Fig. 12 Behavior of the longitudinal load

car enters the curve and continues throughout the curve, as shown in Fig. 12(A), but in a few cases, it vanishes by every rotation of the wheel as shown by Fig. 12(B).

The magnitude of this load is exemplified in Fig. 13, the max. value hereby amounting even to 1.9 tons. The component depending on the tractive force is of course included in these values, but can be considered to be negligible for the short train under consideration. The direction of the loads for each wheel of the truck is as in Fig. 14, based on the analysis of the test results in the upbound and the downbound direction.

4. Conclusion

Reliable data for the lateral and vertical loads

Fig. 13 Magnitude of longitudinal load (Test A, Speed 20 km/hr)

Fig. 14 Direction of the longitudinal load on the wheel

on the wheel were obtained, and the behavior of the longitudinal load on the wheel was clarified for the first time. Thus, the complete information on the fundamental three loads, which is necessary for the dynamic analysis, has been gathered, and the authors intend to embark on the theoretical development of these results.

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The Critical Speed of a Shaft Rotating in Fluid*

By Syūsuke IIDA**

The critical speed of a shaft rotating in fluid is different from that in vacuum. In this paper the effects of fluid upon the critical speed of a shaft rotating in a casing are calculated and compared with the experimental results. Also, taking the gyroscopic effects into consideration, the frequency of the shaft is calculated.

1. Introduction

The vibrations of fluid in which a shaft of circular section vibrates laterally have been analysed by Prof. Kito⁽¹⁾, but they have not been analysed sufficiently in the case of rotational vibrations. The author has analysed the fluid motion in which a circular cylinder revolved eccentrically⁽²⁾. In this report, using the above results, the critical speed of a shaft and its circular

frequencies with the gyroscopic effects are calculated, and in regard to the former the experiments are carried out to verify the theoretical results.

2. Hydrodynamical force on the cylinder⁽²⁾

In Fig. 1, a cylinder of radius r_c which is rotating around its own axis O' with an angular velocity Ω , revolves in fluid around the fixed axis O with an angular velocity ω with an eccentricity d . The hydrodynamical force acting on the cylinder is divided into the component in the direction of the eccentricity and the one, perpendicular to it.

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These values per unit length are as follows :

Case in the wide region,

$$\left. \begin{aligned} \frac{P_L'}{(\rho_w \pi r_i^2) r_i \Omega^2} &= \left(1 - \frac{\omega}{\Omega}\right) \frac{\omega}{\Omega} \frac{d}{r_i} + 4(3B_2 + B_4) \frac{\omega}{\Omega} \frac{d}{r_i} \\ \frac{P_{D_i}'}{(\rho_w \pi r_i^2) r_i \Omega^2} &= \frac{48}{R_i} \left(14 \frac{\omega}{\Omega} \frac{d}{r_i} - 8A_1 - 3A_3\right) - 4(3A_2 + A_4) \frac{\omega}{\Omega} \frac{d}{r_i} \end{aligned} \right\} \dots\dots\dots (1)$$

Case in the cylindrical vessel,

$$\left. \begin{aligned} \frac{P_L'}{(\rho_w \pi r_i^2) r_i \Omega^2} &= \frac{d}{r_i} \left[\left\{ -1 + \left(\frac{r_i}{r_0}\right)^2 \right\} \left(\frac{\omega}{\Omega}\right)^2 + \left\{ 1 + 3\left(\frac{r_i}{r_0}\right)^2 \right\} \frac{\omega}{\Omega} - 2\left(\frac{r_i}{r_0}\right)^2 \right] / \left\{ 1 - \left(\frac{r_i}{r_0}\right)^2 \right\} \\ \frac{P_{D_i}'}{(\rho_w \pi r_i^2) r_i \Omega^2} &= \frac{1}{R} \left(\frac{r_0}{r_i}\right)^2 \left\{ \frac{6c}{(1-c)^3} (A_3 + 4A_5 + 10A_7) - 3 \left(\frac{1}{c} \frac{d}{r_0} - \frac{1}{a} \frac{r_i}{r_0} - \frac{1}{c^2} \frac{1}{a} \frac{r_i}{r_0} \right) \right\} \end{aligned} \right\} \dots\dots\dots (2)$$

where r_0 is the radius of the vessel and these are applicable to small d/r_i and $r_i/r_0 > 0.5$.

The above four equations are applicable to the case when $R_d = \omega d \cdot 2r_i/\nu$ is not so small (larger than 400). These forces have the magnitude that is open to discussion when r_i/r_0 is comparatively large, and have no effects when r_i/r_0 is small (including $r_0 = \infty$). In practical case, there is usually a casing around the shaft or the rotator, so we consider on such cases and calculate using Eq. (2).

When the eccentricity d is small, the hydrodynamical forces P_L' and P_{D_i}' are proportional to d , so they are written as

$$P_L' = m \xi d, \quad P_{D_i}' = m \eta d \dots\dots\dots (3)$$

where m is the mass of unit shaft length. Therefore, their y - and z -components are

$$\left. \begin{aligned} P_{Ly}' &= m \xi y, & P_{Lz}' &= m \xi z; \\ P_{Dy}' &= m \eta z, & P_{Dz}' &= -m \eta y \end{aligned} \right\} \dots\dots\dots (4)$$

Using Eq. (2),

$$\left. \begin{aligned} \xi &= \frac{\rho_w}{\rho} \left[\left\{ -1 + \left(\frac{r_i}{r_0}\right)^2 \right\} \omega^2 + \left\{ 1 + 3\left(\frac{r_i}{r_0}\right)^2 \right\} \omega \Omega \right. \\ &\quad \left. - 2\left(\frac{r_i}{r_0}\right)^2 \Omega^2 \right] / \left\{ 1 - \left(\frac{r_i}{r_0}\right)^2 \right\} \\ \eta &= \frac{\rho_w}{\rho} \frac{1}{R} \left(\eta_1 \frac{\omega}{\Omega} - \eta_2 \right) \Omega^2 \end{aligned} \right\} \dots\dots\dots (5)$$

where ρ is the density of the shaft material and η_1, η_2 are shown in Fig. 2.

3. Critical speed

For simplicity, it is assumed that the deflection line of shaft is parallel to the original center line and in the case of small eccentricity, the

Fig. 1

Fig. 2

resistance is so small that it can be neglected. So taking only P_L' into account, the equation of motion is

$$EI \frac{\partial^4 d}{\partial x^4} = -m \frac{\partial^2 d}{\partial t^2} + m\omega^2 d + P_L' \quad (6)$$

where E is Young's modulus and I , the moment of inertia. When the shaft whirls at the critical speed ω and displays no lateral vibrations, the term $m\partial^2 d/\partial t^2$ is omitted. In this case the solution is assumed to be expressed as

$$d = X(x) \cos \omega t \quad (7)$$

Here we put

$$\xi = \omega^2 k_\omega \quad (8)$$

and assume that the shaft is whirling under the constant relation between ω and Ω (that is, $\omega/\Omega = \text{constant}$). Then k_ω becomes constant for the constant r_i/r_0 and ρ/ρ_w . Introducing Eqs. (7) and (8) into Eq. (6), gives

$$\frac{d^4 X}{dx^4} = n^4 X, \quad n^4 = \frac{m\omega^2(1+k_\omega)}{EI} \quad (9)$$

From this, we have in accordance with the boundary conditions

$$\omega = n^2 \sqrt{\frac{EI}{m(1+k_\omega)}} \quad (10)$$

This critical speed in fluid is denoted by ω_{cw} and that in no fluid, by ω_c (that is, $k_\omega = 0$), then $\omega_c = n^2 \sqrt{EI/m}$ and consequently

$$\omega_{cw}/\omega_c = (1+k_\omega)^{-1/2} \quad (11)$$

From this equation, it is shown that when k_ω is positive, that is, the force acts in the direction that increases the eccentricity, the critical speed in fluid is lower than that in no fluid. Fig. 3 shows this relation.

From Eq. (5), the value of Ω/ω which makes k_ω maximum for a certain ρ/ρ_w is $0.75 + 0.25(r_0/r_i)^2$, so at $r_i/r_0 = 1$, it becomes $\Omega/\omega = 1$, where the drop of the critical speed becomes maximum. Generally speaking, the critical speed in fluid drops in the range $4(r_i/r_0)^2/[1-(r_i/r_0)^2] > \omega/\Omega > 0.5$ and rises in $\omega/\Omega < 0.5$. At $k_\omega = -1$, ω_{cw} becomes infinity, so there is no fear of the critical speed being attained.

For example, the relations between r_i/r_0 and ω_{cw}/ω_c for $\omega/\Omega = 0.5, 1.0, 2.0$ and $\rho/\rho_w = 7.8$ are shown in Fig. 4. In the case of $\omega/\Omega > 0.5$, the critical speed drops as r_i/r_0 tends to unity and at the limit where r_i/r_0 reaches unity, it becomes zero. But in practice, as r_i/r_0 tends to unity, a slight deflection makes the inner cylinder touch the outer vessel and this is an entirely different problem.

As the ratio of density of shaft material to that of fluid ρ/ρ_w becomes small, k_ω becomes large and ω_{cw}/ω_c becomes small, that is, the critical speed of a shaft of light material is lower than

Fig. 3

Fig. 4

that of heavy material under the other conditions being equal.

As it is generally believed that the critical speed occurs at $\omega = \Omega$, it is concluded that it drops in fluid.

To verify the above results, the following experiments are carried out. As shown in Fig. 5, a brass bar of 20 mm diameter is used as an inner cylinder and a transparent acrylic resin pipe (three sizes of 22.7 mm, 25.7 mm and 31.3 mm diameter), as an outer vessel, and water is filled in the gap between them. The bar is supported vertically at both ends with the self-aligning ball bearings and the amplitude is measured at its mid point and the speed of bar is measured at the top by a tachometer. The method of amplitude measurement is as follows. When the light (Zircon lamp—point source—is used) is thrown from the opposite side making it dark around the apparatus, only the

edge of bar glitters and is photographed as a bright line. If a bar vibrates, a bright line becomes a wide band as shown in Fig. 6 which is measured as twice the amplitude. In the Figure, "scale" is a photograph of a scale taken specially with the same focal length and its one graduation which is the unit of amplitude, is 1 mm of the scale. (As a light is emitted from about a point source and refracted in water, 1 mm amplitude of bar is, strictly speaking, unequal to one graduation of "scale" photograph. That is, though the amplitude on the photograph is equal to one graduation of "scale" photograph, it does not mean truly 1 mm. Although these errors are extremely small, unit of mm is omitted to avoid misinterpretation.)

Fig. 7 shows the relations between speed and amplitude obtained from the experiments about three outer vessels. It is shown that the theoretical results agree well with the experimental results.

In these experiments at the value of $\omega = 188.5$ rad/sec, $r_i = 10$ mm, $d > 0.12$ mm and $\nu = 1.146 \times 10^{-6}$ m²/s (at 15°C water); R_d becomes larger than 400, so the theory is now applicable to these experiments.

Fig. 5 Experimental apparatus

4. Vibration under the consideration of gyroscopic effects

Now, we study the vibration of a shaft with an infinitely distributed rotator, taking the gyroscopic effects into consideration. As shown in Fig. 8, we take the center line at no deflection as x -axis and vertically upwards, as z -axis. Taking a shaft element dx , let the components of its deflection be y and z . Let the angles between the shaft and the planes zOx , xOy be β , γ respectively; $\dot{\beta}$ and $\dot{\gamma}$ mean the angular velocities around the z - and y -axis respectively. Then the equations of motion in the direction of y - and z -axis are as follows:

$$m dx \ddot{y} = P_y, \quad m dx \ddot{z} = P_z \quad \dots \dots \dots (12)$$

where P_y and P_z are the external forces and m ,

(case of $r_i/r_0 = 0.88$)

Fig. 6 Amplitude measurement

the mass per unit shaft length including the rotator. The equations of motion around z - and y -axis are as follows :

$$J'dx\ddot{\beta} + Jdx\dot{\Omega}\dot{\gamma} = M_\beta, \quad J'dx\ddot{\gamma} - Jdx\dot{\Omega}\dot{\beta} = M_\gamma \quad (13)$$

where J is the mass polar moment of inertia of unit shaft length and J' , the mass moment of inertia about its diameter ($2J' = J$), and M_β , M_γ , the external moments. In Eq. (13) the second terms are the gyromoments. Furthermore, we put $\dot{\beta} = \dot{y}'$, $\dot{\gamma} = \dot{z}'$ (' indicates the derivative about x); and y - and z -components of the shear force at section x as S_y and S_z respectively, those at section $x+dx$ as S_y' and S_z' ; and the bending moment at the same section as M_y , M_z , M_y' and M_z' . Then the external forces and moments are expressed as follows, including hydrodynamical forces,

$$\left. \begin{aligned} P_y &= S_y' - S_y + m dx \omega^2 y + m dx \xi y + m dx \eta z \\ P_z &= S_z' - S_z + m dx \omega^2 z + m dx \xi z - m dx \eta y \\ M_\beta &= M_y' - M_y + S_y dx \\ M_\gamma &= M_z' - M_z + S_z dx \end{aligned} \right\} \quad (14)$$

Writing $\omega^2 + \xi = \xi'$, $S_y' - S_y = dS_y$, $M_y' - M_y = EI y'''' dx$ and introducing the above relations into Eqs. (12) and (13), we obtain

$$\left. \begin{aligned} m\ddot{y} &= (dS_y/dx) + m\xi'y + m\eta z \\ m\ddot{z} &= (dS_z/dx) + m\xi'z - m\eta y \\ J'(\ddot{y}' + 2\Omega\dot{z}') - EI y'''' &= S_y \\ J'(\ddot{z}' - 2\Omega\dot{y}') - EI z'''' &= S_z \end{aligned} \right\} \quad (15)$$

Eliminating S_y and S_z in the above equations, we obtain

$$\left. \begin{aligned} -m\ddot{y} + J'(\ddot{y}'' + 2\Omega\dot{z}'') - EI y'''' + m\xi'y + m\eta z &= 0 \\ -m\ddot{z} + J'(\ddot{z}'' - 2\Omega\dot{y}'') - EI z'''' + m\xi'z - m\eta y &= 0 \end{aligned} \right\} \quad (16)$$

in which the relations

$$y = B \sin kx e^{i\lambda t}, \quad z = D \sin kx e^{i\lambda t} \quad (17)$$

and

$$J'k^2/m = h, \quad EI k^4/m = \omega_K^2 \quad (18)$$

are introduced. Then they become

$$\left. \begin{aligned} \{(1+h)\lambda^2 - \omega_K^2 + \xi'\} B + (\eta - 2ih\lambda\Omega) D &= 0 \\ -(\eta - 2ih\lambda\Omega) B + \{(1+h)\lambda^2 - \omega_K^2 + \xi'\} D &= 0 \end{aligned} \right\} \quad (19)$$

Eliminating B and D , we obtain

$$\{(1+h)\lambda^2 - \omega_K^2 + \xi'\}^2 + (\eta - 2ih\lambda\Omega)^2 = 0 \quad (20)$$

and solving this about λ , have

$$\left. \begin{aligned} \lambda_1 &= \{h\Omega + \sqrt{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi' + i\eta)}\} / (1+h) \\ \lambda_2 &= \{h\Omega - \sqrt{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi' + i\eta)}\} / (1+h) \\ \lambda_3 &= \{-h\Omega + \sqrt{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi' - i\eta)}\} / (1+h) \\ \lambda_4 &= \{-h\Omega - \sqrt{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi' - i\eta)}\} / (1+h) \end{aligned} \right\} \quad (21)$$

$$\frac{B}{D} = \frac{(1+h)\lambda^2 - \omega_K^2 + \xi'}{\eta - 2ih\lambda\Omega} = e^{\pm(\pi/2)i} \left(\begin{array}{l} + : \text{for } \lambda_1, \lambda_2 \\ - : \text{for } \lambda_3, \lambda_4 \end{array} \right) \quad (22)$$

Here, we put as

$$\left. \begin{aligned} \mu_1 &= [h\Omega + \sqrt{(1+\varepsilon)\{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi')\}}] / (1+h) \\ -\mu_2 &= [h\Omega - \sqrt{(1+\varepsilon)\{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi')\}}] / (1+h) \\ \nu &= \sqrt{\varepsilon\{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi')\}} / (1+h) \\ 2\varepsilon &= \sqrt{1 + \left\{ \frac{\eta(1+h)}{h^2\Omega^2 + (1+h)(\omega_K^2 - \xi')} \right\}^2} - 1 \end{aligned} \right\} \quad (23)$$

Fig. 7 Experimental results

Fig. 8

then Eq. (21) is written as

$$\left. \begin{aligned} \lambda_1 &= \mu_1 + i\nu, \quad \lambda_2 = -\mu_2 + i\nu, \\ \lambda_3 &= \mu_2 + i\nu, \quad \lambda_4 = -\mu_1 + i\nu \end{aligned} \right\} \dots\dots\dots (24)$$

and Eq. (17) becomes with the aid of Eqs. (22) and (24)

$$\left. \begin{aligned} y &= 2D \sin kx e^{-\nu t} (-\sin \mu_1 t + \sin \mu_2 t) \\ z &= 2D \sin kx e^{-\nu t} (\cos \mu_1 t + \cos \mu_2 t) \end{aligned} \right\} \dots\dots\dots (25)$$

The term of circular frequency μ_1 indicates the forward precessional motion and μ_2 , the backward precessional motion.

If ε is neglected as η is small, it becomes

$$\left. \begin{aligned} \frac{\mu_1}{\Omega} &= \frac{1}{1+h} \left\{ \sqrt{h^2 + (1+h) \frac{\omega_K^2 - \xi'}{\Omega^2}} + h \right\} \\ \frac{\mu_2}{\Omega} &= \frac{1}{1+h} \left\{ \sqrt{h^2 + (1+h) \frac{\omega_K^2 - \xi'}{\Omega^2}} - h \right\} \end{aligned} \right\} \dots\dots\dots (26)$$

As ξ' becomes large, that is, the force in the direction increasing the eccentricity acts, μ_1 and μ_2 become small. The limit of the stable vibration is obtained from $\omega_K^2 = \xi'$ and coincides with Eq. (11). Eq. (26) is shown in Fig. 9. As previously noted, when ω/Ω is assumed to be constant, the shaft whirls under the condition $\xi'/\Omega^2 = \text{constant}$. So when ω_K and Ω are given, we first assume ω and then obtain μ_1 and μ_2 along the program, $\omega_K/\Omega \rightarrow \xi'/\Omega^2 \rightarrow \sqrt{\omega_K^2 - \xi'}/\Omega \rightarrow h \rightarrow \mu_{1,2}/\Omega$. If the obtained value of μ_1 is equal to ω and that of μ_2 , to $-\omega$; these μ_1 and μ_2 are the angular velocities of whirl.

5. Conclusion

The critical speed of a shaft rotating in fluid varies according to the ratio of the angular velocity of revolution to that of rotation, ω/Ω , in contrast to the case of no fluid. In the case when a shaft whirls in a casing, the following results are obtained.

- (1) It drops for the range $4(r_i/r_0)^2 / \{1 - (r_i/r_0)^2\} > \omega/\Omega > 0.5$ and has a minimum value at $\omega = \Omega$ when $r_i/r_0 = 1$.
- (2) It drops as r_i/r_0 tends to unity at a certain value of ω/Ω in the above range.
- (3) It rises for $\omega/\Omega < 0.5$.
- (4) It drops as the ratio of the density of shaft material to that of fluid becomes small.

Fig. 9

In the rotational vibration of $\omega = \Omega$ which occurs generally, it drops. And this is verified by the experiments. Taking the gyroscopic effects into consideration, the frequency drops as the hydrodynamical force acts in the direction that increases the eccentricity. So when the vibrations are composed of forward and backward precession, the frequency of the former drops and that of the latter rises.

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