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Optimum Design of Structures with Regard to Their Vibrational Characteristic*

(3rd Report, The Type-2-problems of Rods and Cantilevers, and Reciprocal Relation between Two Basic Types of Problems)

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In the first report, there has been presented a general method of optimum designs of structures in which natural frequencies are involved in objective functions and/or conditions of constraints.

In the second report, the method was demonstrated in the type-1-problems, which is one of basic problems of frequency maximization. Concrete objects were taken from structural elements having 1- and 2-section-freedom**** and numerical results obtained there were discussed in detail from many points of view.

In this report, another basic type-2-problem (i.e. the problem of finding the optimum shapes of structures which give minimum total mass under specified fundamental frequencies) are treated numerically by our same method, taking example of vibrating rods and cantilevers with a point mass.

Moreover, reciprocal relations between type-1-problem and type-2-problem are also clarified analytically in discrete systems and continuous ones, and the relations are verified numerically by making use of the results in rods and cantilevers.

1. Introduction

In the first report, a general method of optimum structural designs was proposed, in which natural frequencies are involved in objective functions and/or conditions of constraints.⁽¹⁾ In the subsequent second report, the type-1-problems of structural elements having 1- and 2-section-freedom were treated as handy illustrative examples, and a number of optimum shapes of structures were found to make fundamental frequencies maximum under given structural mass. Typical numerical results obtained by the method in the first report were demonstrated and investigated from many points of view.⁽²⁾

When we regard total structural weight and vibrational characteristic of structures, another basic types of problem can exist in this field. : the type-2-problem can exist, where optimum shapes of structures are found to make total structural mass minimum under given fundamental frequencies. And the problems can be classified into the problems of minimum weight designs and based on the fundamental idea of optimum designs. Since in this type-2-problems fundamental frequencies

should be given in the first stage of designs, it may bring us difficulty in specifying those fundamental frequencies in some cases. On the other hand, in the type-1-problem we have only to prescribe total structural mass, so we can treat it easily. Consequently, if the relation between two types of problems can be clarified, it may contribute to structural designs to a large extent.

In this report, type-2-problems in extensional vibration of a rod with a point lumped mass and in flexural vibration of a similar cross section cantilever with a point lumped mass were taken up as a typical example of vibration problems in structural elements having 1- and 2-section-freedom which were treated in the second report. And optimum shapes and minimum total structural mass were found by the method in the first report. Moreover we clarified the reciprocal relation between two basic types of problems analytically and examined them numerically by using the results of both types of problems in the second and this reports. Though we can use both the transfer matrix method and the large matrix method for vibration analysis, only the transfer matrix method is adopted here.

2. Nomenclature

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 - **** The writers use the terminology "an element having n-section-freedom" as an abbreviation of "an element having n-degree-of-freedom of section" for brevity.

- A : cross sectional area
- E : Young's modulus
- F : dimensionless axial force (or shearing force)
- I : second moment of inertia
- L : total length of rod (or beam)
- M : point lumped mass
- M_0 : reference mass
- M : dimensionless moment
- R : feasible region
- T : transfer matrix
- λ : dimensionless quantities of natural frequency (proportional to ω^2)

- ϕ : functional
 C : constant
 $f(X)$: objective function
 $g(X)$: function which forms constraint
 k : dynamic stiffness matrix of element
 l : length of element of rod (or beam)
 m : total structural mass of rod (or beam)
 n : number of element
 u : dimensionless extensional displacement
 v : dimensionless flexural displacement
 x : design variable
 α : Lagrange's multipliers
 θ : slope
 λ : square of angular frequency
 μ : mass ratio (point lumped mass/total structural mass) $= M/m$
 μ_G : mass ratio (point lumped mass/reference mass) $= M/M_0$
 μ_R : reciprocal of $\mu = 1/\mu$
 φ : function of design variable
 ω : angular frequency

3. Type-2-problem of a rod and a cantilever

3.1 The type-2-problem of a rod We consider a homogeneous rod with a point lumped mass having density ρ and Young's modulus E , and designate total length and cross sectional area at longitudinal coordinate ξ by L and $A(\xi)$ respectively as shown in Fig.1. In this system, Turner gave the analytically explicit solutions to the type-2-problem as follows⁽³⁾:

$$u_{opt}(\xi) = \sinh\left(\sqrt{A^{(1)}} \frac{\xi}{L}\right) \dots\dots\dots (1)$$

$$m_{opt}^*(\xi) = \frac{1}{2} \left(\frac{m}{L} \right) \frac{\sqrt{A^{(1)}} \sinh 2\sqrt{A^{(1)}} \frac{\xi}{L}}{\cosh^2\left(\sqrt{A^{(1)}} \frac{\xi}{L}\right)} \dots\dots\dots (2)$$

$$m_{opt} = M_0 \mu_G \sinh^2 \sqrt{A^{(1)}} \dots\dots\dots (3)$$

where $u_{opt}(\xi)$, $m_{opt}^*(\xi)$ are vibrational mode and longitudinal mass distribution of optimum rod, and $A = \rho \omega^2 L^2 / E$ is a dimensionless quantity corresponding to the fundamental frequency. And where m_{opt} is the total structural mass of optimum rod (i.e. minimum total structural mass) and $\mu_G = M/M_0$ mass ratio (a point lumped mass/reference mass).

Let the rod shown in Fig.1 be modeled by

total structural mass of rod m density ρ
 Young's modulus E point lumped mass M
 Fig.1 Rod with point lumped mass

Fig.2 Multi-element-model of rod

one having stepwisely varying cross section as shown in Fig.2 and we will find the solutions of the type-2-problem of this system by the method presented in the first report and compare them with analytically explicit ones.

For the convenience of numerical computation, we model the rod by n elements having equal length along it. If the shape function of extensional displacement in each element is assumed by linear function in longitudinal coordinate ξ , the stiffness and inertia matrices of elements can be given by the familiar finite element method. Furthermore assuming steady state harmonic vibration and separating time variable from extensional displacement, we can get dynamic stiffness matrices of elements, where the inertia terms are grouped into the stiffness matrices, and we have the following relation between force and displacement in elements :

$$\begin{Bmatrix} \bar{F}_j^L \\ -\bar{F}_j^R \end{Bmatrix} = x_j \begin{Bmatrix} 1 - \frac{A}{3n^2} & -1 - \frac{A}{6n^2} \\ -1 - \frac{A}{6n^2} & 1 - \frac{A}{3n^2} \end{Bmatrix} \begin{Bmatrix} \bar{u}_j^L \\ \bar{u}_j^R \end{Bmatrix}; \quad (j=1, 2, \dots, n) \dots\dots\dots (4)$$

(superscripts L and R indicate the quantities at left and right sides of elements respectively), where each quantity is expressed in dimensionless forms after being divided by common constant as length l , Young's modulus E and reference mass M_0 :

$\bar{u}_j$ = extensional displacement/ l

x_j = mass ratio (mass of j -th element/reference mass) $= \rho A_j l / M_0$

A = dimensionless quantity of natural frequency $= \rho \omega^2 L^2 / E$

The transfer matrices of elements to transmit the state vectors from left to right of each element can be written from Eq.(4) by:

$$T_j = \begin{Bmatrix} \frac{2(3n^2 - A)}{6n^2 + A} & -\frac{6n^2}{(6n^2 + A)x_j} \\ \frac{(72n^4 - 12n^2A + 5A^2)x_j}{6n^2(6n^2 + A)} & \frac{2(3n^2 + A)}{6n^2 + A} \end{Bmatrix}; \quad (j=1, 2, \dots, n) \dots\dots\dots (5)$$

The transfer matrix T_0 associated with a point lumped mass can be given :

$$T_0 = \begin{Bmatrix} 1 & 0 \\ -\frac{A}{n^2} \mu_G & -1 \end{Bmatrix} \dots\dots\dots (6)$$

Let us here consider the most simple case of 2-element-model as shown in Fig.3. The transfer matrix of the global system T is obtained by subsequent multiplication of each

Fig.3 2-element-model of rod

transfer matrix of elements T_1 , T_2 and that of a lumped mass T_0 :

$$T = T_1 T_2 T_0 \quad (7)$$

Hence if we substitute Eqs.(5),(6) into Eq.(7), T takes the following form:

$$T = \begin{bmatrix} \frac{1}{(6n^2+A)^2} \left\{ 4(3n^2-A)^2 - \frac{x_1}{x_2} (72n^4-12n^2A+5A^2) \right\} & \frac{12^2(3n^2-A)}{(6n^2+A)^2} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) \\ \frac{(3n^2-A)(72n^4-12n^2A+5A^2)}{3n^2(6n^2+A)^2} (x_1+x_2) & -\frac{1}{(6n^2+A)^2} \left\{ 4(3n^2-A)^2 - \frac{x_2}{x_1} (72n^4-12n^2A+5A^2) \right\} \end{bmatrix} \quad (8)$$

Giving boundary conditions:

$$\xi=0; \quad \ddot{u}_2^R=0 \quad (9)$$

$$\xi=L; \quad \ddot{F}_1^L=0 \quad (10)$$

to Eq.(8), we have the following frequency equation:

$$\frac{1}{(6n^2+A)^2} \left\{ 4(3n^2-A)^2 - \frac{x_1}{x_2} (72n^4-12n^2A+5A^2) \right\} + \frac{12(3n^2-A)A\mu_0}{(6n^2+A)^2} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) = 0 \quad (11)$$

On the other hand, as the total structural mass of the rod is expressed by:

$$m = (x_1+x_2)M_0 \quad (12)$$

is identical to unity by taking the point lumped mass as reference mass M_0 . The main condition of constraints and objective function (in the program for computer, minus sign is added to it and that is to be maximized) are

$$A^{(1)} = A^{(1)}(x_1, x_2) = -\bar{A} \quad (13)$$

$$\mu_R = \frac{1}{\mu} = \frac{m}{M} = x_1 + x_2 \quad (14)$$

The landscape of Eqs.(13),(14) is shown in Fig.4 for $A^{(1)} = 3.143$. The optimum point can be defined as a point on which the straight line of objective function in Eq.(13) is tangent to the hyperbola of the main constraint in Eq.(14).

In the case of the multi-element-model, we can treat the system in the same way, and we show the quantities of optimum design concisely in Table 1.

As the optimization is executed by the gradient projection method adopted in the first report, the following equation can be used for the computation of gradients of natural frequencies with respect to design variables as in the second report:

$$\frac{\partial A^{(1)}}{\partial x_j} = \left\{ \begin{array}{c} \ddot{u}_j^L \\ \ddot{u}_j^R \end{array} \right\}^T \left\{ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right\} - \frac{A^{(1)}}{n^2} \left\{ \begin{array}{cc} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{array} \right\} \left\{ \begin{array}{c} \ddot{u}_j^L \\ \ddot{u}_j^R \end{array} \right\}; \quad (j=1, 2, \dots, n) \quad (15)$$

In this type-2-problem, we have to prescribe the fundamental natural frequencies at the first stage of design. That requires us to correct the initial shape of a given rod so that its fundamental frequency becomes equal to the prescribed value. Since the

Table 1 Quantities of optimum design (rod)

design variable	$X = \{x_1, x_2, \dots, x_n\}$ $x_j = \frac{\rho A_j L}{M}; \quad (j=1, 2, \dots, n)$
condition of constraints	$g_j(X) = x_j \geq 0; \quad (j=1, 2, \dots, n)$ $g_{n+1}(X) = A^{(1)}(X) - A_c^{(1)} = 0$ $A_c^{(1)} = \frac{\rho \omega_1^2 L^2}{E} \quad (-\bar{A})$
feasible region	$R = \{X g_k(X) \geq 0; \quad (k=1, 2, \dots, n+1)\}$
objective function	$f(X) = -\sum_{j=1}^n x_j$

hypersurface of main constraint is nonlinear another operation is also required to correct design variable back into the feasible region at each improvement step. Comparing this computation with that of the type-1-problem in the second report, we may find more complexity in it.

We will next show concrete numerical examples. In Fig.5, the ratio of m_{opt} (minimum structural mass) in the optimum rod to m_{unl} in the uniform rod having the same fundamental frequency is shown for various numbers of elements ($n=2 \sim 20$) under constraint $A^{(1)} = 2.0$. It is seen that the ratio approaches to the solution obtained analytically by Turner with the increase of the number of elements. The variation of optimum shapes for various number of elements of this case is also shown in Fig.6 and it is also noted that the increase of the number of elements makes the optimum shapes similar to the analytical solution by Turner.

Fig.4 Landscape of objective function and conditions of constraints

Fig.5 Variation of total mass for various numbers of elements ($A^{(1)} = 2.0$, $\mu_0 = 1$)

3.2 The type-2-problem of a cantilever with a similar cross section We consider a system of a cantilever with a point lumped mass as shown in Fig.7, which has a similar cross section along its length. Besides it is assumed that the cantilever is homogeneous and has the density ρ and Young's modulus E . And its cross sectional area, the second moment of area, total length and total structural mass are denoted by A , I , L and m . As the cross sectional area is similar, the relation between the second moment of area and cross sectional area :

$$I = CA^2 \quad \dots\dots\dots (16)$$

is valid, where c is a common constant in all elements. It is impossible for us to find explicit solutions analytically for type-2-problem of this case. Consequently we will next have the solution by the method in the first report. As it is convenient for numerical calculation, the objective system is modeled by n element with equal length. Assuming flexural deflection in each element by a cubic polynomial of longitudinal coordinate ξ and using usual finite element procedure, we can derive the stiffness and inertia matrices of elements. After the separation of time variable under the assumption of the state of harmonic vibration, we have the following equation in dimensionless form :

$$\begin{Bmatrix} \bar{F}_j^L \\ \bar{M}_j^L \\ -\bar{F}_j^R \\ -\bar{M}_j^R \end{Bmatrix} = x_j^2 \begin{Bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ \text{Sym.} & & 12 & -6 \\ & & & 4 \end{Bmatrix} \begin{Bmatrix} \bar{v}_j^L \\ \bar{\theta}_j^L \\ \bar{v}_j^R \\ \bar{\theta}_j^R \end{Bmatrix} - \frac{Ax_j}{n^3} \begin{Bmatrix} 13 & 11 & 9 & -13 \\ 35 & 210 & 70 & -420 \\ & 1 & 13 & -1 \\ & 105 & 420 & -140 \\ \text{Sym.} & & 13 & 11 \\ & & 35 & -210 \\ & & & 1 \\ & & & 105 \end{Bmatrix} \begin{Bmatrix} \bar{v}_j^L \\ \bar{\theta}_j^L \\ \bar{v}_j^R \\ \bar{\theta}_j^R \end{Bmatrix} \quad (j=1, 2, \dots, n) \quad \dots\dots\dots (17)$$

Fig.6 Variation of optimum shape corresponding to number of elements ($A^{(1)}=2.0$, $\mu_G=1$)

Fig.7 Cantilever with similar cross section

where quantities are divided by such common constants in all elements as length of element l , density ρ , Young's modulus E , reference mass M_0 and c ($=I/A^2$), and have dimensionless forms :

$\bar{v}_j$ = flexural deflection/ l

$\bar{F}_j$ = shearing force/ $(M_0 E c / \rho^2 l^4)$

x_j = mass ratio (mass of j -th element / reference mass)

If we use the partitioned matrices in dynamic stiffness matrix seen in Fig.17, we can get the transfer matrices of elements transmitting state vector from left to right of each element in the following form :

$$T_j = \begin{bmatrix} -\bar{k}_{jB}^{-1} \bar{k}_{jA} & \bar{k}_{jB}^{-1} \\ -\bar{k}_{jB}^T - \bar{k}_{jC} \bar{k}_{jB}^{-1} \bar{k}_{jA} & -\bar{k}_{jC} \bar{k}_{jB}^{-1} \end{bmatrix}; \quad (j=1, 2, \dots, n) \quad \dots\dots\dots (18)$$

The transfer matrix of a point lumped mass can be written by :

$$T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{A}{n^3} \mu_G & 0 & 0 & -1 \end{bmatrix} \quad \dots\dots\dots (19)$$

where rotational inertia of it is neglected. Finally the transfer matrix of the global system is obtained by the following sequential multiplication of each transfer matrix of element and of the lumped mass :

$$T = T_n T_{n-1} \dots T_0 \quad \dots\dots\dots (20)$$

By equating the corresponding determinant to given boundary conditions to zero, we have the frequency equation and it is obvious that the minimum root of it is identical to the fundamental frequency.

The gradient of necessary for optimization with respect to design variables can be computed by the following equation :

$$\frac{\partial A^{(1)}}{\partial x_j} = \begin{Bmatrix} \bar{v}_j^L \\ \bar{\theta}_j^L \\ \bar{v}_j^R \\ \bar{\theta}_j^R \end{Bmatrix}^T \left\{ 2x_j \begin{Bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ \text{Sym.} & & 12 & -6 \\ & & & 4 \end{Bmatrix} - \frac{Ax_j}{n^3} \begin{Bmatrix} 13 & 11 & 9 & -13 \\ 35 & 210 & 70 & -420 \\ & 1 & 13 & -1 \\ & 105 & 420 & -140 \\ \text{Sym.} & & 13 & 11 \\ & & 35 & -210 \\ & & & 1 \\ & & & 105 \end{Bmatrix} \begin{Bmatrix} \bar{v}_j^L \\ \bar{\theta}_j^L \\ \bar{v}_j^R \\ \bar{\theta}_j^R \end{Bmatrix} \right\}; \quad (j=1, 2, \dots, n) \quad \dots\dots\dots (21)$$

The quantities of optimum design in this case are shown concisely in Table 2.

Table 2 Quantities of optimum design (cantilever)

design variable	$X = \{x_1, x_2, \dots, x_n\}$ $x_j = \frac{\rho A_j l}{M}; \quad (j=1, 2, \dots, n)$
condition of constraints	$g_j(X) = x_j \geq 0; \quad (j=1, 2, \dots, n)$ $g_{n+1}(X) = A^{(1)}(X) - A_c^{(1)} = 0$ $A_c^{(1)} = \frac{\rho^2 \omega_1^2 L^5}{MEC}$
feasible region	$R = \{X g_k(X) \geq 0; \quad (k=1, 2, \dots, n+1)\}$
objective function	$f(X) = -\sum_{j=1}^n x_j$

Let us next show concrete numerical examples. In Fig. 8, the ratio of m_{opt} (minimum structural mass) in the optimum cantilever to m_{uni} in the uniform cantilever with the same fundamental frequency is shown for various numbers of elements ($n=2\sim 20$) under main constraint $A^{(1)}=100.0$, where a point lumped mass is taken as reference mass. The ratio of $m_{\text{opt}}/m_{\text{uni}}$ tends to converge to a certain value even in 15-element-element. The variation of optimum shapes of the same case is also shown in Fig. 9 and it is observed that it takes a certain special form with increase of the number of elements.

4. Reciprocal relation between two basic types of problems

Let us investigate the reciprocal relation between the following two basic types of problems in both discrete and continuous systems:

(1) Type-1-problem: The problem of finding optimum shape with maximum fundamental frequency for a given structural mass,

(2) Type-2-problem: The problem of finding optimum shape with minimum structural mass for a given fundamental frequency.

And then, let us demonstrate and confirm the relation by using our numerical results of rods and cantilevers. However, our discussion is restricted only to the necessary conditions of existence of optimum solutions and examinations of sufficient conditions are out of our interest.

Fig. 8 Variation of minimum total mass corresponding to number of elements ($A^{(1)}=100.0$, $\mu_0=1$)

Fig. 9 Variation of optimum shape corresponding to number of elements ($A^{(1)}=100.0$, $\mu_0=1$)

4.1 The reciprocal relation in a discrete system It is assumed that the objective structure is composed of n finite elements and each element is represented by a single variable. If we denote the square of the fundamental frequency and total structural mass by λ and m respectively, they are functions of design variable x and vibrational mode y as follows in general case:

$$\lambda = \lambda(X, Y) \quad \dots\dots\dots (22)$$

$$m = m(X) \quad \dots\dots\dots (23)$$

The necessary conditions for optimality are obtained as shown in Table 3. From this table it is seen that both types of problems reduce to the following type of problem having the same condition.

$$\frac{\partial \lambda}{\partial Y} = 0 \quad \dots\dots\dots (24)$$

$$\frac{\partial \lambda}{\partial X} + \alpha \frac{\partial m}{\partial X} = 0 \quad \dots\dots\dots (25)$$

where α is a constant. Hence in the same objective structure and under the same boundary conditions, there exist the following reciprocal relation between both types of problems.

(1) If we have optimum solution x_1 and maximum fundamental frequency λ_1 in the type-1-problem under the constraint of the total structural mass equal to m_1 , the optimum solution in the type-2-problem under the constraint λ_1 of the same fundamental frequency coincides with x_1 and minimum total structural mass is identical to m_1 .

(2) If we have optimum solution x_2 and minimum total structural mass m_2 in the type-2-problem under the constraint of fundamental frequency equal to λ_2 , the optimum solution in the type-1-problem under the constraint of the same total structural mass m_2 coincides with x_2 and maximum fundamental frequency is identical to λ_2 .

4.2 The reciprocal relation in a continuous system We will designate the square of fundamental angular frequency and total structural mass of the structure by λ and m . We consider first that those are functionals composed of function $x(\xi)$ in a single spatial variable ξ , vibrational mode $y(\xi)$ and their

Table 3 Formulation of two basic types of problems and necessary condition of extremum (discrete system)

Type-1-problem	Type-2-problem
(a) objective function and constraints objective function $\lambda = \lambda(X, Y)$ condition of constraints $m = m(X) = m_c$ (const.) $X \geq 0$ (This condition is considered to be subsidiary)	(a') objective function and constraints objective function $m = m(X)$ condition of constraints $\lambda = \lambda(X, Y) = \lambda_c$ (const.) $X \geq 0$ (This condition is considered to be subsidiary)
(b) objective function using Lagrange's multipliers $\varphi_1 = \lambda(X, Y) + \alpha_1 [m(X) - m_c]$ (α_1 : Lagrange's multiplier)	(b) objective function using Lagrange's multipliers $\varphi_2 = m(X) + \alpha_2 [\lambda(X, Y) - \lambda_c]$ (α_2 : Lagrange's multiplier)
(c) necessary condition of extremum $\frac{\partial \varphi_1}{\partial Y} = 0, \frac{\partial \varphi_1}{\partial X} = 0$ $\rightarrow$ $\frac{\partial \lambda}{\partial Y} = 0,$ $\frac{\partial \lambda}{\partial X} + \alpha_1 \frac{\partial m}{\partial X} = 0$	(c') necessary condition of extremum $\frac{\partial \varphi_2}{\partial Y} = 0, \frac{\partial \varphi_2}{\partial X} = 0$ $\rightarrow$ $\frac{\partial \lambda}{\partial Y} = 0,$ $\frac{\partial m}{\partial X} + \alpha_2 \frac{\partial \lambda}{\partial X} = 0$

derivatives :

$$\lambda = \int_{\xi_0}^{\xi_1} \lambda^* [\xi, x(\xi), y(\xi), y'(\xi), y''(\xi)] d\xi \quad (26)$$

$$m = \int_{\xi_0}^{\xi_1} m^* [\xi, x(\xi)] d\xi \quad (27)$$

where $y'(\xi)$, $y''(\xi)$ are the first and the second derivatives of $y(\xi)$ with respect to ξ , and ξ_0 , ξ_1 are the values at both ends of the domain of function $x(\xi)$ and $y(\xi)$. The expressions (26), (27) are valid for vibration problems of such simple structural members as rods and beams. The formulation of two basic types of problems and necessary conditions of extremum are shown in Table 4. From table 4 the both types of problems reduce to the following type of problem having the same conditions :

$$\frac{d^2}{d\xi^2} \left(\frac{\partial \lambda^*}{\partial y''} \right) - \frac{d}{d\xi} \left(\frac{\partial \lambda^*}{\partial y'} \right) + \frac{\partial \lambda^*}{\partial y} = 0 \quad (28)$$

$$\frac{\partial \lambda^*}{\partial x} + \alpha \frac{\partial m^*}{\partial x} = 0 \quad (29)$$

where α is a constant. The same reciprocal relation as in the discrete system can exist here too if the objective structures and the boundary conditions are identical in them :

(1) If we have optimum solution $x_1(\xi)$ and maximum fundamental frequency λ_1 in the type-1-problem under the constraint of total structural mass equal to m_1 , the optimum solution in the type-2-problem under the constraint of the same fundamental frequency coincides with $x_1(\xi)$ and minimum total structural mass is identical to m_1 .

(2) If we have optimum solution $x_2(\xi)$ and minimum total structural mass m_2 in the type-2-problem under the constraint of fundamental frequency equal to λ_2 , the optimum solution in the type-1-problem under the constraint of the same total structural mass coincides with $x_2(\xi)$ and maximum fundamental frequency is identical to λ_2 .

It should be noted that these reciprocal relations are based on the principle of reciprocity in a general isoperimetric variational problem.⁽⁴⁾

In the case that design variable x and vibrational mode y are functions composed of two spatial coordinates ξ and η ;

$$x = x(\xi, \eta) \quad (30)$$

$$y = y(\xi, \eta) \quad (31)$$

it can be added that we can discuss and derive the reciprocal relation in the same way only by replacing Euler's equation by Ostrogradski's equation (or Euler-Lagrange's equation).

4.3 Concrete demonstration of the reciprocal relation in a discrete system We first consider the system of a rod with a point lumped mass. In the second report dealing with the type-1-problem of the system, the design variable and a point lumped mass were expressed in the dimensionless form :

$$x_j = \frac{\rho A_j l}{m}; \quad (j=1, 2, \dots, n) \quad (32)$$

$$\mu = \frac{M}{m} \quad (33)$$

by taking total structural mass as reference mass. However, in this report handling the type-2-problem of the same system, the total structural mass varies during design procedure, so that they are expressed by the following mass ratio :

$$x_j = \frac{\rho A_j l}{M}; \quad (j=1, 2, \dots, n) \quad (34)$$

$$\mu_0 = \frac{M}{M} = 1 \quad (35)$$

by taking the point lumped mass as reference mass. Such inconsistency makes us inconvenient in case of comparison of both results. Therefore we will hereafter adopt the same dimensionless quantities by taking the lumped mass as reference mass.

The landscape of objective function and optimization process in the both types of problems are shown in Fig.10, where the rod

Fig.10 Optimization process of two basic types of problems (2-element-model of rod)

Table 4 Formulation of two basic types of problems and necessary condition of extremum (continuous system)

Type-1-problem	Type-2-problem
<p>(a) objective functional and condition of constraints objective functional $\lambda = \int_{\xi_0}^{\xi_1} \lambda^* [\xi, x(\xi), y(\xi), y'(\xi), y''(\xi)] d\xi$ condition of constraints $m = \int_{\xi_0}^{\xi_1} m^* [\xi, x(\xi)] d\xi = m_c \text{ (const.)}$ $x(\xi) \geq 0$ (This condition is considered to be subsidiary)</p>	<p>(a)' objective functional and condition of constraints objective functional $m = \int_{\xi_0}^{\xi_1} m^* [\xi, x(\xi)] d\xi$ condition of constraints $\lambda = \int_{\xi_0}^{\xi_1} \lambda^* [\xi, x(\xi), y(\xi), y'(\xi), y''(\xi)] d\xi = \lambda_c \text{ (const.)}$ $x(\xi) \geq 0$ (This condition is considered to be subsidiary)</p>
<p>(b) objective functional using Lagrange's multiplier $\Phi_1 = \int_{\xi_0}^{\xi_1} \{ \lambda^* + \alpha_1 (m^* - m_c) \} d\xi$ (α_1: Lagrange's multiplier)</p>	<p>(b)' objective functional using Lagrange's multiplier $\Phi_2 = \int_{\xi_0}^{\xi_1} \{ m^* + \alpha_2 (\lambda^* - \lambda_c) \} d\xi$ (α_2: Lagrange's multiplier)</p>
<p>(c) necessary condition of extremum $\frac{d^2}{d\xi^2} \left(\frac{\partial \lambda^*}{\partial y''} \right) - \frac{d}{d\xi} \left(\frac{\partial \lambda^*}{\partial y'} \right) + \frac{\partial \lambda^*}{\partial y} = 0,$ $\frac{\partial \lambda^*}{\partial x} + \alpha_1 \frac{\partial m^*}{\partial x} = 0$</p>	<p>(c)' necessary condition of extremum $\frac{d^2}{d\xi^2} \left(\frac{\partial \lambda^*}{\partial y''} \right) - \frac{d}{d\xi} \left(\frac{\partial \lambda^*}{\partial y'} \right) + \frac{\partial \lambda^*}{\partial y} = 0,$ $\frac{\partial m^*}{\partial x} + \alpha_2 \frac{\partial \lambda^*}{\partial x} = 0$</p>

is modeled by 2-element-model. In the type-1-problem, taking $x_1=5$, $x_2=5$ (the values of uniform rod) as an initial point and giving the constraints, we found the optimum fundamental frequency $A=3.143$ and optimum solution $x_1=2.130$, $x_2=7.870$. On the other hand, in the type-2-problem taking the same initial value and the same optimum solution $x_1=2.130$, $x_2=7.870$. The optimum solution of both cases are given as a contact point of $A^{(1)}=3.143$ on $x_1+x_2=10$.

Let us examine the reciprocal relation in the case of multi-element-model. The optimum shape of the rod of 20-element-model in the type-2-problem is shown in Fig.11 for constraints of $A^{(1)}=2.0$ and $\mu_G=1$. If we takes the obtained minimum structural mass $\sum_{j=1}^{20} x_j=3.745$, $\mu_G=1.0$ for constraints in the type-1-problem, we have the same optimum shape as Fig.11. Hence we can recognize the validity of the reciprocal relation.

In the case of the system of a cantilever with a point lumped mass, we can also confirm the reciprocal relation by comparing the results of both types of problems. The optimum shape of a cantilever of 20-element-model in the type-2-problem is shown in Fig.12 for the constraints $A^{(1)}=100.0$, $\mu_G=1$. It is also observed that the optimum shape in the type-1-problem almost coincides with one in Fig.12 for the constraints of $\sum_{j=1}^{20} x_j=5.422$, $\mu_G=1$ obtained in the type-1-problem.

5. Conclusion

As handy and simple examples of basic type of problem in the optimum structural design with respect to their vibrational characteristic and weight, we took the type-2-problems of rods and cantilevers with a

point lumped mass. i.e. We found optimum shapes of them to make the total structural mass minimum for given fundamental frequencies. And we had the following conclusion:

(1) In the case of rods, the total structural mass was reduced to 41.9% of that of uniform rods with the same fundamental frequencies for the constraint of $A^{(1)}=2.0$, $\mu_G=1$. On the other hand, in the case of cantilevers the total structural mass was reduced to 49.6% of that of uniform cantilevers with the same fundamental frequencies for the constraints of $A^{(1)}=100.0$ and $\mu_G=1$.

(2) The numerical results of rods obtained here coincides fairly well with Turner's analytical solution.

We clarified the following reciprocal relation between both discrete and continuous systems, and demonstrated it concretely by using the numerical results obtained in the second and this reports:

- (1) The optimum solution in the type-2-problem coincides with one in the type-1-problem if the fundamental frequency obtained in the type-1-problem is taken for the constraint there.
- (2) The optimum solution in the type-1-problem coincides with one in the type-2-problem if the fundamental frequency obtained in the type-2-problem is taken for the constraint there.

In this study, we used the computer systems of IBM 7090 in Waseda University and HITAC 5020 in Tokyo University. The author wishes to express his gratitude to Mr. Okuda and Yasukawa (they are now in HITACHI Ltd.) who carried out numerical computations.

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Fig.11 Optimum shape of rod (20-element-model)

Fig.12 Optimum shape of cantilever (20-element-model)