{Symmetric} & & & \end{bmatrix} \\ Q_1 = (1+K^2)/2, \quad Q_2 = (1-K^2)/4, \\ B_k = 2\left\{1 - p\left(1 - \frac{2k}{N}\right)\right\}, \quad C_k = -1 + p\left(1 - \frac{2k+1}{N}\right) \end{aligned} \quad (6.d)$$

Matrix K_0 is equivalent to $K(0)$ when we write $K_1 = K(p)$, regarding matrix K_1 as a function of p . Further, introducing the parameter λ ,

$$\lambda = \frac{E_d}{T_m} \left(\frac{A}{PN} \right)^2 \quad (7)$$

and defining the following relation

$$\mathbf{Y} = \mathbf{y}/A = [Y_1, Y_2, \dots, Y_{N-1}] \quad (8)$$

Eq.(6) becomes as follows:

$$\begin{aligned} \Omega^2 \left(\frac{\pi}{N} \right)^2 M \mathbf{Y}'' + K_1 \mathbf{Y} + \frac{\lambda N}{2} \sum_{j=1}^N (Y_j - Y_{j-1})^2 K_0 \mathbf{Y} \\ = \left[\Omega^2 \left(\frac{\pi}{N} \right)^2 \frac{1-K^2}{4} + 1 + p \left(\frac{1}{N} - 1 \right) + \frac{\lambda N}{2} \sum_{j=1}^N (Y_j - Y_{j-1})^2 \right] \cos \tau \end{aligned} \quad (9)$$

The notations $\mathbf{y}''$ and $\mathbf{Y}''$ in Eqs.(6) and (9) represent the second derivatives with respect to τ . Among the steady solutions of Eq.(9), we take note of the subharmonic oscillations of order 1/3 in which the frequency of the chain is equal to one-third of the frequency of the forced displacement. In this report, we apply the method of the harmonic balance that presented the good approximation in the harmonic vibration. Neglecting the effect of friction in Eq.(9), we assume the following expression as the first approximation to the subharmonic oscillations of order 1/3 with symmetric modes:

$$\mathbf{Y} = \mathbf{Y}_{1/3} \cos \frac{\tau}{3} + \mathbf{Y}_{1/1} \cos \tau \quad (10)$$

(b)

Fig.2

Substituting Eq.(10) into Eq.(9), then using the identities,

$$\cos^2\tau = (3\cos\tau + \cos 3\tau)/4 \quad (11)$$

,etc. and equating the coefficients of terms containing $\cos\tau/3$ and $\cos\tau$ to zero respectively yields

$$\begin{aligned} \cos\frac{\tau}{3}: & \left[-\frac{\Omega^2}{9} \left(\frac{\pi}{N}\right)^2 M + K_1 + \frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{3}{4} (Y_{1/3,j} - Y_{1/3,j-1})^2 \right. \right. \\ & \left. \left. + \frac{1}{2} (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) + \frac{1}{2} (Y_{1/1,j} - Y_{1/1,j-1})^2 \right\} K_0 \right] Y_{1/3} \\ & + \left[\frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{1}{4} (Y_{1/3,j} - Y_{1/3,j-1})^2 + (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) \right\} K_0 \right] Y_{1/1} \\ & = \left[\frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{1}{4} (Y_{1/3,j} - Y_{1/3,j-1})^2 + (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) \right\} \right] \quad (12) \\ & \quad 0 \end{aligned}$$

$$\begin{aligned} \cos\tau: & \left[-\Omega^2 \left(\frac{\pi}{N}\right)^2 M + K_1 + \frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{1}{2} (Y_{1/3,j} - Y_{1/3,j-1})^2 + \frac{3}{4} (Y_{1/1,j} - Y_{1/1,j-1})^2 \right\} K_0 \right] Y_{1/1} \\ & + \left[\frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{1}{4} (Y_{1/3,j} - Y_{1/3,j-1})^2 + (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) \right\} K_0 \right] Y_{1/3} \\ & = \left[\Omega^2 \left(\frac{\pi}{N}\right)^2 \frac{1-K^2}{4} + 1 + p \left(\frac{1}{N} - 1\right) + \frac{\lambda N}{2} \sum_{j=1}^N \left\{ \frac{1}{2} (Y_{1/3,j} - Y_{1/3,j-1})^2 + \frac{3}{4} (Y_{1/1,j} - Y_{1/1,j-1})^2 \right\} \right] \quad (13) \\ & \quad 0 \end{aligned}$$

From the boundary conditions of Eq.(5), the following relations hold:

$$Y_{1/1,0} = 1, Y_{1/3,0} = Y_{1/3,N} = Y_{1/1,N} = 0 \quad (14)$$

Equations (12) and (13) are a set of non-linear simultaneous algebraic equations. The relations between Ω and $Y_{1/1}, Y_{1/3}$ are given by solving Eqs.(12) and (13) for the given parameters N, p, λ and K . We can obtain the resonance curves from the relations between the specified elements among the elements of $Y_{1/3}, Y_{1/1}$ (for example, the element that possesses the maximum amplitude in resonance.) and Ω/Ω_0 obtained in such a way (Ω_0 , the linear natural frequency ratio, has already been obtained in the form $\Omega_0 = \Omega_0(N, K, p)$.) for the given parameters N, K, p and λ . On the other hand, we can also obtain the resonance curves from the relations between the specified elements among $Y_{1/3}, Y_{1/1}$ and λ for the given parameters N, K, p and Ω . The former corresponds to the relation between the amplitude of the chain and Ω/Ω_0 under the constant forced displacement, and the latter to the relation between the amplitude of the chain and λ under the constant forced frequency Ω/Ω_0 . Using the values of $Y_{1/1}, Y_{1/3}$ obtained in such a way, the increment of tension during vibration from the initial tension with the help of Eq.(3) becomes as follows:

$$\begin{aligned} \frac{E_d A^2}{2P^2 N} \sum_{j=1}^N (Y_j - Y_{j-1})^2 &= \frac{\lambda N T_m}{2} \sum_{j=1}^N (Y_j - Y_{j-1})^2 \\ &= \frac{\lambda N T_m}{2} \sum_{j=1}^N \left[(Y_{1/3,j} - Y_{1/3,j-1}) \cos\frac{\tau}{3} + (Y_{1/1,j} - Y_{1/1,j-1}) \cos\tau \right]^2 \quad (15) \end{aligned}$$

The above expression has the absolute maximum at $\tau=0$ in our problem. Defining the maximum by $\Delta T_{\max}$, the following relation holds:

$$\begin{aligned} \Delta T = \frac{\Delta T_{\max}}{T_m} &= \frac{\lambda N}{2} \sum_{j=1}^N [(Y_{1/3,j} - Y_{1/3,j-1}) \\ &+ (Y_{1/1,j} - Y_{1/1,j-1})]^2 \quad (16) \end{aligned}$$

Making use of Eq.(16), we can obtain the relation between ΔT and Ω/Ω_0 or the relation between ΔT and λ .

Equating $Y_{1/3}$ to the zero vector in Eqs.(12) and (13), Eq.(13) becomes the equations determining the displacement $Y_{1/1}$ of the harmonic vibration. This harmonic solution has already been obtained and compared with the experimental results.

4. Numerical results

We solve numerically the set of non-linear simultaneous equations in Eqs.(12) and (13) by the step by step procedure, i.e., the Newton-Raphson method for the given parameters $N, K, p, \Omega(\Omega/\Omega_0)$ and λ . Once a solution has been set up, a sequence of steady solutions with a slight change of Ω or λ are obtained successively by regarding

the solutions at the preceding step as the initial values. From the numerical computation, are obtained the relations between the amplitude of the chain and Ω/Ω_0 , the relation ΔT versus Ω/Ω_0 under the constant λ , the relations between the amplitude of the chain and λ , and relation ΔT versus λ under the constant Ω/Ω_0 for the given parameters N, K and p . The results of subharmonic oscillation of order 1/3 obtained in such a way are shown in Figs. 2~8. We use the displacement at the mid-span of the chain as the amplitude. The resonance curves (HV) in the vicinity of $\Omega/\Omega_0=1$ in the graphs of the relation between the amplitude of the chain and Ω/Ω_0 and of the relation ΔT versus Ω/Ω_0 , indicate those of the harmonic vibration. The resonance curves (SHV) lying in the region where the frequency ratio Ω/Ω_0 is more than three, represent those of the sub-

Fig.3

harmonic oscillation of order $1/3$. There are six kinds of steady solutions in a certain limited range of the frequencies Ω/Ω_0 and of the values of λ . The three kinds in which the fundamental harmonic components of the subharmonic oscillations are all positive, we call A, B and C curves respectively in a decreasing order of the maximum amplitude. The other three kinds in which the fundamental harmonic components of the subharmonic oscillations are all negative, we call D, E and F curves respectively in a decreasing order of the absolute value of maximum amplitude. Further, the notation $1/3$ is put down in the fundamental component, and the notation $1/1$ in the third higher harmonic component. Hence, the curves in a group with the mark "1/1" in the region of $\Omega/\Omega_0 > 3$ in figures indicate the third higher harmonic components of subharmonic oscillation of order $1/3$, namely, the strained components.

Point P on C- $1/1$ curves corresponding to the value of Ω/Ω_0 at which $Y_{1/3} = 0$ on C- $1/3$ curve is a point of intersection with the nonlinear resonance curve of the third harmonic vibration, and is called the bifurcation point (cf. Fig. 9).

Table 1 Numerical examples

Fig.	Ω/Ω_0	λ	p	N	K
2	variable	0.01	0.0	40	1.0
3	variable	0.005	0.0	40	1.0
4	4.5	variable	0.0	10	1.0
5	variable	0.01	0.0	10	1.0
6	variable	0.01	0.5	40	1.0
7	variable	0.01	0.5	10	1.0
8	4.5	variable	0.5	10	1.0

Fig. 4

The influence of each parameter on the nonlinear vibration is summarized as follows:

Effect of λ (cf. Figs. 2, 3 and 4) :

When the forced frequency is changed, the amplitude curves incline to the right side as λ increases, but the curve Ω/Ω_0 versus ΔT doesn't change in the slope, the distance between the two lines becoming wider. The boundary point Q between A and B curves moves in the increasing direction of frequency, while the boundary point R between D and E curves moves in the decreasing direction of frequency, as λ increases. Further, the point S on C- $1/3$ and F- $1/3$ curves corresponding to zero values of the fundamental harmonic components, moves to the increasing direction of the frequency as λ increases. (The fact that C and F curves have a point of intersection at

Fig. 5

Fig. 6

point S, may be confirmed by the numerical results.) The disposition of each point is indicated by $\Omega(S) < \Omega(R) < \Omega(Q)$, namely, the distribution is in the order, S, R, Q from the left hand side in figures, when the forced frequency is changed under the constant λ .

Effect of p (cf. Figs.3,6 and Figs.4, 8): When the forced frequency is changed under the constant λ , the amplitudes of the chain decrease as if their curves were moved parallel downward as p increases. Where λ is changed, the points Q, R and S move in the increasing direction of λ as p increases.

Effect of N (cf. Figs.3,5): When the forced frequency is changed, points Q and R move in the increasing direction of frequency, but point S moves in the opposite direction as N increases. There is not any great change of the amplitudes of A and D curves with the value of N , and the amplitude curves of C and F move parallel upward as N increases.

Effect of K : In general, the influence of K on the nonlinear character is small.

Next, let's examine the vibration mode and fluctuation pattern of tension during vibration of the chain. In Fig.10, a vibration mode in the resonance curve indicated in Fig.2 is shown, divided into two groups, the fundamental harmonic components (1/3) and the third higher harmonic components (1/1). The modes formed by the third higher harmonics on A and D curves are the fourth harmonic resonance modes in phase with the forced displacement, those on B and E curves are the third harmonic resonance modes out of phase with the forced displacement and those on C and F curves are the third harmonic modes in phase with the forced displacement. The elapses of the displacement of the roller at the mid-span with time τ are described in Fig.11. Figs.12 indicate the elapses of the vibra-

tion over the span of chain on A and C curves with time. The fluctuating patterns of tension during vibration indicated in Figs.12 are shown in Fig.13. These vibration modes and fluctuating patterns of tension vary quantitatively by virtue of the parameters N , K , p , λ and Ω/Ω_0 , but the tendency mentioned above does not change generally.

5. Stability of periodic oscillations

All the six kinds of the subharmonic oscillations of order 1/3 obtained in Sec.4 are not necessarily realized. The appearance of these steady solutions confirms the stability conditions of the periodic oscillations. In this report, by setting up the variational equations from the equations of motion, we investigate the first approximate stability of the steady oscillations from the resultant simultaneous Hill's equations. Substituting the following term with a small variation η_i

$$Y_i + \eta_i = Y_{i/3, i} \cos \tau/3 + Y_{i/1, i} \cos \tau + \eta_i \quad (i=1, 2, \dots, N-1) \quad (17)$$

into Eq.(9) instead of Y_i , yields the linear variational equations,

$$\begin{aligned} Q^2 \left(\frac{\pi}{N} \right)^2 M \eta'' + K \eta + \frac{\lambda N}{2} \sum_{j=1}^N (Y_j - Y_{j-1})^2 K \eta \\ + \lambda N \sum_{j=1}^N (Y_j - Y_{j-1}) (\eta_j - \eta_{j-1}) K \eta = \left[\lambda N \sum_{j=1}^N (Y_j - Y_{j-1}) (\eta_j - \eta_{j-1}) \right] \cos \tau \end{aligned} \quad (18)$$

where

$$\eta_0 = \eta_N = 0 \quad (18.a)$$

We define the following relations,

$$\alpha_i = -Y_{i/3, i-1} + 2Y_{i/3, i} - Y_{i/3, i+1}, \quad \beta_i = -Y_{i/1, i-1} + 2Y_{i/1, i} - Y_{i/1, i+1} \quad (19)$$

Making use of the above relations, Eq.(18) becomes the following simultaneous Hill's equations.

$$Q^2 \left(\frac{\pi}{N} \right)^2 M \eta'' + [D_0 + D_1 \cos \frac{2}{3} \tau + D_2 \cos \frac{4}{3} \tau + D_3 \cos 2\tau] \eta = 0 \quad (20)$$

where

Fig.7

Fig.8

Fig.9 Bifurcation point P

$$\left. \begin{aligned}
 D_0 &= K_1 + \frac{\lambda N}{4} \left\{ \sum_{j=1}^N (Y_{1/3,j} - Y_{1/3,j-1})^2 + \sum_{j=1}^N (Y_{1/1,j} - Y_{1/1,j-1})^2 \right\} K_0 + \frac{\lambda N}{2} [\alpha_i \beta_j + \beta_i \alpha_j] \\
 D_1 &= \frac{\lambda N}{2} \left\{ \frac{1}{2} \sum_{j=1}^N (Y_{1/3,j} - Y_{1/3,j-1})^2 + \sum_{j=1}^N (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) \right\} K_0 \\
 &\quad + \frac{\lambda N}{2} [\alpha_i \alpha_j + \beta_i \alpha_j + \alpha_i \beta_j] \\
 D_2 &= \frac{\lambda N}{2} \sum_{j=1}^N (Y_{1/3,j} - Y_{1/3,j-1})(Y_{1/1,j} - Y_{1/1,j-1}) K_0 + \frac{\lambda N}{2} [\beta_i \alpha_j + \alpha_i \beta_j] \\
 D_3 &= \frac{\lambda N}{4} \sum_{j=1}^N (Y_{1/1,j} - Y_{1/1,j-1})^2 K_0 + \frac{\lambda N}{2} [\beta_i \beta_j]
 \end{aligned} \right\} \dots (20-a)$$

, notation [] represents the matrix and the letter within square brackets is (i,j) element of the matrix. Putting $\phi = \tau/3$, Eq. (20) becomes as follows:

$$\left(\frac{\lambda N}{3} \right)^2 \mathbf{M} \frac{d^2 \boldsymbol{\eta}}{d\varphi^2} + [D_0 + D_1 \cos 2\varphi + D_2 \cos 4\varphi + D_3 \cos 6\varphi] \boldsymbol{\eta} = \mathbf{0} \dots (21)$$

According to Floquet's theory, a particular solution of Eq.(21) is given by

$$e^{i\varphi} \boldsymbol{\phi}(\varphi) \dots (22)$$

where $\boldsymbol{\phi}(\varphi)$ is a periodic function of φ with period π or 2π . Since Eq.(21) is unchanged if $-\varphi$ is written for φ ,

$$e^{-i\varphi} \boldsymbol{\phi}(-\varphi) \dots (23)$$

is another independent solution. Hence, the general solution of Eq.(21) may be given by linear combination of Eqs.(22) and (23). The number μ is called the characteristic exponent of the system (21). Substituting Eq. (22) into Eq.(21) and rearranging yields

$$\left(\frac{\pi \Omega}{3N} \right)^2 \mathbf{M} \frac{d^2 \boldsymbol{\phi}}{d\varphi^2} + 2\mu \left(\frac{\pi \Omega}{3N} \right)^2 \mathbf{M} \frac{d\boldsymbol{\phi}}{d\varphi} + [D_0 + \mu^2 \left(\frac{\pi \Omega}{3N} \right)^2 \mathbf{M} + D_1 \cos 2\varphi + D_2 \cos 4\varphi + D_3 \cos 6\varphi] \boldsymbol{\phi} = \mathbf{0} \dots (24)$$

According to Whittaker⁽⁹⁾, the first approximations of the function $\boldsymbol{\phi}(\varphi)$ in the nth unstable region are given by

$$\boldsymbol{\phi}(\varphi) = \begin{bmatrix} a_1 \sin(n\varphi - \sigma_1) \\ a_{N-1} \sin(n\varphi - \sigma_{N-1}) \end{bmatrix} \dots (25)$$

By substituting Eq.(25) into Eq.(24) and equating the coefficients of the terms containing $\sin n\varphi$, $\cos n\varphi$ to zero, we will investigate the condition of stability in the nth unstable region where $n=1,2$ and 3.

Fig.10 Fundamental harmonics (1/3) and third higher harmonics (1/1) at $\Omega/\Omega_0=6.5$ in Fig.2

Fig.11 Displacement patterns (at $k=20$) at $\Omega/\Omega_0=6.5$ in Fig.2

Fig.12 Vibration modes at $\Omega/\Omega_0=6.5$ in Fig.2

(i) Stability in the first unstable region $n=1$

Substitute $\phi = \{a_i \sin(\phi - \sigma_i)\}$ into Eq.(24) and equate the coefficients of the terms containing $\sin\phi$, $\cos\phi$ to zero. Then eliminating $\sin\phi$, $\cos\phi$ from the resultant homogeneous equations yields,

$$\Delta_1(\mu) = \begin{vmatrix} -\left(\frac{\pi Q}{3N}\right)^2 M + D - \frac{D_1}{2} & 2\mu\left(\frac{\pi Q}{3N}\right)^2 M \\ 2\mu\left(\frac{\pi Q}{3N}\right)^2 M & \left(\frac{\pi Q}{3N}\right)^2 M - D - \frac{D_1}{2} \end{vmatrix} = 0 \quad (26)$$

where

$$D = D_0 + \mu^2 \left(\frac{\pi Q}{3N}\right)^2 M$$

If μ satisfying Eq.(26) is purely imaginary ($\mu^2 < 0$), η is bounded and, therefore, the solution Y is stable. Since $\mu^2 = 0$ at the boundary between the stable and the unstable regions, we have the following stable condition:

$$(-1)^n \Delta_1(0) < 0 \quad (27)$$

where

$$\Delta_1(0) = -\left(\frac{\pi Q}{3N}\right)^2 M + D_0 - \frac{D_1}{2} \times \left| \left(\frac{\pi Q}{3N}\right)^2 M - D_0 - \frac{D_1}{2} \right| \quad (28)$$

(ii) Stability in the second unstable region $n=2$

Stable condition is

$$(-1)^n \Delta_2(0) < 0 \quad (29)$$

where

$$\Delta_2(0) = -\left(\frac{2\pi Q}{3N}\right)^2 M + D_0 - \frac{D_2}{2} \times \left| \left(\frac{2\pi Q}{3N}\right)^2 M - D_0 - \frac{D_2}{2} \right| \quad (30)$$

(iii) Stability in the third unstable region $n=3$

Stable condition is

$$(-1)^n \Delta_3(0) < 0 \quad (31)$$

where

$$\Delta_3(0) = -\left(\frac{\pi Q}{N}\right)^2 M + D_0 - \frac{D_3}{2} \times \left| \left(\frac{\pi Q}{N}\right)^2 M - D_0 - \frac{D_3}{2} \right| \quad (32)$$

Since the system has multiple degrees of freedom, it is not possible to draw the boundary curves of stability with ease, but we compute the values of $\Delta_1(0)$, $\Delta_2(0)$ and $\Delta_3(0)$ and can determine the stability of the corresponding steady vibration according to the sign (positive or negative) of their values by use of $Y_{1/3}$, $Y_{1/1}$ obtained in Sec.4. The following became clear after the investigation on the stability condition. B curve is unstable in the first or the third unstable region, and D, E and F curves are unstable in the first unstable region. Any curve is not unstable in the second unstable region. But A and C curves possess the stable and unstable regions. The region between points Q and Q_A on A curve and the region between points S and S_C on C curve in the fundamental harmonic component are stable. (The corresponding stable regions in the third harmonic are between points Q' and Q'_A and between point P and S'_C respectively.) Since the stability condition is the first approximation, there is a part that becomes simultaneously unstable on B curve in both the first and the third unstable regions, and we can not confirm that the point Q at which A resonance curve possesses the vertical tangency, is just at the boundary between the stable and the unstable regions: It is

Fig.13 Fluctuation patterns of tension at $\Omega/\Omega_0=6.5$ in Fig.2

well known that the higher we approximate the function $\phi(\psi)$ in the unstable region, the more exactly we can investigate the stability conditions⁽⁹⁾.

Now, in a different method from the above mentioned, we can determine whether the solution is stable or unstable by solving numerically the differential equations in Eq.(20) under certain initial conditions. As a result of the numerical computations, we can directly verify the resultant, stable and unstable regions determined above. But the only boundary between the stable and the unstable regions near the point Q or Q_A on A curve is not clear, namely, the variations don't necessarily diverge definitely so remarkably. After all, it became clear from the first approximate stability conditions that two kinds of the subharmonic oscillations of order 1/3 with symmetric modes might be realized, and each stable region possessed the upper and the lower bounds.

6. Characteristics of subharmonic oscillations of order 1/3

To begin with, we consider the vibration characteristics of the displacement at the midspan and the fluctuation of tension during vibration, when the amplitude of the forced displacement is constant but the forced frequency varies (cf. Fig.14). We now start at point 1 on C curve of the fundamental harmonic component under the appropriate initial condition. As Ω/Ω_0 is decreased gradually, the fundamental harmonic component decreases and while the third higher harmonic component increases relatively along C curve, but the compound amplitude becomes wholly smaller, and finally the fundamental harmonic component

disappears. The fluctuation of tension during vibration ΔT also decreases gradually along C curve. Namely, the process 1-4-S (1'-4'-P) is followed. On the other hand, upon performing the experiment in the other direction from the point 1 (1'), both the amplitude and ΔT increase until the point S_c (or S'_c) is reached. On crossing this point, since λ is held constant and no stable solutions may exist for $\Omega(Q_A) < \Omega(S_c)$, a further increase in Ω/Ω_0 would require a jump from point S_c (S'_c) to point 2, i.e., the vibration except for the subharmonic oscillation of order 1/3, for example the linear harmonic vibration of higher order, with a sudden decrease in both the amplitude and ΔT . Namely, the process 1-5- $S_c \rightarrow 2$ (1'-5'- $S'_c \rightarrow 2$) may result, where the marks $\downarrow$ and $\uparrow$ represent an accompanying decrease in the fluctuation of tension and in the amplitude respectively.

Next, we start at point 3 (3') on A curve. As Ω/Ω_0 is decreased gradually, the amplitude and ΔT decrease as far as the point Q. We observe that either the jump from point Q to point 4 (4') on C curve with smaller amplitude but with larger fluctuation of tension at the same frequency, i.e. (3-4), or the jump from point Q to point 7, i.e., the vibration except for the subharmonic oscillation of order 1/3, takes place with a decreasing Ω/Ω_0 . In the experiment performed in the other direction from point 3 (3') both the amplitude and the fluctuation of tension follow A curve and slowly increase as far as the point Q_A (Q'_A). After crossing the point, the jump from point Q_A to point 6 may take place when $\Omega(Q_A) > \Omega(S_c)$, namely, the process 3- $Q_A \rightarrow 6$ (3'- $Q'_A \rightarrow 6$) may occur. But the jump to point 5 or to point 6 may occur when $\Omega(Q_A) < \Omega(S_c)$. If the jump to point 5 takes place, afterward the same thing as the mentioned above holds.

In Fig.15, we indicate the vibration

Fig.14 Characteristics of subharmonic oscillation of order 1/3 when $\Omega(Q_A) < \Omega(S_c)$

Fig.15 Characteristics of subharmonic oscillation of order 1/3 when $\lambda(S_c) < \lambda(Q_A)$

characteristics where the amplitude of the forced displacement varies but the forced frequency is constant, and can give similar explanations to Fig.14 about the vibration characters but they may be omitted here. But we should note that as λ is increased, the amplitude of the chain doesn't decrease but the ratio of the amplitude of the chain to the forced displacement amplitude decreases, and of course, the amplitude of the chain increases in practice.

7. Conclusions

Since the roller chain possesses not only a high modulus for elongation but also its mean line density is comparatively large, it is expected that the influences of the fluctuation of tension due to vibration and of the weight of the chain on the forced vibration, especially the subharmonic oscillations of order 1/3 will be remarkable. In this report, the authors have investigated these effects theoretically by regarding the chain as a multiple-degree-of-freedom system. The results are summarized as follows.

- (1) The subharmonic oscillations are excited by the fluctuations of tension, namely of extension of the chain during vibration.
- (2) The authors have shown that the aspect of the subharmonic oscillation of order 1/3 is mainly controlled by the parameters λ and p under the boundary conditions of one end being subject to the forced displacement and the other end being fixed, and that it makes a complicated behavior under combined effects of the two parameters.
- (3) There exist in practice only two kinds of the subharmonic oscillations of order 1/3 with symmetric modes and they have different vibration modes from each other.

- (4) The regions in which they can be realized are not common to each other and have limits even in the analysis neglecting the friction. The jump phenomena also take place.

The comparison of the theoretical results in this report with the experimental results will be presented in the next paper.

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