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A Basis for Computer Generation of the Drawings of Machine Parts*

(Part 1. Topological Generating Rules of Convex Polyhedra
corresponding to Dual Graphs having Hamiltonian Cycles)

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It is pointed out that, in making automatic the processes of machine design by means of man-machine interaction, the machine lacks the capability of generating the shape of machine parts. To overcome this difficulty in principle, the paper presents a topological method of generating convex polyhedra. As an instruction from man the edge-vertex table $(k, \lambda_k | k=3, 4, \dots, r, \text{ where } k: \text{number of edges gathering into a vertex, } \lambda_k: \text{number of } k\text{-edged vertex})$ is proposed. When the machine receives the edge-vertex table, it generates all of the mutually different linear dual graphs on convex polyhedra, where the graphs have Hamiltonian cycles. The fundamental theory for the above process is contained.

1. Introduction

Artificial intelligences have been developed in various branches of science and technology. As an example, we have gradually become enthusiastic to make automatic the processes of machine design.^{(1)~(3)} It seems for the author that the present automatic design techniques lack the ability to generate the drawings of machine structures or machine parts. This fact may retard the development of the field. A somewhat standardized machine part allows the drawings to be filed up on a magnetic tape. Infinitely many machine parts will, however, exist which are different from each other in shape and function. Drawings of these parts could not be filed in any sense. In designing these particular parts, if it is necessary for man to write an ordinary FORTRAN program or to input lines by means of lightpen, the existence of the automaton will be almost of no value. It will be valuable to overcome the inconvenience, if the automaton itself can generate the drawings of a machine part with requested shape, on the basis of a general program, and can receive any correction order from man, such as "chamfer" or "round the edge."

For a basis of above technique, a topological generating rule of polyhedra is developed. As the command for describing the desired polyhedron from man to the automaton, an edge vertex table is proposed. When the automaton receives the command, it generates all of the convex polyhedra whose dual graph has Hamiltonian cycles. The paper presents basic rules for above procedure. Counting up problem of convex polyhedra seems still not to have been solved. This paper does not solve the problem but only presents its numerical technique.

2. Some new definitions and known theorems

In this chapter some definitions and known theorems for the polyhedra and their graphs are presented.

2.1 Convex polyhedra

A polyhedron is called positive if its interior is solid and called negative if its exterior is solid. A polyhedron is called convex if it can be convex.

2.2 Simple polyhedra and complex polyhedra

A polyhedron is called simple if it consists of a set of connected edges. A polyhedron is called complex if it consists of a family of sets $\{S_i | i=1, 2, \dots, n; n>1\}$, where S_i and S_j are disconnected but S_i is

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a set of connected edges. Discussions in this paper are limited to simple polyhedra.

2.3 The name of vertices

A vertex of a polyhedron is called k edged vertex if it has k edges. k is called the degree of the vertex. Three edged vertex is called α -vertex, four edged β -vertex and so on as in Table 1.

2.4 The name of polyhedra

A polyhedron is called α -polyhedron if it consists of α -vertices only. A polyhedron is called $\alpha 4$ -polyhedron if it consists of four α -vertices. In the same manner, it is called $\alpha 8 \beta 1$ -polyhedron if it consists of eight α -vertices and a β -vertex. A polyhedron is called $(m \times \xi_m | m=3, 4, \dots, p)$ -polyhedron if it consists of ξ_m of m -edged polygons where $m=3, 4, \dots, p$. For example, if $\xi_3=4$, $\xi_5=3$, $\xi_9=1$, then it is called $(3 \times 4, 5 \times 3, 9 \times 1)$ -polyhedron.

2.5 Polyhedral cycles

A polyhedron is described by a set of polyhedral cycles. The closed chain of edges which make a face of a polyhedron is called polyhedral cycle or p -cycle. A p -cycle has a direction. It is called a front p -cycle, if we see the cycle from the space at its neighbourhood. It is called a rear p -cycle, if we see the cycle through the solid interior of the polyhedron. A front p -cycle has the direction, whereas the p -cycle has its face at the left side of it. A p -cycle is called positive if the cycle has its face inside. A p -cycle is called negative if the cycle has its face outside.

2.6 Dual graph of polyhedron

The dual graph of polyhedron is a graph with its point corresponding to the face of the polyhedron and its line, connecting two points, corresponding to the existence of the edge of the polyhedron. The dual graph of a convex polyhedron is drawn as the plane graph. It is called spread dual graph if its

points spread in a plane, and is called linear dual graph if its points lie on a straight line, as in Fig. 1.

2.7 Some properties of dual graphs of polyhedra

If the dual graph of a polyhedron is plane, then a region of the dual graph corresponds to a vertex of the polyhedron. The ordered chain of the names of the region around a point V_g of a plane dual graph in a fixed direction at the neighbourhood of V_g is the p -cycle of the polyhedron. The number of lines which make a region of a plane dual graph is equal to the degree of the corresponding vertex of the polyhedron.

2.8 The existence theorems of convex polyhedra

In a convex polyhedron having λ_k vertices of k -th degree ($k=3, 4, \dots, r$), let μ be the total number of edges and v the total number of faces, then

$$\sum_{k=3}^r \lambda_k = \mu - v + 2, \quad \sum_{k=3}^r k \lambda_k = 2\mu \dots (1)$$

(Euler's theorem)

It is clear from Eqs.(1) that, if an edge-vertex table $(k, \lambda_k | k=3, 4, \dots, r)$ is given, then the values of μ and v are uniquely decided respectively.

Necessary condition for existence of the convex polyhedron corresponding to a given edge-vertex table $(k, \lambda_k | k=3, 4, \dots, r)$ is

$$\sum_{k=3}^r \lambda_k (1 - \frac{k}{6}) \geq 2 \dots \dots \dots (2)$$

(Harary⁽⁴⁾)

From which it is clear that there exist no convex polyhedra which consist of vertices of degree 6 or greater.

Necessary conditions for existence of the convex polyhedron corresponding to a given face table $(m, \xi_m | m=3, 4, \dots, p)$ are

Table 1 The degree of vertex and its symbol

deg. of vertex	symbol	deg. of vertex	symbol
3	α	0	ω
4	β	1	θ
5	γ	2	κ
6	δ	10	$\theta\omega$
7	ϵ	11	$\theta\theta$
8	ζ	12	$\theta\kappa$
9	η	:	:
		:	:

Fig. 1 An example of polyhedron and its dual graphs

$$\sum_{m=3}^p m \xi_m = 2\mu, \quad \sum_{m=3}^p \xi_m = v \dots \dots \dots (3)$$

$$\max(m) = p \leq v - 1 \dots \dots \dots (4)$$

According to Steinitz and Rademacher⁽⁵⁾, the necessary and sufficient condition for the existence of a convex polyhedron P is that the graph G of P is planar and three tuply connected. The theorem is expanded to the dual graph $\bar{G}$.

3. Generation of dual graphs of polyhedra

For a given edge vertex table $(k, \lambda_k | k=3, 4, \dots, r)$, one or more face tables $\{(m, \xi_m | m=3, 4, \dots, p)\}$ are obtained, but the face tables give only the necessary conditions for the existence of convex polyhedra. On the other hand, a three tuply connected plane dual graph gives the necessary and sufficient condition for the existence of convex polyhedron. Hence, if all the dual graphs satisfying a given edge-vertex table are enumerated, then all the corresponding convex polyhedra are obtained.

A numerical enumeration process is given in this chapter. It is, however, not still possible to enumerate all of the convex polyhedra, but only ones whose dual graphs have Hamiltonian cycles. It is convenient to utilize the linear dual graph for the purpose.

3.1 Linear dual graph

In a linear dual graph, a straight line connecting adjacent points is called the centre line, the other lines being called wing lines. The centre lines lie horizontal on a paper. So, the wing above the centre line is called upper wing and the wing beneath the centre line is called lower wing. The wing line connecting the left end and right end points is called the greatest wing line. From the definition the following fact is deduced.

[Fact 1] If a linear dual graph has a Hamiltonian cycle, then the dual graph is drawn such that the centre lines are connected and there exists the greatest wing.

If a linear dual graph is drawn in the above manner, it is called standardized.

[Example 1] The linear dual graph of a $\beta 9$ -convex polyhedron is illustrated in Fig.2. The graph has no Hamiltonian cycles.

3.2 Generation of linear dual graphs having Hamiltonian cycles

The generating rules of stand-

ardized linear dual graphs having Hamiltonian cycles are given in the following.

If there is a path of length k , as $V_1-V_2-\dots-V_{k+1}$, in a graph, then the length is called span. If there is a cycle of length k , then it is called that one of the lines looks about span $k-1$. If the path consists of only the centre lines in a linear graph, then it is called basic span. A wing line is called basic wing line, if it looks about only the basic span. In the linear dual graph of a convex v -faced polyhedron having Hamiltonian cycles, the total centre lines have span $v-1$. This is called total span. Let S_t be total span, then $S_t = v-1$.

The ordered chain of basic spans corresponding to one of the wings, for example upper wing, is called the chain of basic spans, where the wing has the greatest wing line. In case where the total span is given, it is called the partition of span, to generate the chain of basic spans.

When a basic span or a span is parenthesized, the pair of parentheses corresponds to a wing line. Hence, a pair of the parentheses has a span 1 when it is looked over from outside of the parentheses. The parenthesized chain of basic spans is called the chain of wing lines. The number d_i of stacked row of parentheses between two adjacent spans or at the ends corresponds to the number of wing lines incident with i -th point in the dual graph. The ordered chain $\{d_i | i=1, 2, \dots, v\}$ is called the line number chain in a wing.

Let two distinct chains of wing lines, with the same total span, be T and B . Let the chain of wing lines be T' when the outermost pair of parentheses is removed from T . Let $C = (1)(1)\dots(1)(1)$ be the chain of centre lines. When stacking T' , C and B from top to bottom, this is called the table of the chain of wing lines. The stack of line number chains for the upper, centre

Fig. 2 An example of convex polyhedron whose dual graph has no Hamiltonian cycles ($\beta 9$), $(3 \times 8, 4 \times 3)$

and lower wings corresponding to the table of the chain of wing lines is called the table of line numbers.

The table is a matrix with v columns and 3 rows. The ordered chain $\{m_i | i=1,2,\dots,v\}$ of the sum of factors in each column is called the total line number chain. There exist no polyhedra if $\{m_i\}$ contains $m_i \leq 2$.

[Example 2] An example is given in Fig.3, for the processes from total span to a linear dual graph.

If a linear dual graph $\bar{G}$, obtained by the above procedure, is three tuply connected, then there exists a convex polyhedron corresponding to $\bar{G}$. The connectivity of $\bar{G}$ is confirmed by the following theorem, which is easily proved.

[Theorem 2] Necessary and sufficient condition that the standardized linear dual graph $\bar{G}$ having Hamiltonian cycles is three tuply connected, is that, for points $V_1, V_2, \dots, V_v$ of $\bar{G}$ and for all $V_i, V_j | j > i+1$, there exists at least a wing line $V_p - V_q$ such that

$$i < p < j, q < i \text{ or } q > j \dots (5)$$

Now, for a given edge-vertex table $(k, \lambda_k | k=3,4,\dots,r)$ all the distinct and existent sets of basic spans, chain of basic span, chains of wing lines, line number chain in a wing line, tables of line numbers and total line number chain are generated as the following.

[Procedure 1] Confirm that the given edge-vertex table satisfies Eq.(2). Calculate total span S_t from

$$S_t = v - 1 = \sum_{k=3}^r \left(\frac{k}{2} - 1 \right) \lambda_k + 1$$

If S_t is an integer, then go to procedure 2, otherwise end.

[Procedure 2] Partition S_t into $\{S_{ei} | \sum S_{ei} = S_t\}$ satisfying the following theorem, which is easily proved.

[Theorem 3] Necessary condition for the existence of wing line, which satisfies given $(k, \lambda_k | k=3,4,\dots,r)$, is that, $S_{ei} \leq \max(k) - 1$ and there exists at least a basic span $S_{ei} = k - 1 | k \in (k, \lambda_k | k=3,4,\dots,r)$.

[Procedure 3] Generate all but distinct ordered chain, called the chain of basic span, from a set $\{S_{ei} | \sum S_{ei} = S_t\}$. Do above operation with all sets.

[Procedure 4] Generate all possible chains of wing lines from a chain of basic span. The procedure is as follows: Make basic wing lines first. Other wing lines are produced successively. Give No.0 to a basic wing line or other combination, which are reckoned as the kernel. To the wing lines produced successively outside of the preceding ones, give No.1,2,...

Make all distinct enumeration, satisfying the following condition:

(a) Let S be a basic span or a span. If there exists $S = k - 1$ for given $\{k, \lambda_k\}$, then we can make a basic wing line or a wing line looking about S , and (b) if there exists $S_e = \max(k) - 1$ for given $\{k, \lambda_k\}$, then we must make a basic wing line looking about S_e .

[Procedure 5] Make edge-vertex table $\{k, \lambda'_k\}$ for a wing according to the obtained chain of wing lines for a wing. If there exists $\lambda'_k > \lambda_k$, then reject the chain of wing lines.

[Procedure 6] Make line number chain.

[Procedure 7] If the sum of edge-vertex tables of two distinct chains of wing lines for a wing is equal to the given edge-vertex table, then go to procedure 8, otherwise reject the combination. Make all combinations.

[Procedure 8] Make total line number chain $\{m_i | i=1,2,\dots,v\}$. If $m_i \geq 3$, then go to procedure 9, otherwise reject the chain.

The above procedures generate all plane linear dual graphs having Hamiltonian cycles. Two problems remain: to test if the graph is three tuply connected, and the rejection of isomorphic graphs. It may be convenient to utilize adjacency matrix for solving these problems.

3.3 Isomorphic conversion of linear dual graphs

Two graphs G_1 and G_2 are iso-

Chain of basic span = 1,1,2 3,1
Chain of wing line = (1(1(2))) ((3)1)
Line number chain = 1,1,1,0,3 2,0,0,1,1

Table of wing lines
Chain of upper wing lines = 1(1(2))
Chain of centre lines = (1)(1)(1)(1)(1)
Chain of lower wing lines = ((3)1)

Table of line numbers
Upper wing line number chain = 0,1,1,0,2
Centre line number chain = 1,2,2,2,1
Lower wing line number chain = 2,0,0,1,1 (+)
Total line number chain = 3,3,3,3,4

Fig. 3 An example of generating the dual graph

morphic (written $G_1 \cong G_2$) if there exists a 1-1 correspondence between their point sets which preserves adjacency.

[Theorem 4] In a standardized linear dual graph of convex polyhedron, having Hamiltonian cycles, without loss of isomorphic relations, (a) a point can move to another position if the adjacency, plane and standardized condition are preserved, (b) an end point can move to the other end, and (c) the greatest wing line can move to the other wing.

The conversion of graphs by (a), (b) and (c) of the above theorem is called isomorphic conversion, and conversion (b) is called circulation.

[Procedure 9] Reject a total line number chain $\{m_1, m_2, \dots, m_v\}$ if $m_1 < m_i | i \geq 2$. Go to procedure 11.

This procedure corresponds to the rejection of isomorphic graphs which are equalized by circulation.

3.4 Asymmetric adjacency matrix of linear dual graph

It is called asymmetric adjacency matrix of linear dual graph if a matrix satisfies the following conditions:

- (a) $a_{ij} = 0 | i = 1, 2, \dots, v$,
- (b) in a standardized linear dual graph of convex polyhedron having Hamiltonian cycles
 $a_{i, i+1} = 1 | i = 1, 2, \dots, v-1$,
 $a_{v, 1} = 1$,
- (c) if $a_{ij} = 1$, then $a_{ji} = 0$,
- (d) a factor corresponding to the upper (or lower) wing line has the value 1 above (or under) the main diagonal.

From the definition the following fact is induced.

[Fact 5] In an asymmetric adjacency matrix of linear dual graph, the total sum of factors of i -th column and i -th row is equal to the order of i -th point in the dual graph.

If a linear dual graph $\bar{G}$ is plane, asymmetric adjacency matrix of $\bar{G}$ is called plane. The following theorem is easily proved.

[Theorem 6] Necessary and sufficient condition that an asymmetric adjacency matrix of linear dual graph is plane, is

- (1) If $a_{ij} = 1$ for $j \geq i+2$, then $a_{kl} = 0$ for $i < k < j$, $l > j$
- and (2) If $a_{ij} = 1$ for $i \geq j+2$, then $a_{kl} = 0$ for $k > i$, $j < l < i$.

3.5 Isomorphic conversion of asymmetric adjacency matrix

Let $[M_1]$ and $[M_2]$ be asymmetric adjacency matrices of linear dual graph $\bar{G}_1$ and $\bar{G}_2$ respectively. If $\bar{G}_1 \cong \bar{G}_2$, then we write $[M_1] \cong [M_2]$. If $\bar{G}_1$

has Hamiltonian cycles, then it is said that $[M_1]$ has Hamiltonian cycles.

Suppose, there exist two plane asymmetric adjacency matrices $[A] = [a_{ij} | i, j = 1, 2, \dots, v]$ and $[B] = [b_{kl} | k, l = 1, 2, \dots, v]$ which have Hamiltonian cycles, but $[A] \neq [B]$. If there is a 1-1 correspondence $\{C\}$, such that $a_{ij} = b_{kl}$, between i and k and between j and l , then an isomorphic conversion from $[A]$ to $[B]$ is possible and vice versa.

The isomorphic conversion is made as follows.

[Procedure 10] In an asymmetric adjacency matrix $[A] = [a_{ij} | i, j = 1, 2, \dots, v]$ of plane dual graph having Hamiltonian cycles, if there exists a 1-1 correspondence $\{C\}$ between i and k and between j and l such that $a_{ij} = b_{kl}$ for all i and j , and $b_{k, k+1} = b_{v, 1}$ for $k = 1, 2, \dots, v-1$, then:

- (1) At $[A]$, shift j -th row to k -th row for $i = 1, 2, \dots, v$. Denote the new matrix $[A'] = [a'_{kj}]$.
- (2) At $[A']$, shift j -th column to l -th column for $j = 1, 2, \dots, v$. Denote the new matrix $[A''] = [a''_{kl}]$.
- (3) Exchange $a''_{k, k+1}$ and $a''_{k+1, k}$ if the condition $a''_{k, k+1} = a''_{v, 1}$ for $k = 1, 2, \dots, v-1$ is not satisfied.
- (4) Exchange $a''_{k, 1}$ and $a''_{1, k}$ so that the conditions in Theorem 6 are not satisfied.

[Example 3] In the following asymmetric adjacency matrix,

$$[A] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for example, $a_{7,1}$, $a_{6,7}$, $a_{6,8}$, $a_{3,8}$, $a_{2,3}$, $a_{4,2}$, $a_{4,5}$, $a_{5,1}$ have the value 1, then the graph has a Hamiltonian cycle

1-7-6-8-3-2-4-5-1

Hence there is a 1-1 correspondence between i and k and between j and l , such as

i \ j	1	7	6	8	3	2	4	5	
k \ l	1	1	2	3	4	5	6	7	8

Following this table of correspondence, shifting 7th row to 2nd row, 6th to 3rd at $[A]$, we have

$$[A'] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Shifting 7th column to 2nd column, 6th to 3rd, ... at $[A']$, we have

$$[A"] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

At $[A"]$, we perform exchanges $(2,1)$ $\Rightarrow a_{2,1}^* a_{1,2}^*$, $(3,2)$, $(5,4)$, $(6,5)$ and $(7,6)$, which correspond to Procedure 10-(3). Next, we perform the exchanges according to Procedure 10-(4). Fix $a_{4,1}^*$ to unity, which enforces $(4,3)$, $(7,3)$ and $(8,3)$, which in turn enforces $(1,5)$, $(1,6)$ and $(2,4)$. The last exchange enforces $(3,1)$ and the final result is

$$[B] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Transposition $[M^T]$ (exchange of row and column) corresponds to the exchange of upper and lower wings in the linear graph. By the definition $[M] = [M^T]$.

[Theorem 7] There exists unique isomorphic conversion of a given plane asymmetric adjacency matrix $[A]$ of 3 tuply connected graph, if the conversion is performed by Procedure 10 with existence of 1-1 correspondence $\{C\}$.

[Proof] At $[A] = [a_{ij}] | i, j = 1, 2, \dots, v$, if there exists a 1-1 correspondence $\{C\}$ between i and k and between j and l such that $a_{ij} = b_{kl}$ for all i and j and $b_{k,k+1} = b_{v,1} = 1$ for $k = 1, 2, \dots, v-1$, then there exists a Hamiltonian cycle C_B which passes $k = 1, 2, \dots, v, 1$ in this order. Hence, there exists a linear graph G_B having C_B as its centre lines and the greatest wing line. Clearly $G_B = G_A$, and G_A is unique for given $[A]$ by the definition.

G_B is standardized 3 tuply connected linear graph having at least two Hamiltonian cycles. Suppose, there is a wing line $V_k - V_1$, not the greatest wing line, in G_B . It is impossible to move $V_k - V_1$ to the other wing preserving plane and the order of the vertices, because there exists at least a wing line $V_p - V_q$ in the other wing such as $k < p < 1$ and $\alpha < k$ or $\alpha > 1$, so, if $V_k - V_1$ is moved to the other wing, it intersects with $V_p - V_q$, and G_B becomes non-plane. It is possible to exchange upper and lower wings wholly. This, however, is isomorphic with the original one. Hence G_B is unique.

[Procedure 11] Transform the table of wing lines into an asymmetric adjacency matrix $[M]$.

[Procedure 12] Go to Procedure 13 if $[M]$ is three tuply connected, otherwise reject $[M]$.

[Procedure 13] For all of $[M]$ having the same face table, perform all possible isomorphic conversion of Procedure 10, and if two $[M]$'s are isomorphic to each other, then reject one of them.

The preceding procedures generate all distinct convex polyhedra whose dual graphs have Hamiltonian cycles, for given edge-vertex table. [Example 4] All convex polyhedra of $(\alpha 4 \beta 2)$, whose dual graphs have Hamiltonian cycles are generated as follows. From Eqs.(1), we have $\mu = 10$ and $v = 6$. Hence $S_t = 5$. Then, the sets of basic spans are $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$. According to Procedure 3 to 6, we have the chains of basic span, edge-vertex tables for a wing, chains of wing lines and line number chains as in Table 2. Thirty eight chains of wing lines are obtained in this example. According to Procedure 7 to 9, tables of line numbers, total line number chains and linear dual graphs are generated as in Fig.4. In the figure, graphs h and k are rejected because they are two tuply connected. Other ten graphs have a common face table $(4 \times 2, 3 \times 4)$, hence it is anticipated that they are all isomorphic, it is checked whether asymmetric adjacency matrices $[b]^\sim$ [1] are isomorphic with $[a]$. We have, for example,

$$[a] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[b] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Possible Hamiltonian cycles in $\bar{G}_B$ are the following four:

A: 1-5-4-2-3-6-1
A': 4-5-1-6-3-2-4
B: 1-2-4-3-6-5-1
B': 4-2-1-5-6-3-4

for the purpose of converting $[b]$ and equalizing with $[a]$. Performing isomorphic conversion for $[b]$ following A', $[b]$ becomes equal to $[a]$. Hence, $[b]^\sim [a]$. In the similar way, it is confirmed that all other matrices are isomorphic with $[a]$. Then, there exists only a convex polyhedron shown in Fig.5 in this example.

[Example 5] Computer generated convex polyhedra, whose dual graphs have Hamiltonian cycles, are given in Table 3. The output is in the form of asymmetric adjacency matrices. It is still not possible for the author to output the illustrations as shown in Fig.6 from the computer. It is clear from Fig.4

Fig. 4 Linear dual graph and line number chain of $\alpha 4\beta 2$ convex polyhedra

that it is possible to generate only up to a part of octahedron, even if a large machine is utilized. Improvements may raise the limit but perhaps not exceed decahedron. To overcome this difficulty, it will be necessary to consider limitation techniques for the polyhedra requested from man.

4. Consideration about the input

Only the edge-vertex table is proposed in this paper as the input. The automata can, however, receive other forms of input describing polyhedra. For example, if the number of faces v is input, the automaton generates all the edge-vertex table first, and then generates the polyhedra. If, for another example, a face table is input, then the

Fig. 5 $\alpha 4\beta 2$ convex polyhedron
The figure corresponds to the linear dual graph in Fig. 4a.

Table 2 Generation of line number chain in Example 3

Chain of basic span	Generating order of the chain of wing lines	Edge-vertex table for a wing	Chain of wing line	Line number chain	No.
3, 2	$\begin{bmatrix} 3, 2 \\ 0 \ 0 \\ 1 \ 1 \\ 0 \ 1 \end{bmatrix}$	$\alpha 2\beta 1$	((3) (2))	200202	1
		$\alpha 0\beta 2$	((3) (2))	200101	2
2, 3	$\begin{bmatrix} 2, 3 \\ 0 \ 0 \\ 1 \ 1 \\ 1 \ 0 \end{bmatrix}$	$\alpha 2\beta 1$	((2) (3))	202002	3
		$\alpha 0\beta 2$	((2) (3))	101002	4
⋮	⋮	⋮	⋮	⋮	⋮
1, 2, 1, 1	$\begin{bmatrix} 1, 2, 1, 1 \\ 1 \ 0 \ 1 \ 2 \\ 1 \ 0 \ 2 \ 3 \\ 2 \ 0 \ 1 \ 3 \\ 3 \ 0 \ 1 \ 2 \\ 2 \ 0 \ 1 \ 2 \end{bmatrix}$	$\alpha 2\beta 1$	((1(2)1)1)	210111	28
		$\alpha 4\beta 0$	((1(2)1)1)	310211	29
		$\alpha 4\beta 0$	((1((2)1))1)	220121	30
		$\alpha 4\beta 0$	(1((2)1)1)	130112	31
		$\alpha 2\beta 1$	(1((2)1)1)	120111	32
1, 1, 2, 1	$\begin{bmatrix} 1, 1, 2, 1 \\ 2 \ 1 \ 0 \ 1 \\ 2 \ 1 \ 0 \ 2 \\ 2 \ 1 \ 0 \ 3 \\ 3 \ 1 \ 0 \ 2 \\ 3 \ 2 \ 0 \ 1 \end{bmatrix}$	$\alpha 2\beta 1$	(1(1(2)1))	111012	33
		$\alpha 2\beta 1$	(1(1(2))1)	111021	34
		$\alpha 4\beta 0$	((1(1(2)))1)	211031	35
		$\alpha 4\beta 0$	(1((1(2)1))1)	121022	36
		$\alpha 4\beta 0$	(1(1((2)1)))	112013	37
1, 1, 1, 2	$\begin{bmatrix} 1, 1, 1, 2 \\ 3 \ 2 \ 1 \ 0 \end{bmatrix}$	$\alpha 4\beta 0$	(1(1(1(2))))	111104	38

Numbering rule of the generating order of the chain of wing lines is as written in Procedure 4. Write the chain of basic span at first. Write No. 0 directly under the basic span which a basic wing line, to be made, looks about. Write the number of the order of wing lines successively generated outside, under the basic span corresponding to both ends of the wing line, but to one end if there is no confusion. Write the numbers of order stacking successively under the preceding ones.

Table 3 Some results on computer generated convex polyhedra

Number of faces	Edge-vertex table	Required time before checking isomorphism	Number of polyhedra whose isomorphism is to be checked	Time required for checking isomorphism	Number of distinct polyhedra
4	α_4	0.009	1	0.009	1
5	α_6	0.024	3	0.012	1
5	$\alpha_4\beta_1$	0.042	1	0.011	1
6	α_8	0.128	14	0.208	2
6	$\alpha_6\beta_1$	0.331	23	0.459	2
6	$\alpha_4\beta_2$	0.291	10	0.057	1
7	α_{10}	1.133	65	4.97	5
7	$\alpha_8\beta_1$	3.269	177	22.452	8
7	$\alpha_6\beta_2$	4.614	163	23.322	9
8	α_{12}	11.623	521	285.72	14
8	$\alpha_{10}\beta_1$	47.777	1748	CPU time overflow(600s)	
8	$\alpha_8\beta_2$	75.07	1991	ditto	
9	α_{14}	132.59	4061	ditto	
9	$\alpha_{12}\beta_1$	CPU time overflow (number of one wing=1522)			
9	$\alpha_{10}\beta_2$	Poor memory (number of polyhedra whose isomorphism is to be checked=over 12000,number of one wing=2290)			
10	α_{16}	ditto (	ditto		1430)
10	$\alpha_{14}\beta_1$	ditto (	ditto		5630)
10	$\alpha_{12}\beta_2$	ditto (	ditto		over 8000)

Computer utilized : HITAC 8800/8700, OS7 system at the Univ. Tokyo
 Input : Edge-vertex tables, Output : Asymmetric adjacency matrices and other quantities cited in the above tables

Fig. 6 All of α_{12} convex polyhedra whose dual graph have Hamiltonian cycles

automaton obtains edge-vertex tables satisfying

$$\sum_k \lambda_k = \sum_m \left(\frac{m}{2} - 1 \right) \xi_m + 2$$

$$\sum_k \lambda_k = \sum_m \xi_m$$

where the edge-vertex tables are

probably unique. It is only necessary to check the isomorphism among polyhedra satisfying both the above edge-vertex table and the given face table, which cuts the processing time remarkably.

5. Conclusions

Topological generating rules of convex polyhedra are developed, aiming to have a basis for the computer generation of machine drawings. The important points are

(1) As the input describing polyhedra, the edge-vertex table is proposed and shown to be useful for the computer generation of polyhedra.

(2) Procedures and necessary theorems for the generation of all but distinct convex polyhedra, whose dual graphs have Hamiltonian cycles, satisfying the given edge-vertex table, are presented.

The remaining problems are the improvement of processing time, the generating rules of convex polyhedra, whose dual graphs have no Hamiltonian cycles, and the generating rules of non-convex polyhedra.

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