tension is expressed by Eq. (6)'. It is because the phenomenon shown in Fig. 8 takes place under tension. But when the material yields under tension, this phenomenon takes place at the point P in Fig. 10, and the yielding is another phenomenon prior to this. Thus the condition of yielding expressed by Eq. (12) is a different problem from this theory of plasticity, and of course there is no contradiction between them.

Fig. 11 shows the experimental results on the yielding and the plastic deformation beyond that, in the cases of pure shear, and tension. The shearing stress τ is taken in ordinate and the longitudinal strain ϵ in abscissa. Material is a mild steel of 0.39% carbon content. Test pieces are hollow cylinders of outer diameter 18 mm and inner diameter 16 mm. The stress ratio between the plastic pure shear and plastic tension does not satisfy Eq. (11) exactly, as this material is considered to be not isotropic enough. And yet from this figure we will be able to understand what was explained above.

Conclusions

A new theory of plasticity was advanced. The experiments on the stress relation between plastic pure shear and plastic tension conformed closely with this theory. Moreover, hysteresis loops and anisotropy due to plastic deformation will also be accounted for by this theory.

As for the yielding of mild steel, it is already known that the yielding spreads under a constant shearing stress, independent of the kind of load, when the stress distribution is uniform. It was shown that this fact did not conflict with the theory, though the yielding was the first step of plastic deformation.

We wish to express our thanks to Mr. F. Nagai for his assistance in carrying out the experiments and computations.

Reference

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On the Transverse Flexure of an Infinite Strip with Semicircular Notches on Both Edges*

By Osamu TAMATE** and Shunsuke SHIOYA***

In this paper, the effects of a pair of symmetrical semicircular notches on the transverse flexure of an infinite strip under the state of plain bending are investigated by use of the Poisson-Kirchhoff theory of thin plates. The unknown coefficients included in the solution are determined by the method of perturbation.

In order to clarify the effect of making the semicircular notches, the distributions of bending couples and the stress concentration factor k_t are calculated and plotted versus the ratio of notch diameter to strip width and Poisson's ratio of the plate material. The deflections of the strip are also calculated and shown graphically.

1. Introduction

The problem of the stress distribution in an infinite strip containing notches and subjected to transverse bending has already received some theoretical attention in certain special cases. Lee⁽¹⁾

has investigated the influence of two symmetrically disposed hyperbolic notches on the transverse flexure of thin plate. Neuber⁽²⁾ also has discussed similar problems.

In a previous paper by the authors⁽³⁾, the theoretical solution has been given for a semi-infinite plate with a semicircular notch under plain bending. The present paper gives a theoretical solution for an infinite strip containing two symmetrically placed semicircular notches under plain bending.

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The analysis is developed on the basis of the Poisson-Kirchhoff theory⁽⁴⁾ of thin plates. Numerical examples also are worked out in some detail, in order to clarify the effect of the notches.

2. Preliminary results

We consider an infinitely long thin strip of isotropic, elastic material and take its middle plane as the x, y plane. Let the strip be bounded in the plane by the lines $x = \pm b$. The strip is to be notched by two semicircular notches each of radius a with their centers at the points $(\pm b, 0)$ as shown in Fig. 1. Polar co-ordinates (r, θ) also will be used, which are defined by

$$z = x + iy = (r \cos \theta - b) + ir \sin \theta \dots (1)$$

We also will write

$$x/b = \xi, y/b = \eta, r/b = \rho, a/b = \lambda \dots (2)$$

The strip will be considered subjected to the limitations of the Poisson-Kirchhoff theory of thin plates. Accordingly, the transverse deflection w of the middle plane of the strip is governed by the equation

$$\nabla^4 w = p/D \quad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \dots (3)$$

where p is the pressure applied to the surface of the strip and D is the flexural rigidity of the strip.

The bending moments M_r and M_θ , twisting moment $M_{r\theta}$, and shearing forces Q_r and Q_θ , all per unit of arc length with reference to the polar co-ordinates (r, θ) , are expressible by the formulas;

$$\left. \begin{aligned} M_r &= -\frac{D}{b^2} \left\{ \frac{\partial^2 w}{\partial \rho^2} + \nu \left(\frac{\partial w}{\rho \partial \rho} + \frac{\partial^2 w}{\rho^2 \partial \theta^2} \right) \right\} \\ M_\theta &= -\frac{D}{b^2} \left\{ \frac{\partial w}{\rho \partial \rho} + \frac{\partial^2 w}{\rho^2 \partial \theta} + \nu \frac{\partial^2 w}{\partial \rho^2} \right\} \\ M_{r\theta} &= -M_{\theta r} = \frac{D}{b^2} (1-\nu) \frac{\partial}{\partial \rho} \left(\frac{\partial w}{\rho \partial \theta} \right) \\ Q_r &= -\frac{D}{b^3} \frac{\partial}{\partial \rho} \nabla^2 w, \quad Q_\theta = -\frac{D}{b^3} \frac{\partial}{\rho \partial \theta} \nabla^2 w \end{aligned} \right\} \dots (4)$$

If t and n denote the direction tangential and

$$w_{00} = -\frac{M_0 b^2}{D(1-\nu^2)} \frac{1}{2} (\eta^2 - \nu \xi^2) = -\frac{M_0 b^2}{D(1-\nu^2)} \left\{ \frac{\nu}{2} (2\rho \cos \theta - 1) + \rho^2 \left(\frac{1-\nu}{4} - \frac{1+\nu}{4} \cos 2\theta \right) \right\} \dots (8)$$

w_{00} satisfies only the conditions on the straight edges and at infinity. Two auxiliary biharmonic functions w' and w'' , even in both x and y , as required by symmetry, are to be constructed such that the corresponding stresses vanish at infinity as well as on the straight boundaries. The series of these functions can be derived by differentiation from two fundamental solutions for an unnotched strip having singularities at the center of the notches. Of these, the following are required for the present solution.

$$w_{2n} = -\frac{1}{(2n+2)!} \int_0^\infty u^{2n} \frac{2\{(\epsilon^{-1}-1)s-2uc\}C+4u\xi sS}{\Delta} \cos u\eta du \dots (9)$$

$$w_{2n}' = \frac{1}{(2n+1)!} \int_0^\infty u^{2n-1} \frac{\{2(\epsilon^{-1}+1)c+4us\}C-4u\xi cS}{\Delta} \cos u\eta du \dots (10)$$

where

$$\begin{aligned} C &= \cosh u\xi, \quad S = \sinh u\xi, \quad c = \cosh u, \quad s = \sinh u, \\ \Delta &= \sinh 2u - 2\epsilon u, \quad \epsilon = (1-\nu)/(3+\nu) \end{aligned}$$

Fig. 1

normal to the boundary of the strip, respectively, the boundary conditions for a free boundary are generally given by

$$M_n = 0, \quad V_n = Q_n - \frac{\partial M_{n,t}}{\partial s} = 0 \dots (5)$$

where s denotes the arc length along the boundary of the strip.

With these formulas, we shall seek the solution for the case where the strip, having a pair of semicircular notches, is subjected to the plain bending of intensity M_0 , about x -axis.

In this instance, since the middle plane of the strip is free from lateral loads, p is equal to zero; w is then a plane, single-valued, biharmonic function. We also assume that the straight edges of the strip and the rims of the notches are both stress free.

3. Method of solution

Consider the biharmonic function

$$w = w_{00} + w' + w'' \dots (6)$$

where

$$w' = \sum_{n=0}^{\infty} a_{2n} w_{2n}, \quad w'' = \sum_{n=0}^{\infty} b_{2n} w_{2n}' \dots (7)$$

The basic biharmonic function w_{00} corresponds to the deflection of the strip, without notches, subjected to plain bending and can be written as

The boundary conditions at the rims of the notches are then satisfied by adjusting the unknown coefficients a_{2n} and b_{2n} attached to these functions.

To apply the boundary conditions at the rim of one of the symmetrical notches, it is convenient to express the biharmonic function w in terms of polar co-ordinates (ρ, θ) with the origin at the point o_2 $(-b, o)$. Considering Eq. (1), we have

$$w = -\frac{M_0 b^2}{D(1-\nu)} \left[\frac{1}{1+\nu} \rho^2 \left\{ \frac{1-\nu}{4} - \frac{1+\nu}{4} \cos 2\theta \right\} + \sum_{n=0}^{\infty} \frac{a_{2n}}{2n+2} \left[\left\{ 1 - \frac{1}{2n+1} \left(\frac{1}{\epsilon} - 1 \right) \right\} \frac{\cos(2n+1)\theta}{\rho^{2n+1}} \right. \right. \\ \left. \left. + \frac{\cos(2n+3)\theta}{\rho^{2n+1}} \right] + b_0 \left\{ - \left(\frac{1}{\epsilon} + 1 \right) \log \rho + \cos 2\theta \right\} + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n+1)} \left[\left\{ 1 + \frac{1}{2n} \left(\frac{1}{\epsilon} + 1 \right) \right\} \frac{\cos 2n\theta}{\rho^{2n}} \right. \right. \\ \left. \left. + \frac{\cos(2n+2)\theta}{\rho^{2n}} \right] + l_0 \rho^2 + \sum_{n=1}^{\infty} \left[\frac{1}{2n-1} \left\{ 1 - \frac{1}{2n} \left(\frac{1}{\epsilon} + 1 \right) \right\} l_{2n-2} + \frac{1}{2n+1} l_{2n} \rho^2 \right] \rho^{2n} \cos 2n\theta \right. \\ \left. + \frac{1}{2} m_1 \rho^3 \cos \theta + \sum_{n=1}^{\infty} \left[\frac{1}{2n} \left\{ 1 + \frac{1}{2n+1} \left(\frac{1}{\epsilon} - 1 \right) \right\} m_{2n-1} + \frac{1}{2n+2} m_{2n+1} \rho^2 \right] \rho^{2n+1} \cos(2n+1)\theta \right] \quad \dots\dots\dots (11)$$

where

$$l_{2n} = \sum_{s=0}^{\infty} \{ {}^{2n} \alpha_{2s} a_{2s} + {}^{2n} \beta_{2s} b_{2s} \}, \quad m_{2n+1} = \sum_{s=0}^{\infty} \{ {}^{2n+1} \kappa_{2s} a_{2s} + {}^{2n+1} \mu_{2s} b_{2s} \} \quad \dots\dots\dots (12)$$

$${}^{2n} \alpha_{2s} = -2\epsilon I_{2n+2s+3} / (2s+2)! (2n)!, \quad {}^{2n+1} \kappa_{2s} = - \{ I_{2n+2s+3} - 2\epsilon I_{2n+2s+4} - J_{2n+2s+3} \} / (2s+2)! (2n+1)!, \\ {}^{2n} \beta_{2s} = - \{ I_{2n+2s+1} + 2\epsilon I_{2n+2s+2} + J_{2n+2s+1} \} / (2s+1)! (2n)!, \quad {}^{2n+1} \mu_{2s} = 2\epsilon I_{2n+2s+3} / (2s+1)! (2n+1)! \quad \dots\dots\dots (13)$$

$$I_m = \int_0^{\infty} (u^m / \Delta) du, \quad J_m = \int_0^{\infty} (u^m e^{-2u} / \Delta) du \quad \dots\dots\dots (14)$$

As w is an even function in both x and y , if the boundary conditions on the edge of a semicircular notch are satisfied, the conditions on the other notch are also satisfied automatically. From Eq. (5), the boundary conditions at $\rho = \lambda$ are expressed as follows.

$$\left. \begin{aligned} M_r &= -\frac{D}{b^2} \left\{ \frac{\partial^2 w}{\partial \rho^2} + \nu \left(\frac{\partial w}{\rho \partial \rho} + \frac{\partial^2 w}{\rho^2 \partial \theta^2} \right) \right\}_{\rho=\lambda} = 0, \\ V_r &= - \left\{ \frac{D}{b^3} \frac{\partial}{\partial \rho} \nabla^2 w + \frac{1}{b^3} \frac{\partial M_{r\theta}}{\rho \partial \theta} \right\}_{\rho=\lambda} = 0 \text{ or } \{ D\mathcal{Q} + M_{r\theta} \}_{\rho=\lambda} = \text{constant} \end{aligned} \right\} \quad \dots\dots\dots (15)$$

where $\mathcal{Q}$ is a conjugate function of $\nabla^2 w$.

Substituting Eq. (11) into Eq. (15), the boundary conditions at the rim of the notch give

$$\begin{aligned} & \frac{1}{2} - \frac{1}{2} \cos 2\theta + \frac{a_0}{\lambda^3} \left(2 - \frac{1}{\epsilon} \right) \cos \theta + \sum_{p=1}^{\infty} \left\{ \frac{a_{2p-2}}{\lambda^{2p+1}} \left(2p+2 - \frac{1}{\epsilon} \right) + \frac{a_{2p}}{\lambda^{2p+3}} \left(2p+2 - \frac{1}{\epsilon} \right) \right\} \cos(2p+1)\theta \\ & + \frac{b_0}{\lambda^2} \left(1 + \frac{1}{\epsilon} \right) + \sum_{n=1}^{\infty} \left\{ \frac{b_{2n-2}}{\lambda^{2n}} \left(2n+1 - \frac{1}{\epsilon} \right) + \frac{b_{2n}}{\lambda^{2n+2}} \left((2n+1) + \frac{1}{\epsilon} \right) \right\} \cos 2n\theta + \left(\frac{1}{\epsilon} - 1 \right) l_0 \\ & + \sum_{n=1}^{\infty} \left\{ \lambda^{2n-2} l_{2n-2} \left(2n-1 - \frac{1}{\epsilon} \right) + \lambda^{2n} l_{2n} \left(2n-1 + \frac{1}{\epsilon} \right) \right\} \cos 2n\theta + \frac{1}{\epsilon} \lambda m_1 \cos \theta \\ & + \sum_{p=1}^{\infty} \left\{ \lambda^{2p-1} m_{2p-1} \left(2p + \frac{1}{\epsilon} \right) + \lambda^{2p+1} m_{2p+1} \left(2p + \frac{1}{\epsilon} \right) \right\} \cos(2p+1)\theta = 0 \quad \dots\dots\dots (16) \\ & - \frac{1}{2} \sin 2\theta - \frac{a_0}{\lambda^3} \left(2 - \frac{1}{\epsilon} \right) \sin \theta - \sum_{p=1}^{\infty} \left\{ \frac{a_{2p-2}}{\lambda^{2p+1}} \left(2p+2 + \frac{1}{\epsilon} \right) + \frac{a_{2p}}{\lambda^{2p+3}} \left(2p+2 - \frac{1}{\epsilon} \right) \right\} \sin(2p+1)\theta \\ & - \sum_{n=1}^{\infty} \left\{ \frac{b_{2n-2}}{\lambda^{2n}} \left(2n+1 + \frac{1}{\epsilon} \right) + \frac{b_{2n}}{\lambda^{2n+2}} \left(2n+1 + \frac{1}{\epsilon} \right) \right\} \sin 2n\theta + \sum_{n=1}^{\infty} \left\{ \lambda^{2n-2} l_{2n-2} \left(2n-1 - \frac{1}{\epsilon} \right) \right. \\ & \left. + \lambda^{2n} l_{2n} \left(2n-1 + \frac{1}{\epsilon} \right) \right\} \sin 2n\theta - \lambda m_1 \frac{1}{\epsilon} \sin \theta + \sum_{p=1}^{\infty} \left\{ \lambda^{2p-1} m_{2p-1} \left(2p + \frac{1}{\epsilon} \right) \right. \\ & \left. + \lambda^{2p+1} m_{2p+1} \left(2p - \frac{1}{\epsilon} \right) \right\} \sin(2p+1)\theta = 0 \quad \dots\dots\dots (17) \end{aligned}$$

Moreover, sines and cosines of odd multiples of θ can be expressed in Fourier series of sines or cosines, respectively, of even multiples of θ valid between $\theta = -\pi/2$ and $\theta = \pi/2$:

$$\left. \begin{aligned} \cos(2p+1)\theta &= \frac{4}{\pi} \left\{ \frac{(-1)^p}{2(2p+1)} + \sum_{n=1}^{\infty} \frac{(-1)^{n+p} (2p+1)}{(2p+2n+1)(2p-2n+1)} \cos 2n\theta \right\}, \\ \sin(2p+1)\theta &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+p} 2n}{(2p+2n+1)(2p-2n+1)} \sin 2n\theta \end{aligned} \right\} \quad \dots\dots\dots (18)$$

Substituting Eq.(18) into Eqs.(16) and (17), and then equating to zero the coefficients of $\cos 2n\theta$ and $\sin 2n\theta$, respectively, we get

$$\left. \begin{aligned} & \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)(2n+5)} \{2(2n+5) - (2n+1)\varepsilon - (2n+1)\varepsilon^2\} a_{2n}' \\ &= -\frac{\varepsilon^2}{2} - (1-\varepsilon)\varepsilon l_0' + (1+\varepsilon)\varepsilon l_2' + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)(2n+3)} \\ &\quad \times \{2(2n-1) - (2n+3)\varepsilon - (2n+3)\varepsilon^2\} m_{2n+1}', \\ & \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+p}}{(2n-2p+1)(2n+2p+1)(2n+2p+3)(2n+2p+5)} \{(2n+2p+1)(2n+2p+5) + 2(2n-2p+1)\varepsilon \\ &\quad - (2n-2p+1)(2n-2p+3)\varepsilon^2\} a_{2n}' = -\frac{\varepsilon^2}{4} \delta_{1,p} - \frac{1}{2} \{1 - (2p-1)\varepsilon\} \varepsilon l_{2p-2}' + 2p\varepsilon^2 l_{2p}' \\ &\quad + \frac{1}{2} \{1 + (2p+1)\varepsilon\} \varepsilon l_{2p+2}' + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+p}}{(2n-2p-1)(2n-2p+1)(2n-2p+3)(2n+2p+3)} \\ &\quad \times \{(2n-2p-1)(2n-2p+3) - 2(2n+2p+3)\varepsilon - (2n+2p+1)(2n+2p+3)\varepsilon^2\} m_{2n+1}' \quad (p \geq 1) \end{aligned} \right\} \dots (19)$$

$$\begin{aligned} b_{2p}' &= -\frac{1}{2} \varepsilon \delta_{0,p} - \{1 - (2p+1)\varepsilon\} l_{2p}' + (2p+1)\varepsilon l_{2p+2}' - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+p}(2p+1)}{(2n-2p+1)(2n+2p+3)(2n+2p+5)} \\ &\quad \times \{(2n+2p+5) - (2n-2p+1)\varepsilon\} a_{2n}' + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+p}(2p+1)}{(2n-2p-1)(2n-2p+1)(2n+2p+3)} \\ &\quad \times \{(2n-2p-1) - (2n+2p+3)\varepsilon\} m_{2n+1}' \dots (20) \end{aligned}$$

where $\delta_{i,j}$ is Kronecker's delta and

$$a_{2n}' = a_{2n}/\lambda^{2n+3}, \quad b_{2n}' = b_{2n}/\lambda^{2n+2}, \quad l_{2n}' = \lambda^{2n} l_{2n}, \quad m_{2n+1}' = \lambda^{2n+1} m_{2n+1} \dots (21)$$

$$l_{2n}' = \sum_{s=0}^{\infty} \{2^n \alpha_{2s}' a_{2s}' + 2^n \beta_{2s}' b_{2s}'\}, \quad m_{2n+1}' = \sum_{s=0}^{\infty} \{2^{n+1} \kappa_{2s}' a_{2s}' + 2^{n+1} \mu_{2s}' b_{2s}'\} \dots (22)$$

$$\left. \begin{aligned} 2^n \alpha_{2s}' &= \lambda^{2n+2s+3} 2^n \alpha_{2s}, & 2^n \beta_{2s}' &= \lambda^{2n+2s+2} 2^n \beta_{2s}, & 2^{n+1} \kappa_{2s}' &= \lambda^{2n+2s+4} 2^{n+1} \kappa_{2s}, \\ 2^{n+1} \mu_{2s}' &= \lambda^{2n+2s+3} 2^{n+1} \mu_{2s} \end{aligned} \right\} \dots (23)$$

To solve these equations, we may apply a method of perturbation, in which ε is the perturbation parameter. As a first step in the solution, the integrals I_m and J_m are to be expanded into power series in ε ; that is

$$I_m = \int_0^{\infty} \frac{u^m}{A} du = \int_0^{\infty} \frac{u^m}{\sinh 2u} \left(1 - \frac{2\varepsilon u}{\sinh 2u}\right)^{-1} du = \frac{1}{2^{m+1}} \sum_{s=0}^{\infty} \varepsilon^s \int_0^{\infty} \frac{x^{m+s}}{(\sinh x)^{s+1}} dx \dots (24)$$

Therefore, we obtain

$$I_m = \sum_{s=0}^{\infty} i_{m,s} \varepsilon^s \dots (25)$$

where

$$\left. \begin{aligned} i_{m,0} &= \frac{m!}{2^m} \left(1 - \frac{1}{2^{m+1}}\right) S_{m+1}, & i_{m,1} &= \frac{(m+1)!}{2^{2m+1}} S_{m+1}, \\ i_{m,s} &= \frac{(m+s)(m+s-1)}{s(s-1)} i_{m,s-2} - \frac{4(s-1)}{s} i_{m+2,s-2}, & (s \geq 2) \\ S_m &= \sum_{n=0}^{\infty} n^{-m} \end{aligned} \right\} \dots (26)$$

In a similar way, we have

$$J_m = \sum_{s=0}^{\infty} j_{m,s} \varepsilon^s \dots (27)$$

where

$$\left. \begin{aligned} j_{m,0} &= \frac{m!}{2^{m+1}} S_{m+1}, & j_{m,1} &= \frac{(m+1)!}{2^m} \left\{ S_{m+1} - S_{m+2} - \frac{1}{2^{m+1}} \left(S_{m+1} - \frac{1}{2} S_{m+2} \right) \right\}, \\ j_{m,s} &= \frac{(m+s)(m+s-1)}{s(s-1)} j_{m,s-2} - \frac{4(m+s)}{s(s-1)} j_{m+1,s-2} - \frac{4(s-2)}{s-1} j_{m+2,s-2} & (s \geq 2) \end{aligned} \right\} \dots (28)$$

On Substituting Eq. (14), (25) and (27) into Eq. (23), we get

$$\left. \begin{aligned} 2^n \alpha_{2p}' &= \lambda^{2n+2p+3} \sum_{s=1}^{\infty} 2^n \alpha_{2p}^{(s)} \varepsilon^s, & 2^n \beta_{2p}' &= \lambda^{2n+2p+2} \sum_{s=0}^{\infty} 2^n \beta_{2p}^{(s)} \varepsilon^s, \\ 2^{n+1} \kappa_{2p}' &= \lambda^{2n+2p+4} \sum_{s=0}^{\infty} 2^{n+1} \kappa_{2p}^{(s)} \varepsilon^s, & 2^{n+1} \mu_{2p}' &= \lambda^{2n+2p+3} \sum_{s=1}^{\infty} 2^{n+1} \mu_{2p}^{(s)} \varepsilon^s \end{aligned} \right\} \dots (29)$$

where

$$\left. \begin{aligned} {}^{2n}\beta_{2p}^{(0)} &= -(i_{2n+2p+1}, 0 + j_{2n+2p+1}, 0) / (2n)! (2p+1)!, \\ {}^{2n+1}\kappa_{2p}^{(0)} &= -(i_{2n+2p+3}, 0 + j_{2n+2p+3}, 0) / (2n+1)! (2p+2)! \end{aligned} \right\} \dots\dots\dots (30)$$

$$\left. \begin{aligned} {}^{2n}\alpha_{2p}^{(s)} &= -2i_{2n+2p+3, s-1} / (2n)! (2p+2)!, \\ {}^{2n}\beta_{2p}^{(s)} &= -(i_{2n+2p+1, s} + 2i_{2n+2p+2, s-1} + j_{2n+2p+1, s}) / (2n)! (2p+1)!, \\ {}^{2n+1}\kappa_{2p}^{(s)} &= -(i_{2n+2p+3, s} - 2i_{2n+2p+4, s-1} - j_{2n+2p+3, s}) / (2n+1)! (2p+2)!, \\ {}^{2n+1}\mu_{2p} &= 2i_{2n+2p+3, s-1} / (2n+1)! (2p+1)! \end{aligned} \right\} \dots\dots\dots (31) \quad (s \geq 1)$$

We then seek a solution to Eq.(19) and (20) in the form

$$a_{2n}' = {}^{(0)}a_{2n} + \sum_{r=1}^{\infty} {}^{(r)}a_{2n}, \quad b_{2n}' = {}^{(0)}b_{2n} + \sum_{r=1}^{\infty} {}^{(r)}b_{2n} \dots\dots\dots (32)$$

where

$${}^{(r)}a_{2n} = \sum_{s=2}^{\infty} {}^{(r)}a_{2n}^{(s)} \epsilon^s, \quad {}^{(r)}b_{2n} = \sum_{s=1}^{\infty} {}^{(r)}b_{2n}^{(s)} \epsilon^s \dots\dots\dots (33)$$

In a similar way

$$l_{2n}' = \sum_{r=1}^{\infty} {}^{(r)}l_{2n}, \quad m_{2n+1}' = \sum_{r=1}^{\infty} {}^{(r)}m_{2n+1} \dots\dots\dots (34)$$

where

$$\left. \begin{aligned} {}^{(r)}l_{2n} &= \sum_{s=1}^{\infty} {}^{(r)}l_{2n}^{(s)} \epsilon^s, \quad {}^{(r)}m_{2n+1} = \sum_{s=2}^{\infty} {}^{(r)}m_{2n+1}^{(s)} \epsilon^s, \\ {}^{(r)}l_{2n}^{(s)} &= \sum_{p=0}^{\infty} \left[\lambda^{2n+2p+4} \sum_{u=1}^{s-2} {}^{2n}\alpha_{2p}^{(u)} ({}^{(r-1)}a_{2p}^{(s-u)} + \lambda^{2n+2p+2} \sum_{u=0}^{s-1} {}^{2n}\beta_{2p}^{(u)} ({}^{(r-1)}b_{2p}^{(s-u)}) \right] \\ {}^{(r)}m_{2n+1}^{(s)} &= \sum_{p=0}^{\infty} \left[\lambda^{2n+2p+4} \sum_{u=0}^{s-2} {}^{2n+1}\kappa_{2p}^{(u)} ({}^{(r-1)}a_{2p}^{(s-u)} + \lambda^{2n+2p+3} \sum_{u=1}^{s-1} {}^{2n+1}\mu_{2p}^{(u)} ({}^{(r-1)}b_{2p}^{(s-u)}) \right] \end{aligned} \right\} \dots\dots\dots (35)$$

where ${}^{(r)}a_{2n}^{(s)}$, ${}^{(r)}b_{2n}^{(s)}$, ${}^{(r)}l_{2n}^{(s)}$, and ${}^{(r)}m_{2n+1}^{(s)}$ are independent of ϵ . Substituting these equations into Eq. (19) and (20), and then equating the coefficients of the same power of ϵ on both sides, we obtain

$$\sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(2)} = -\pi/16, \quad \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(2)} = -(\pi/16)\delta_{1,p} \quad (p \geq 1) \dots\dots\dots (36)$$

$$\sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(3)} = \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{I} {}^{(0)}a_{2k}^{(2)}, \quad \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(3)} = \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{I} {}^{(0)}a_{2k}^{(2)} \quad (p \geq 1) \dots\dots\dots (37)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(s)} &= \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{I} ({}^{(0)}a_{2k}^{(s-1)} + {}^{(0)}a_{2k}^{(s-2)}), \\ \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{I} {}^{(0)}a_{2n}^{(s)} &= \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{I} {}^{(0)}a_{2k}^{(s-1)} - \sum_{k=0}^{\infty} (2k-2p+3) {}^{2p}\Psi_{2k} \text{I} {}^{(0)}a_{2k}^{(s-2)} \quad (p \geq 1) \end{aligned} \right\} \dots\dots\dots (38) \quad (s \geq 4)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(r)}a_{2n}^{(2)} &= \frac{\pi}{8} (-{}^{(r)}l_0^{(1)} + {}^{(r)}l_2^{(1)}) + \sum_{k=0}^{\infty} {}^0\phi_{2k} \text{I} {}^{(r)}m_{2k+1}^{(2)}, \\ \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{I} {}^{(r)}a_{2n}^{(2)} &= \frac{\pi}{8} (-{}^{(r)}l_{2p-2}^{(1)} + {}^{(r)}l_{2p+2}^{(1)}) + \sum_{k=0}^{\infty} {}^{2p}\phi_{2k} \text{I} {}^{(r)}m_{2k+1}^{(2)} \quad (p \geq 1) \end{aligned} \right\} \dots\dots\dots (39) \quad (r \geq 1)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(r)}a_{2n}^{(3)} &= \frac{\pi}{8} (-{}^{(r)}l_0^{(2)} + {}^{(r)}l_0^{(1)} + {}^{(r)}l_2^{(2)} + {}^{(r)}l_2^{(1)}) + \sum_{k=0}^{\infty} {}^0\phi_{2k} \text{I} {}^{(r)}m_{2k+1}^{(3)} \\ &\quad - \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{II} {}^{(r)}m_{2k+1}^{(2)} + \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{I} {}^{(r)}a_{2k}^{(2)}, \\ \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{I} {}^{(r)}a_{2n}^{(3)} &= \frac{\pi}{8} \{ -{}^{(r)}l_{2p-2}^{(2)} + (2p-1) {}^{(r)}l_{2p-2}^{(1)} + 4p {}^{(r)}l_{2p}^{(1)} + {}^{(r)}l_{2p+2}^{(2)} + (2p+1) {}^{(r)}l_{2p+2}^{(1)} \} \\ &\quad + \sum_{k=0}^{\infty} {}^{2p}\phi_{2k} \text{I} {}^{(r)}m_{2k+1}^{(3)} - \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{II} {}^{(r)}m_{2k+1}^{(2)} + \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{I} {}^{(r)}a_{2k}^{(2)} \quad (p \geq 1) \end{aligned} \right\} \dots\dots\dots (40) \quad (r \geq 1)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} {}^0\phi_{2n} \text{I} {}^{(r)}a_{2n}^{(s)} &= \frac{\pi}{8} (-{}^{(r)}l_0^{(s-1)} + {}^{(r)}l_0^{(s-2)} + {}^{(r)}l_2^{(s-1)} + {}^{(r)}l_2^{(s-2)}) + \sum_{k=0}^{\infty} {}^0\phi_{2k} \text{I} m_{2k+1}^{(s)} \\ &\quad - \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{II} ({}^{(r)}m_{2k+1}^{(s-1)} + {}^{(r)}m_{2k+1}^{(s-2)}) + \sum_{k=0}^{\infty} {}^0\Psi_{2k} \text{I} ({}^{(r)}a_{2k}^{(s-1)} + {}^{(r)}a_{2k}^{(s-2)}), \end{aligned} \right\}$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{ I } {}^{(r)}a_{2n}^{(s)} &= \frac{\pi}{8} \{ - {}^{(r)}l_{2p-2}^{(s-1)} + (2p-1) {}^{(r)}l_{2p-2}^{(s-2)} + 4p {}^{(r)}l_{2p}^{(s-2)} + {}^{(r)}l_{2p+2}^{(s-1)} \\ &+ (2p+1) {}^{(r)}l_{2p+2}^{(s-2)} \} + \sum_{k=0}^{\infty} {}^{2p}\phi_{2k} \text{ I } {}^{(r)}m_{2k+1}^{(s)} - \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{ II } {}^{(r)}m_{2k+1}^{(s-1)} \\ &- \sum_{k=0}^{\infty} (2k+2p+1) {}^{2p}\Psi_{2k} \text{ II } {}^{(r)}m_{2k+1}^{(s-2)} + \sum_{k=0}^{\infty} {}^{2p}\Psi_{2k} \text{ I } {}^{(r)}a_{2k}^{(s-1)} - \sum_{k=0}^{\infty} (2k-2p+3) {}^{2p}\Psi_{2k} \text{ I } {}^{(r)}a_{2k}^{(s-2)} \end{aligned} \right\} \begin{aligned} &\dots (41) \\ &(r \geq 1) \\ &(s \geq 4) \\ &(p \geq 1) \end{aligned}$$

$$\left. \begin{aligned} {}^{(0)}b_{2p}^{(1)} &= -\frac{1}{2} \delta_{0,p}, \quad {}^{(0)}b_{2p}^{(2)} = -\frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) {}^{2p}\phi_{2n} \text{ I } {}^{(0)}a_{2n}^{(2)} = \frac{1}{4} \delta_{0,p} + \frac{3}{4} \delta_{1,p}, \\ {}^{(0)}b_{2p}^{(s)} &= -\frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) {}^{2p}\phi_{2n} \text{ I } {}^{(0)}a_{2n}^{(s)} + \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) (2n+2p+7) {}^{2p+2}\Psi_{2n} \text{ I } {}^{(0)}a_{2n}^{(s-1)} \end{aligned} \right\} \dots (42) \quad (s \geq 3)$$

$$\left. \begin{aligned} {}^{(r)}b_{2p}^{(1)} &= - {}^{(r)}l_{2p}^{(1)}, \quad {}^{(r)}b_{2p}^{(2)} = - {}^{(r)}l_{2p}^{(2)} + (2p+1) {}^{(1)}l_{2p}^{(1)} + (2p+1) {}^{(r)}l_{2p+2}^{(1)} \\ &- \frac{4}{\pi} \sum_{n=0}^{\infty} {}^{2p}\phi_{2n} \text{ I } (2p+1) {}^{(r)}a_{2n}^{(2)} + \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) {}^{2p}\phi_{2n} \text{ I } {}^{(r)}m_{2n+1}^{(2)}, \\ {}^{(r)}b_{2p}^{(s)} &= - {}^{(r)}l_{2p}^{(s)} + (2p+1) {}^{(r)}l_{2p}^{(s-1)} + (2p+1) {}^{(r)}l_{2p+2}^{(s-1)} - \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) {}^{2p}\phi_{2n} \text{ I } {}^{(r)}a_{2n}^{(s)} \\ &+ \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) (2n+2p+7) {}^{2p+2}\Psi_{2n} \text{ I } {}^{(r)}a_{2n}^{(s-1)} + \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) {}^{2p}\phi_{2n} \text{ I } {}^{(r)}m_{2n+1}^{(s)} \\ &+ \frac{4}{\pi} \sum_{n=0}^{\infty} (2p+1) (2n-2p-3) {}^{2p+2}\Psi_{2n} \text{ II } {}^{(r)}m_{2n+1}^{(s-1)} \end{aligned} \right\} \begin{aligned} &\dots (43) \\ &(r \geq 1) \\ &(s \geq 3) \end{aligned}$$

where

$$\left. \begin{aligned} {}^{2p}\phi_{2n} \text{ I} &= \frac{(-1)^{n+p}}{(2n-2p+1)(2n+2p+3)} \quad (p \geq 0), \\ {}^{0}\Psi_{2n} \text{ I} &= \frac{(-1)^n}{2(2n+3)(2n+5)}, \quad {}^{2p}\Psi_{2n} \text{ I} = \frac{(-1)^{n+p+1}}{(2n+2p+1)(2n+2p+3)(2n+2p+5)} \quad (p \geq 1) \\ {}^{0}\Psi_{2n} \text{ II} &= \frac{(-1)^n}{2(2n-1)(2n+1)}, \quad {}^{2p}\Psi_{2n} \text{ II} = \frac{(-1)^{n+p}}{(2n-2p-1)(2n-2p+1)(2n-2p+3)} \quad (p \geq 1) \end{aligned} \right\} \dots (44)$$

With these relations, the coefficients ${}^{(r)}a_{2n}^{(s)}$ and ${}^{(r)}b_{2n}^{(s)}$ in the Series (33) can be determined successively for $r=0, 1, 2, \dots$ and $s=1, 2, 3, \dots$. In particular, the coefficients ${}^{(0)}a_{2n}^{(s)}$ and ${}^{(0)}b_{2n}^{(s)}$, which correspond to the solution for the semi-infinite plate, notched by a single semicircular notch, were already obtained by the authors⁽³⁾.

It may be noted that in the infinite sets of Eqs. (36)~(41) the coefficients of ${}^{(r)}a_{2n}^{(s)}$ are common for all values of r and s . Hence, if we solve these equations once, the solutions in successive cycles are obtained only by the calculations about the constant terms.

The foregoing solution is valid in so far as the Series (32) is convergent. For brevity, no proof of convergence will be pursued here, but it will be illustrated by the numerical examples which follow.

4. Numerical examples

The foregoing solution is worked out in some detail. For the three cases of the ratio $a/b=0.1, 0.2$, and 0.3 , the calculations of ${}^{(r)}a_{2n}^{(s)}$ and ${}^{(r)}b_{2n}^{(s)}$ are carried out until $r=3$ and $s=7$, but their values are not shown here for brevity. With these values, it is now rather straightforward to calculate the deflection at the middle plane of the strip and the bending moments in the neighbourhood of the notches.

On working out the values of M_r and V_r at

Fig. 2 Bending couple M_θ at the rim of the notch

Fig. 3 Bending couple M_y on the straight edge

the rim of the notch, by using the values of coefficients obtained, the greatest values found were $(M_r)_{\theta=\pi/2}=0.02M_0$, $(M_r)_{\theta=\pi/2}=0.02M_0$, and $(M_r)_{\theta=\pi/2}=0.02M_0$ for $a/b=0.1, 0.2$, and 0.3 , respectively. Therefore the values of coefficients calculated are thought to be sufficient to get the stresses with a sufficient degree of approximation.

The values of (i) M_θ on the edges of a semi-circular notch, (ii) M_y on the straight edges, (iii) M_x and M_y along the line of symmetry, ox , are calculated and shown graphically in Figs. 2, 3, and 4 respectively.

The maximum bending couple occurs at the apex of the notch. Since $M_r=0$ on the same edge, the maximum couple given by

$$M_{\max}/M_0 = 1 - 2(1-\varepsilon) \sum_{s=1}^{\infty} M_s' \varepsilon^{s-1} \dots \dots \dots (45)$$

$$\left. \begin{aligned} M_1' &= {}^{(0)}b_0^{(1)} + \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} {}^{(r)}b_{2n}^{(1)} + \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} {}^{(r)}l_{2n}^{(1)}, \\ M_2' &= {}^{(0)}b_0^{(2)} + {}^{(0)}b_2^{(2)} + \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} {}^{(r)}a_{2n}^{(2)} + \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} {}^{(r)}b_{2n}^{(2)} + \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} ({}^{(r)}l_{2n}^{(2)} + {}^{(r)}m_{2n+1}^{(2)}), \\ M_s' &= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} ({}^{(r)}a_{2n}^{(s)} + {}^{(r)}b_{2n}^{(s)}) + \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} ({}^{(r)}l_{2n}^{(s)} + {}^{(r)}m_{2n+1}^{(s)}) \end{aligned} \right\} \dots \dots \dots (46) \quad (s \geq 3)$$

Fig. 5 Stress concentration factor k_b versus ε

Fig. 6 Stress concentration factor k_b versus λ

Fig. 4 Bending couples M_x and M_y on the minimum section

Fig. 7 Distribution of the deflection on the straight edge

Fig. 8 Distribution of the deflection on the minimum section

By use of this equation, the stress concentration factor $k_b = (1-\lambda)M_{\max}/M_0$ is calculated and plotted versus ϵ in Fig. 5. The stress concentration factor k_b is also plotted against λ in Fig. 6. When λ tends to unity, the stress concentration factor k_b may be given approximately by the results obtained for the case of two symmetrically disposed hyperbolic notches. These approximate values also are shown by dotted lines in Fig. 6.

The distribution of the deflection at the middle plane of the strip is now calculated and shown in Figs. 7, 8, and 9. In these figures, the deflection at the point $O(0,0)$ is assumed to be zero in every case.

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Fig. 9 Distribution of the deflection at the rim of the notch

throughout the progress of the present investigation.

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Stresses in a Circular Disk with an Eccentric Circular Hole Fixed at its Center and under a Tangential Force on the Outer Edge*

By Kazyu MIYAO**

In a previous paper, the author has reported "Stresses in a Circular Disk with an Eccentric Circular Hole under Radial Forces." In this paper, the stresses in a circular disk containing an eccentric circular hole fixed at its center and under the action of a tangential force were analysed. The method of solution was the same as that used in the previous paper, and the bipolar coordinates were used in the solution. The complete stress-function in the disk with a hole is constructed in the form, in which the basic stress-function is added to the auxiliary one. From the results obtained, the solution of the problem in a semi-infinite plate under a concentrated tangential force on its straight edge and containing an unstressed circular hole was derived.

Symbols

In this paper the following symbols are used:

- x, y : rectangular coordinates
 r, θ, r, θ : polar coordinates
 a : real positive length
 α, β : orthogonal curvilinear coordinates

$1/h$: stretch ratio

$\widehat{\alpha\alpha}, \widehat{\beta\beta}$: normal stress components in curvilinear coordinates

$\widehat{\alpha\beta}$: shearing stress in curvilinear coordinates

χ : complete stress function

χ_0 : basic stress function

χ_1 : auxiliary stress function

$\rho, \sigma, \tau, \alpha_1, \alpha_2$: constants

F : applied force, ν : Poisson's ratio,

ξ : $\alpha - \alpha_2$, ξ_1 : $\alpha_1 - \alpha_2$

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