539.4.013:539.38.092

Paper No. 139- 3

A Crack Problem on a Slab with an Annular Crack Subjected to Pressure*

By Takashi KOIZUMI**, Toshikazu SHIBUYA***, Ichiro NAKAHARA**

and Shuho TANAKA****

Although the stress analysis of a body containing cracks has been widely investigated concerning the stress intensity factor well used in fracture mechanics, few papers are reported on slab problems in spite of their availability.

In the present paper, a slab with a pressurised annular crack on the middle plane is treated on the basis of three-dimensional theory of elasticity. The displacement normal to the crack faces is supposed to be expressed by the Fourier series. Thus the triple integral equation is replaced by a set of the infinite simultaneous equations and its solution is much simplified. Additionally, displacements, stresses and stress intensity factors are also shown by those coefficients of Fourier series and their numerical results are easily obtained. It is concluded that displacements and stress intensity factors become large as the slab thickness is small under the same conditions.

1. Introduction

In fracture mechanics, although many problems have been analyzed to determine stress intensity factors, they are mostly the results of two-dimensional problems, and few results of three-dimensional analysis have been reported except an axis-symmetrical one. In axis-symmetrical crack problems, Sneddon⁽¹⁾ has treated an infinite elastic body containing a penny-shaped crack subjected to pressure on its free surfaces, and obtained a closed form solution. Lowengrub⁽²⁾ has analyzed the problem on a slab with a penny-shaped crack on the middle plane. Moss et al.⁽³⁾ have obtained an approximate solution on stress intensity factors in an infinite body with a pressurised annular crack by using the result for a penny-shaped crack. The authors⁽⁴⁾ also have strictly investigated the same problem, obtaining stresses and stress intensity factors. By the way, the problem on a slab with an annular crack has not been treated until now in spite of its availability.

In the present paper, the authors analyze the axis-symmetrical problem on a slab with a pressurised annular crack on the middle plane. Here it is assumed in analytical method that the normal displacement on the crack surface is expressed by the Fourier series⁽⁴⁾. Thus the problem is simplified to a solution of the infinite simultaneous equations instead of a complicated integral equation.

2. Stress functions

An annular crack pressurised with p_0 exists on the middle plane of an infinitely wide slab with thickness $2h$. Let's use the cylindrical co-ordinates (r, θ, z) as shown in Fig.1. If (u_r, v_θ, w_z) denotes each component of displacement vector, the solution of the equations of elasticity can be written as follows, by using Boussinesq's stress functions φ_0 and φ_3 ,

$$\left. \begin{aligned} u_r &= \frac{\partial \varphi_0}{\partial r} + z \frac{\partial \varphi_3}{\partial r}, & v_\theta &= 0 \\ w_z &= \frac{\partial \varphi_0}{\partial z} + z \frac{\partial \varphi_3}{\partial z} - (3-4\nu)\varphi_3 \end{aligned} \right\} \dots\dots\dots (1)$$

where ν is Poisson's ratio, and

$$\nabla^2 \varphi_0 = \nabla^2 \varphi_3 = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

We can easily obtain the stresses from those stress functions on the basis of Hooke's law.

3. Boundary conditions

We know at a sight of Fig.1 that the boundary conditions are expressed by the following equations,

$$(\tau_{rz})_{z=0} = 0 \quad (0 \leq r < \infty) \quad \dots\dots\dots (2)$$

$$(\sigma_z)_{z=0} = -p_0 \quad (r_1 < r < r_0) \quad \dots\dots\dots (3)$$

Fig.1. A slab containing an annular crack subjected to pressure.

* Received 27th March, 1975.

** Professor, Faculty of Engineering, Tokyo Institute of Technology, Meguro-ku, Tokyo.

*** Associate Professor, T.I.T.

**** Ishikawajima-Harima Industry Co.

$$\begin{aligned}
(w_z)_{z=0} &= 0 & (0 \leq r \leq r_1, r_0 \leq r < \infty) & \dots (4) \\
(\sigma_z)_{z=\pm h} &= 0 & (0 \leq r < \infty) & \dots (5) \\
(\tau_{rz})_{z=\pm h} &= 0 & (0 \leq r < \infty) & \dots (6) \\
(\text{displacements, stresses})_{r \rightarrow \infty} &\rightarrow 0 & \dots (7)
\end{aligned}$$

4. Analysis

Considering Eq.(7), we will take the functions φ_0 and φ_s in the following forms,

$$\begin{aligned}
2G\varphi_0 &= \int_0^\infty [A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z] J_0(\lambda r) d\lambda \\
2G\varphi_s &= \int_0^\infty [C(\lambda) \sinh \lambda z + D(\lambda) \cosh \lambda z] J_0(\lambda r) d\lambda
\end{aligned} \dots (8)$$

where $J_0(\lambda r)$ is a Bessel function of the first kind in order zero, and G denotes the shear modulus. $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are unknown functions with respect to λ .

We can get displacements and stresses from Eqs.(1) and (8) now.

$$\begin{aligned}
2Gu_r &= - \int_0^\infty \lambda J_1(\lambda r) [A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z + C(\lambda) z \sinh \lambda z + D(\lambda) z \cosh \lambda z] d\lambda \\
2Gw_r &= \int_0^\infty J_0(\lambda r) [A(\lambda) \lambda \sinh \lambda z + B(\lambda) \lambda \cosh \lambda z + C(\lambda) \{ \lambda z \cosh \lambda z - (3-4\nu) \sinh \lambda z \} \\
&\quad + D(\lambda) \{ \lambda z \sinh \lambda z - (3-4\nu) \cosh \lambda z \}] d\lambda \\
\sigma_r &= \int_0^\infty \left[A(\lambda) \left\{ \frac{\lambda}{r} J_1(\lambda r) - \lambda^2 J_0(\lambda r) \right\} \cosh \lambda z + B(\lambda) \left\{ \frac{\lambda}{r} J_1(\lambda r) - \lambda^2 J_0(\lambda r) \right\} \sinh \lambda z \right. \\
&\quad + C(\lambda) \left\{ \left(\frac{1}{r} J_1(\lambda r) - \lambda J_0(\lambda r) \right) \lambda z \sinh \lambda z - 2\nu \lambda J_0(\lambda r) \cosh \lambda z \right\} \\
&\quad + D(\lambda) \left\{ \left(\frac{1}{r} J_1(\lambda r) - \lambda J_0(\lambda r) \right) \lambda z \cosh \lambda z - 2\nu \lambda J_0(\lambda r) \sinh \lambda z \right\} \Big] d\lambda \\
\sigma_\theta &= - \int_0^\infty \left[A(\lambda) \frac{\lambda}{r} J_1(\lambda r) \cosh \lambda z + B(\lambda) \frac{\lambda}{r} J_1(\lambda r) \sinh \lambda z \right. \\
&\quad + C(\lambda) \left\{ \frac{1}{r} J_1(\lambda r) \lambda z \sinh \lambda z + 2\nu \lambda J_0(\lambda r) \cosh \lambda z \right\} \\
&\quad + D(\lambda) \left\{ \frac{1}{r} J_1(\lambda r) \lambda z \cosh \lambda z + 2\nu \lambda J_0(\lambda r) \sinh \lambda z \right\} \Big] d\lambda \\
\sigma_z &= \int_0^\infty \lambda J_0(\lambda r) [A(\lambda) \lambda \cosh \lambda z + B(\lambda) \lambda \sinh \lambda z + C(\lambda) \{ \lambda z \sinh \lambda z - 2(1-\nu) \cosh \lambda z \} \\
&\quad + D(\lambda) \{ \lambda z \cosh \lambda z - 2(1-\nu) \sinh \lambda z \}] d\lambda \\
\tau_{rz} &= \int_0^\infty \lambda J_1(\lambda r) [-A(\lambda) \lambda \sinh \lambda z - B(\lambda) \lambda \cosh \lambda z \\
&\quad + C(\lambda) \{ (1-2\nu) \sinh \lambda z - \lambda z \cosh \lambda z \} + D(\lambda) \{ (1-2\nu) \cosh \lambda z - \lambda z \sinh \lambda z \}] d\lambda
\end{aligned} \dots (9)$$

Substituting τ_{rz} into the condition (2), we get

$$\int_0^\infty \lambda J_1(\lambda r) [-B(\lambda) \lambda + D(\lambda) (1-2\nu)] d\lambda = 0 \dots (11)$$

Thus

$$B(\lambda) \lambda - D(\lambda) (1-2\nu) = 0 \dots (12)$$

In a similar manner, substituting σ_z and τ_{rz} into the conditions (5) and (6), we can get the simultaneous equations with respect to unknown functions.

$$\begin{aligned}
A(\lambda) \lambda \cosh \lambda h + B(\lambda) \lambda \sinh \lambda h + C(\lambda) [\lambda h \sinh \lambda h - 2(1-\nu) \cosh \lambda h] \\
+ D(\lambda) [\lambda h \cosh \lambda h - 2(1-\nu) \sinh \lambda h] = 0 \dots (13)
\end{aligned}$$

$$\begin{aligned}
A(\lambda) \lambda \sinh \lambda h + B(\lambda) \lambda \cosh \lambda h - C(\lambda) [(1-2\nu) \sinh \lambda h - \lambda h \cosh \lambda h] \\
- D(\lambda) [(1-2\nu) \cosh \lambda h - \lambda h \sinh \lambda h] = 0 \dots (14)
\end{aligned}$$

The unknown functions $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ can be expressed by $D(\lambda)$ as follows:

$$\begin{aligned}
\lambda B(\lambda) &= (1-2\nu) D(\lambda), \quad \lambda A(\lambda) = - \frac{\lambda^2 h^2 + (1-2\nu) \sinh^2 \lambda h}{\lambda h + \sinh \lambda h \cosh \lambda h} D(\lambda) \\
C(\lambda) &= - \frac{\sinh^2 \lambda h}{\lambda h + \sinh \lambda h \cosh \lambda h} D(\lambda)
\end{aligned} \dots (15)$$

Additionally, the conditions (3) and (4) can be rewritten by using Eq.(15)

$$(\sigma_z)_{z=0} = \int_0^\infty \lambda D(\lambda) \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} J_0(\lambda r) d\lambda = -p_0 \quad (r_1 < r < r_0) \dots (16)$$

$$(w_z)_{z=0} = - \frac{1-\nu}{G} \int_0^\infty D(\lambda) J_0(\lambda r) d\lambda = 0 \quad (r \leq r_1, r \geq r_0) \dots (17)$$

Being able to determine the unknown function $D(\lambda)$ in the triple integral equations (16) and (17), we will get a solution satisfying all the boundary conditions. However, it is very difficult to obtain the above solution directly.

Then we first assume that the displacement on the crack surface, $(w_z)_{z=0}$; $(r_1 \leq r \leq r_0)$, is expressed by the Fourier series. We exchange the variables now,

$$r_i = r_c - b, \quad r_0 = r_c + b, \quad 2r_c b \cos \phi = r_c^2 + b^2 - r^2 \quad \dots\dots\dots(18)$$

where r_c and $2b$ denote the mean crack radius and the crack width, respectively. The displacement $(w_z)_{z=0}$ is zero in the intervals $0 \leq r \leq r_i$ and $r_0 \leq r$, and it takes a definite value in the remaining one $r_i < r < r_0$. By the way, as the interval $r_i \leq r \leq r_0$ corresponds to $0 \leq \phi \leq \pi$; ($r = r_i$ is equal to $\phi = 0$, and $r = r_0$ is $\phi = \pi$), $(w_z)_{z=0}$ in $r_i \leq r \leq r_0$ may be expressed by the Fourier series of ϕ . Here, noting that $(w_z)_{z=0}$ is equal to zero at $\phi = 0$ and π , and also taking a finite value in $0 < \phi < \pi$, we may suppose $(w_z)_{z=0}$ by using unknown coefficients a_n' as

$$2G(w_z)_{z=0} = \frac{2}{\pi r_c b} \sum_{n=1}^{\infty} a_n' \sin n\phi \quad \dots\dots\dots(19)$$

Remembering the next integral formula⁽⁵⁾

$$I_n = \int_0^{\infty} \lambda J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda = \begin{cases} 0, & (r < r_i, r > r_0) \\ \frac{1}{\pi r_c b} \frac{\cos n\phi}{\sin \phi}, & (r_i < r < r_0) \end{cases} \quad \dots\dots\dots(20)$$

and subtracting I_{n+1} from I_{n-1} , we get

$$\int_0^{\infty} J_0(\lambda r) G_n(\lambda) d\lambda = \begin{cases} 0, & (r \leq r_i, r \geq r_0) \\ \frac{2}{\pi r_c b} \sin n\phi, & (r_i \leq r \leq r_0) \end{cases} \quad \dots\dots\dots(21)$$

where

$$G_n(\lambda) = \lambda [J_{n-1}(\lambda r_c) J_{n-1}(\lambda b) - J_{n+1}(\lambda r_c) J_{n+1}(\lambda b)]$$

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} J_0(\lambda r) G_n(\lambda) d\lambda = 1, \quad (r_i \leq r \leq r_0) \quad \dots\dots\dots(27)$$

Eq.(27) must be satisfied independently of values of r . Then we represent $J_0(\lambda r)$ by the Fourier series in the interval $r_i \leq r \leq r_0$ by using the relation (18).

$$J_0(\lambda r) = J_0(\lambda \sqrt{r_c^2 + b^2 - 2r_c b \cos \phi}) = J_0(\lambda r_c) J_0(\lambda b) + 2 \sum_{m=1}^{\infty} J_m(\lambda r_c) J_m(\lambda b) \cos m\phi \quad \dots\dots\dots(28)$$

Substituting Eq.(28) into Eq.(27), we can obtain

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} G_n(\lambda) \left[J_0(\lambda r_c) J_0(\lambda b) + 2 \sum_{m=1}^{\infty} J_m(\lambda r_c) J_m(\lambda b) \cos m\phi \right] d\lambda = 1 \quad \dots\dots\dots(29)$$

Here, if we equate the coefficients of $\cos m\phi$ in both sides of Eq.(29), we can introduce a set of the infinite simultaneous equations with respect to unknown coefficients a_n ,

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} G_n(\lambda) J_m(\lambda r_c) J_m(\lambda b) d\lambda = \delta_{0,m}, \quad (m=0, 1, 2, \dots) \quad \dots\dots\dots(30)$$

where $\delta_{0,m}$ denotes the Kronecker delta.

We can readily recognize the asymmetry of the matrix of the simultaneous equation (30). Then it is not always possible to compute a_n by those equations. For the simplicity of numerical treatment, taking the difference between the m -th and the $(m+2)$ th equations, we get the following result that has the symmetric matrix,

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} G_n(\lambda) G_m(\lambda) d\lambda = \delta_{1,m} \quad (m=1, 2, \dots) \quad \dots\dots\dots(31)$$

The unknown coefficients a_n are determined by solving Eq.(31). Thereafter the function $D(\lambda)$ is given by the equation

$$-D(\lambda) = p_0 \sum_{n=1}^{\infty} a_n G_n(\lambda) \quad \dots\dots\dots(32)$$

Substituting Eqs.(15) and (32) into Eq.(10), we can get displacements and stresses. Especially the displacement $(w_z)_{z=0}$ and the stress $(\sigma_z)_{z=0}$ on the crack surface are important, and they are given as

$$\frac{2G}{p_0} (w_z)_{z=0} = \begin{cases} 0 & (0 \leq r \leq r_i, r_0 \leq r) \\ \frac{4(1-\nu)}{\pi r_c b} \sum_{n=1}^{\infty} a_n \sin n\phi & (r_i \leq r \leq r_0) \end{cases} \quad \dots\dots\dots(33)$$

Summing up all the terms after the multiplication by a_n' on both sides of Eq.(21), and considering Eqs.(4) and (19), we get the resulting series for the displacement on the crack surface.

$$2G(w_z)_{z=0} = \sum_{n=1}^{\infty} a_n' \int_0^{\infty} J_0(\lambda r) G_n(\lambda) d\lambda, \quad (0 \leq r < \infty) \quad \dots\dots\dots(22)$$

On the other hand, substituting the relation(15) into Eq.(9), and now putting $z=0$, we get

$$2G(w_z)_{z=0} = - \int_0^{\infty} 2(1-\nu) D(\lambda) J_0(\lambda r) d\lambda, \quad (0 \leq r < \infty) \quad \dots\dots\dots(23)$$

Here, comparing $(w_z)_{z=0}$ in Eq.(22) with that in Eq.(23), we obtain consequently

$$-2(1-\nu) D(\lambda) = \sum_{n=1}^{\infty} a_n' G_n(\lambda) \quad \dots\dots\dots(24)$$

Further, from the substitution of Eq.(24) into Eq.(16), we get

$$(\sigma_z)_{z=0} = \frac{1}{2(1-\nu)} \sum_{n=1}^{\infty} a_n' \int_0^{\infty} \frac{\lambda^2 h^2 - \sinh^2 \lambda h}{\lambda h + \sinh \lambda h \cosh \lambda h} \times J_0(\lambda r) G_n(\lambda) d\lambda = -p_0, \quad (r_i \leq r \leq r_0) \quad \dots\dots\dots(25)$$

In the above expression, let's replace the coefficients a_n' by

$$[1/2(1-\nu)p_0] a_n' = a_n, \quad (n=1, 2, 3, \dots) \quad \dots\dots\dots(26)$$

Then Eq.(25) is rewritten as follows:

$$\frac{(\sigma_z)_{z=0}}{p_0} = \sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \left[1 - \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} \right] G_n(\lambda) J_0(\lambda r) d\lambda - \sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda G_n(\lambda) J_0(\lambda r) d\lambda \quad \dots (34)$$

Since the first term integrand of the righthand side in Eq.(34) converges rapidly to zero as the value of λ becomes large, the integral can be easily evaluated numerically. Moreover, the second term of Eq.(34) can be evaluated by the method explained later when $r < r_c$ and $r > r_0$. Now using the relationship

$$G_n(\lambda) = \frac{2n}{r_c b} \frac{\partial}{\partial \lambda} \{ J_n(\lambda r_c) J_n(\lambda b) \} \quad \dots (35)$$

we get from the integration by

$$\int_0^{\infty} \lambda J_0(\lambda r) G_n(\lambda) d\lambda = -\frac{2n}{r_c b} \left[\int_0^{\infty} J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda + r \frac{\partial}{\partial r} \int_0^{\infty} J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda \right] \quad \dots (36)$$

Now let us introduce the next notation

$$I_0^n = \int_0^{\infty} J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda \quad \dots (37)$$

Thus Eq.(34) can be rewritten as follows:

$$\frac{(\sigma_z)_{z=0}}{p_0} = \sum_{n=1}^{\infty} a_n \int_0^{\infty} \lambda \left\{ 1 - \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} \right\} G_n(\lambda) J_0(\lambda r) d\lambda + \frac{2}{r_c b} \sum_{n=1}^{\infty} a_n \left(I_0^n + r \frac{\partial I_0^n}{\partial r} \right) \quad \dots (38)$$

In order to evaluate the integral in Eq.(38), we utilize the integral formula⁽⁵⁾ as

$$\int_0^{\infty} J_\nu(\lambda y) J_\mu(\lambda x) J_\nu(a\lambda) d\lambda = \frac{\Gamma((1/2)(1+\sigma+\rho))}{\Gamma(\mu+1)\Gamma(\xi+1)\Gamma((1/2)(1-\sigma+\rho))} \frac{1}{a} \left(\frac{x}{a} \right)^\mu \left(\frac{y}{a} \right)^\xi \\ \times F\left(\frac{1+\sigma-\rho}{2}, \frac{1+\sigma+\rho}{2}; \mu+1; \sin^2 \phi \right) F\left(\frac{1+\sigma-\rho}{2}, \frac{1+\sigma+\rho}{2}; \xi+1; \sin^2 \phi \right) \quad \dots (39)$$

where

$$\sigma = \mu + \xi, \quad x = a \sin \phi \cos \phi, \quad y = a \cos \phi \sin \phi, \quad a > 0, \quad 0 < \phi, \quad \phi < \pi/2, \quad R_\mu(\mu + \xi + \rho) > -1 \quad y+x = a \sin(\phi + \phi), \quad y-x = a \sin(\phi - \phi)$$

$$\phi = \frac{1}{2} \left[\sin^{-1} \frac{y+x}{a} + \sin^{-1} \frac{y-x}{a} \right], \quad \phi = \frac{1}{2} \left[\sin^{-1} \frac{y+x}{a} - \sin^{-1} \frac{y-x}{a} \right]$$

and $F(\alpha, \beta; \gamma; z)$ is the Gaussian hypergeometric function.

Eq.(39) must be applied to each case of $r < r_c$ or $r > r_0$ in accordance with the magnitude of r .

(a) The case of $0 \leq r < r_c$. Since $r < r_c$, in this case, we put

$$\rho = n, \quad \mu = n, \quad \xi = 0 \quad \sigma = n \quad a = r_c, \quad r = r_c \cos \phi \sin \phi (=y), \quad b = r \sin \phi \cos \phi, \quad \left(\frac{\phi}{\phi} \right) = \frac{1}{2} \left[\sin^{-1} \frac{r+b}{r_c} \pm \sin^{-1} \frac{r-b}{r_c} \right]$$

Then we get

$$I_0^n = \int_0^{\infty} J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda \\ = \frac{\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)} \frac{1}{r_c} \left(\frac{b}{r_c} \right)^n F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2 \phi \right) F\left(\frac{1}{2}, n+\frac{1}{2}; 1; \sin^2 \phi \right) \quad \dots (40)$$

Considering the following relation and differentiating Eq.(40) with respect to r ,

$$\frac{\partial}{\partial \phi} F(\alpha, \beta; \gamma; \sin^2 \phi) = \frac{\alpha \beta}{\gamma} \sin 2\phi F(\alpha+1, \beta+1; \gamma+1; \sin^2 \phi) \\ \left(\frac{\partial \phi}{\partial r} \right) = \frac{1}{2} \left[\frac{1}{\sqrt{(r_0+r)(r_1-r)}} \pm \frac{1}{\sqrt{(r_0-r)(r_1+r)}} \right]$$

we can get the differentiation of I_0^n with respect to r ,

$$\frac{\partial I_0^n}{\partial r} = \frac{\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)} \frac{1}{r_c} \left(\frac{b}{r_c} \right)^n \frac{1}{4} \left(n + \frac{1}{2} \right) \\ \times \left[\frac{\sin 2\phi}{n+1} F\left(\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2 \phi \right) F\left(\frac{1}{2}, n+\frac{1}{2}; 1; \sin^2 \phi \right) \left\{ \frac{1}{\sqrt{(r_0+r)(r_1-r)}} - \frac{1}{\sqrt{(r_0-r)(r_1+r)}} \right\} \right. \\ \left. + \sin 2\phi F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2 \phi \right) F\left(\frac{3}{2}, n+\frac{3}{2}; 2; \sin^2 \phi \right) \left\{ \frac{1}{\sqrt{(r_0+r)(r_1-r)}} + \frac{1}{\sqrt{(r_0-r)(r_1+r)}} \right\} \right] \quad \dots (41)$$

(b) The case of $r > r_0$. Since $r > r_0$, in this case, we put

$$\rho = 0, \quad \xi = n, \quad \mu = n \quad \sigma = 2n \quad a = r, \quad r_c = r \cos \phi \sin \phi (=y), \quad b = r \sin \phi \cos \phi (=x), \quad \left(\frac{\phi}{\phi} \right) = \frac{1}{2} \left[\sin^{-1} \frac{r_0}{r} \pm \sin^{-1} \frac{r_1}{r} \right]$$

Then we get

$$I_0^n = \frac{(-1)^n}{\pi} \left[\frac{\Gamma(n+(1/2))}{\Gamma(n+1)} \right]^2 \frac{1}{r} \left(\frac{r_0 b}{r^2} \right)^n F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2 \phi \right) F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2 \phi \right) \quad \dots (42)$$

Thereafter, considering the following relation

$$\left(\frac{\partial\phi}{\partial r}\right) = -\frac{1}{2r} \left[\frac{r_0}{\sqrt{r^2-r_0^2}} \pm \frac{r_i}{\sqrt{r^2-r_i^2}} \right]$$

and differentiating Eq.(42) with respect to r , we can get the result

$$\begin{aligned} \frac{\partial I_0^n}{\partial r} = & \frac{(-1)^{n+1}}{2\pi} \left[\frac{\Gamma(n+(3/2))}{\Gamma(n+1)} \right]^2 \frac{1}{n+1} \frac{1}{r^2} \left(\frac{r_0 b}{r^2} \right)^n \left[\frac{4(n+1)}{n+1/2} F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\phi\right) F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\phi\right) \right. \\ & + \sin 2\phi F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\phi\right) F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\phi\right) \left\{ \frac{r_0}{\sqrt{r^2-r_0^2}} - \frac{r_i}{\sqrt{r^2-r_i^2}} \right\} \\ & \left. + \sin 2\phi F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\phi\right) F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\phi\right) \left\{ \frac{r_0}{\sqrt{r^2-r_0^2}} + \frac{r_i}{\sqrt{r^2-r_i^2}} \right\} \right] \dots\dots\dots(43) \end{aligned}$$

5. Stress intensity factors

It is generally known that the stress along the extension of the crack axis into material region has a singularity $1/\sqrt{\rho}$, where ρ is a distance from a crack tip. It can be easily shown that the first term of Eq.(38) takes a finite value, and also it is evident from Eqs.(40) and (42) that I_0^n in the second term of the right-hand side in Eq.(38) has no singularity at a crack tip. In contrast with those, it is found that the expression $\partial I_0^n/\partial r$ has the singular parts such as $1/\sqrt{r_i-r}$ and $1/\sqrt{r-r_0}$. Consequently, the stress intensity factors corresponding to the opening mode at the inner and the outer edges of the crack are defined by the expressions

$$K_{Ii} = \lim_{r \rightarrow r_i-0} \sqrt{2(r_i-r)} (\sigma_z)_{z=0}, \quad K_{Io} = \lim_{r \rightarrow r_0+0} \sqrt{2(r-r_0)} (\sigma_z)_{z=0} \dots\dots\dots(44)$$

6. Numerical calculations

In numerical calculations we must solve a set of the simultaneous equations(31). For this purpose, first, we must evaluate the infinite integrals in Eq.(31). Let us put A_{mn} as the coefficient at the m -th row and the n -th column.

$$A_{mn} = A_{nm} = \int_0^\infty \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} G_n(\lambda) G_m(\lambda) d\lambda \dots\dots\dots(45)$$

Since the magnitude of $(\sinh^2 \lambda h - \lambda^2 h^2)/(\lambda h + \sinh \lambda h \cosh \lambda h)$ converges to unity rapidly with an increasing value of λ , A_{mn} may be expressed approximately by the equation

$$A_{mn} \approx \int_0^{\lambda_0} \frac{\sinh^2 \lambda h - \lambda^2 h^2}{\lambda h + \sinh \lambda h \cosh \lambda h} G_n(\lambda) G_m(\lambda) d\lambda + A_{mn}' \dots\dots\dots(46)$$

where λ_0 is a very large value, and A_{mn}' is represented by the equation

$$A_{mn}' = \int_{\lambda_0}^\infty G_n(\lambda) G_m(\lambda) d\lambda \dots\dots\dots(47)$$

In the present paper the first term in Eq.(46) is evaluated by the second rule of Simpson's numerical integral formula, taking $\lambda_0=500$. Meanwhile the asymptotic formula of the Bessel function

$$J_m(\lambda r) \approx \sqrt{\frac{2}{\lambda r \pi}} \cos\left(\lambda r - \frac{2m+1}{4} \pi\right)$$

is used to evaluate the second term when λ is large enough. Thus

$$A_{mn}' \approx \frac{4mn}{\pi^2 r_0^2 b^2} \int_{\lambda_0}^\infty \frac{1}{\lambda^2} [r_i^2 \sin^2 \lambda r_i - \{(-1)^m + (-1)^n\} r_i r_0 \sin \lambda r_i \cos \lambda r_0 + (-1)^{m+n} r_0^2 \cos^2 \lambda r_0] d\lambda \dots\dots\dots(48)$$

From the integration of Eq.(48) by parts, introducing the integral sine and cosine functions $si(x)$, $ci(x)$, we can obtain the approximate formula for A_{mn}'

$$\begin{aligned} A_{mn}' \approx & \frac{4mn}{\pi^2 r_0^2 b^2} \left[r_i^2 \left\{ \frac{\sin^2 \lambda_0 r_i}{\lambda_0} - r_i si(2\lambda_0 r_i) \right\} + (-1)^{m+n} r_0^2 \left\{ \frac{\cos^2 \lambda_0 r_0}{\lambda_0} + r_0 si(2\lambda_0 r_0) \right\} \right. \\ & \left. - \{(-1)^m + (-1)^n\} r_0 r_i \left\{ \frac{\sin \lambda_0 r_i \cos \lambda_0 r_0}{\lambda_0} - r_i ci(2\lambda_0 r_i) + b ci(2\lambda_0 b) \right\} \right] \dots\dots\dots(49) \end{aligned}$$

We obtain the coefficients A_{mn} in accordance with the above-mentioned method and get only the first n roots of a set of the simultaneous equations. Here we have preliminarily checked that we can get numerically good results, taking $n=15$ in a case of $r_i/r_0 \leq 0.2$ and $n=10$ in $r_i/r_0 > 0.2$.

Fig.2. The distributions of w_z for $r_i=0.5, r_o=1$ and $h=1$.

Fig.3. The distributions of σ_z for $r_i=0.5, r_o=1$ and $h=1$.

Fig.4. The distributions of $(w_z)_{z=0}$ and $(\sigma_z)_{z=0}$ for $r_o=1$ and $r_i=0.5$.

7. Numerical results

In the present results we put $\nu=0.3$. Figs.2 and 3 show the radial distributions of w_z and σ_z . In the same figures, each result corresponding to the dashed-line shows the one for an infinite medium $h \rightarrow \infty$. The displacement on the crack surface for the slab is always larger than that for the infinite body. When an external force acts on a body containing a crack, it is important to know the displacement and the stress on the crack surface. Then we check the effects of the slab thickness upon the stresses and the displacements. Fig.4 shows the distributions of w_z and σ_z on the plane $z=0$ for various thicknesses of slabs under the same dimension of the crack ($r_i=0.5, r_o=1.0$). Under the same pressure P_0 , $(w_z)_{z=0}$ increases as a slab becomes thin and, inversely, decreases with an increasing slab thickness, tending to approach the value for the infinite body. The stress $(\sigma_z)_{z=0}$ becomes infinite at a crack edge. It

Fig.5. The effect of a crack dimension on $(w_z)_{z=0}$ for $r_o=1$ and $h=1$.

Fig.6. The effect of a crack dimension on $(\sigma_z)_{z=0}$ for $r_o=1$ and $h=1$.

Fig.7. The variations of stress intensity factors with a slab thickness for $r_i/r_o=0.5$.

seems, however, that an asymptotic manner to infinity is predominant as a slab becomes thin.

Figs.5 and 6 show the radial distributions of $(w_z)_{z=0}$ and $(\sigma_z)_{z=0}$ for various values of the inner radius r_i under the same values of the slab thickness $h=1.0$ and the outer radius $r_o=1.0$.

Fig.7 shows the stress intensity factors at the inner and the outer edges of the crack in the case of the ratio of an inner radius to an outer one being equal to 0.5. In the same figure, the results for an infinite body are also shown.

Fig.8. The relationship between stress intensity factors and r_i/r_o .

Fig.8 shows the variations of stress intensity factors K_{Ii} and K_{Io} with the ratio r_i/r_o in the case of the ratio of slab thickness to crack outer radius, h/r_o , being unity. The results for an infinite body are also given.

8. Conclusions

The crack problem of a slab having an annular crack on the middle plane being subjected to pressure on its surface, is treated on the basis of the three-dimensional theory of elasticity. In analysis, the solution of triple integral equations is extremely simplified by solving a set of infinite simultaneous equations through the Fourier series representation of the displacement normal to the crack surface. In addition, displacements, stresses and stress intensity factors are also represented by the Fourier series, and they are evaluated and shown numerically.

From those results, it is found that the displacement on the crack surface becomes large as the slab thickness decreases under the same magnitude of pressure, and that the stress in the direction of the slab thickness becomes infinite at the crack edge, but the asymptotic tendency to infinity is predominant for a small thickness of the slab. Concerning the stress intensity factors, the values at an inner edge, K_{Ii} , are always larger than those at an outer edge, K_{Io} , and they become small as the slab becomes thick under the same crack dimension, converging to the values for an infinite body.

References

- (1) Sneddon, I.N., "The distribution of stress in the neighborhood of a crack in an elastic solid," Proc. Royal Soc. Lond., Ser.A, 187(1946), 229-260.
- (2) Lowengrub, M., "Stress in the vicinity of a crack in a thick elast. plate," Quart. Appl. Math., 19-2(1961-4), 119-126.
- (3) Moss, L.W. and Kobayashi, A.S., "Approximate analysis of axisymmetric problems in fract. mech. with application to a flat toroidal crack," Inter. J. of Fract. Mech., 7-1(1971-3), 89-99.
- (4) Shibuya, T. et al., "Axisymmetric stresses in an infinite elastic solid containing a flat annular crack under internal pressure," ZAMM, 55-7(1975-7), 395-402.
- (5) Erdelyi, A.(editor), "Tables of integral transforms," 1st ed. Vol.2, McGraw-Hill, New York, (1954).