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A METHOD FOR THE RAPID SOLUTION OF THE DISPLACEMENT OF THE KINEMATIC LINKAGE*

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In this paper the techniques are presented which allow one to solve rapidly the displacement equation of the kinematic linkage. Especially the following two techniques make a significant contribution to the development of the method.

(1) The new method of formulation of the problem saves one from the cumbersome manipulation of the equations, which is usually required to translate the statement about a given linkage into an analytical one.

(2) A set of displacement equations is solved by the Newton-Raphson method, which is remarkable in convergence speed but is apt to be affected by the choice of the starting value. Owing to the closer estimation by the parabolic extrapolation through the last three solutions for the preceding three input crank angles, the convergence rate is about twice that of the usually chosen starting value or the solution corresponding to the preceding input crank angle.

I. INTRODUCTION

Analysis of the displacement of the kinematic linkage requires the solution of a closed-loop equation or higher-order simultaneous equations. As it cannot be solved analytically except for the simple linkage, such as a planar 4-bar linkage, many methods of the numerical solution have been presented so far. In general, however, they demand a large amount of algebraic manipulation for the formulation of the problems^(1,2)

For example, Hartenberg's iterative method⁽³⁾ has, in addition to the defect that the solution time is comparatively large, the difficulty that a large amount of time is spent in preparing the input data because of its very generality in formulation. While the analysis method utilizing the optimization scheme^(4,5) provides the advantage of the brevity of the formulation, it is considered to be impractical for the reason of the large computational time.

On the other hand, development of a high-speed analysis method is desired particularly when the displacement analysis should be frequently repeated as in the case of optimizing design parameters of the linkage by means of the mathematical programming methods.

The analysis method presented in this paper is endowed with quickness in computation through exploitation of the rapid convergence characteristic of the Newton-Raphson method.

Additionally, the brevity in the formulation of the problem, which is implemented through the use of revolute matrix, saves us from the cumbersome mathematical manipulation.

2. NOMENCLATURE

i, j, k	: orthogonal base vectors. The planar linkage moves in the planes perpendicular to k .
$\hat{t}, \hat{u}, \hat{v}, \hat{w}$	: unit vectors which stand for revolute axes.
θ, ϕ, ψ, χ	: relative angular displacements between adjacent links.
x	: solution vector $(\phi, \psi, \chi)^T$.
$R(\theta, \hat{t})$	: revolute matrix (see appendix).

3. THE METHOD FOR THE DISPLACEMENT ANALYSIS OF THE LINKAGE

3.1 The fundamental consideration

For the sake of brevity, the planar 4-bar linkage, shown in Fig.1 will be used to illustrate the conception of the present method. The unit vectors $\hat{t}, \hat{u}, \hat{v}$, and $\hat{w}$ stand for the orientation of the axes of the revolute pairs joining the links 1, 2, 3, and 0 to the links 0, 1, 2, and 3, respectively. Of course all these unit vectors in this case coincide with a unit vector k but they are discriminated for convenience of explanation.

First, at the point P the closed loop of the linkage is cut into two open loops. The point P is now considered a double point representing both the point P₂ and the point P₃.

Fig.1 Planar 4-bar linkage

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The reference vectors $\overrightarrow{OQ_{10}}$, $\overrightarrow{Q_{20}P_{20}}$, and $\overrightarrow{O_1P_{30}}$ for the respective links 1, 2, and 3 are established as shown in Fig. 2. They are used as bases to be referred to when the relative displacements among the adjacent links are measured. Therefore, in general, they need not always form a closed loop as shown in Fig. 2.

In case of a spatial open loop connected with revolute pairs, illustrated in Fig. 3, the kinematic variables θ , ϕ , and ψ represent the angular displacements of the links relative to adjacent links nearer to the frame link. The positional vector $\overrightarrow{OP}$ in the open loop, such as that of Fig. 3 is calculated in the following manner.

The initial link vectors, $a\hat{a}_0$, $b\hat{b}_0$, and $c\hat{c}_0$, which serve also as the reference link vectors, are rotated by the angular displacement θ about the axis $\hat{t}$ into the following forms.

$$\left. \begin{aligned} a\hat{a} &= R(\theta, \hat{t})a\hat{a}_0 & \hat{u} &= R(\theta, \hat{t})\hat{u}_0 \\ b\hat{b}' &= R(\theta, \hat{t})b\hat{b}_0 & \hat{v}' &= R(\theta, \hat{t})\hat{v}_0 \\ c\hat{c}' &= R(\theta, \hat{t})c\hat{c}_0 \end{aligned} \right\} (1)$$

The above expressions are transformed by the further angular displacement ϕ about the axis $\hat{u}$ as follows.

$$\left. \begin{aligned} b\hat{b} &= R(\phi, \hat{u})b\hat{b}' & \hat{v} &= R(\phi, \hat{u})\hat{v}' \\ c\hat{c}'' &= R(\phi, \hat{u})c\hat{c}' \end{aligned} \right\} (2)$$

The vector $c\hat{c}''$ is transformed by the further rotation ψ of the link 3 relative to the link 2 into the following form

$$c\hat{c} = R(\psi, \hat{v})c\hat{c}'' \quad (3)$$

After calculation of the net orientation of each link in the open loop with respect to the relative rotations θ , ϕ , and ψ , the positional vector of the point P is expressed as

$$\overrightarrow{OP} = \mathbf{P} = a\hat{a} + b\hat{b} + c\hat{c} \quad (4)$$

The reference vectors $a\hat{a}_0$, $b\hat{b}_0$, and $c\hat{c}_0$ may be chosen arbitrarily as shown in Fig. 3, where they are chosen to be collinear for simplicity.

The procedure above described is summarized in Table 1, named in this paper "displacement sequence diagram".

Fig. 2 Illustration of the present method for the analysis of the 4-bar linkage

Fig. 3 Relative angular displacement of links and the reference configuration for the linkage

By referring to this table, the positional vector of the point P, where the closed loop is cut into two open loops, can be evaluated almost mechanically.

Returning to the planar 4-bar linkage shown in Fig. 2, the displacement analysis is performed in the following steps;

(Step 1) When both the relative angular displacements and the displacement sequence diagram are given as shown in Table 2, positional vectors $\overrightarrow{OP_{21}}$ and $\overrightarrow{OP_{30}}$ for the dis-jointed points P_{21} and P_{30} are determined by the above-mentioned procedure. In order for two open loops to form a closed loop, it is required that the point P_{21} coincide with the point P_{30} , that is, the relation $\overrightarrow{OP_{21}} = \overrightarrow{OP_{30}}$ hold.

(Step 2) The links 2 and 3 are rotated ϕ and ψ about the respective revolute axes $\hat{u}$ and $\hat{v}$ until the condition

$$P_{32} = \overrightarrow{OP_{32}} - \overrightarrow{OP_{22}} = \mathbf{f}(\phi, \psi) = \mathbf{f}(\mathbf{X}) = \mathbf{0} \quad (5)$$

$$\text{or} \quad h(\mathbf{X}) = |\mathbf{P}_{32}|^2 = |\overrightarrow{OP_{32}} - \overrightarrow{OP_{22}}|^2 = 0 \quad (6)$$

is satisfied.

(Step 3) When Eq. (5) or (6) is realized, two open loops in question are closed to form a quadrilateral $OQ_{11}P_{22}O_1$, and namely the displacement equation of the linkage is thought to have been solved for a given position of the input link $a\hat{a}$.

(Step 4) The input crank angle θ is advanced by $d\theta$ and the preceding steps are iterated for the updated input crank angle $\theta + d\theta$.

The flow chart for the analysis is shown in Fig. 4. The number written in the chart corresponds to that of the above-mentioned step.

3.2 Computational techniques

First in this section, the method for solution of $\mathbf{X}$ mentioned in Step 2 is to be described.

The problem which appears in Step 2 may be considered either that of solving the equation $\mathbf{f}(\mathbf{X}) = \mathbf{0}$ or that of finding the

Table 1 Displacement sequence diagram

	$\hat{t}$	$\hat{a}$	$\hat{u}$	$\hat{b}$	$\hat{v}$	$\hat{c}$
θ	Δ	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
ϕ	$\times$	$\times$	Δ	$\bigcirc$	$\bigcirc$	$\bigcirc$
ψ	$\times$	$\times$	$\times$	$\times$	Δ	$\bigcirc$

Δ axis of rotation

$\bigcirc$ axis affected by the rotation about the axis specified with Δ

$\times$ axis unaffected by the rotation about the axis specified with Δ

Table 2 Displacement sequence diagram for the analysis of the planar 4-bar linkage

	$\hat{t}$	$\hat{a}$	$\hat{u}$	$\hat{b}$	$\hat{v}$	$\hat{c}$
θ	Δ	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\times$	$\times$
ϕ	$\times$	$\times$	Δ	$\bigcirc$	$\times$	$\times$
ψ	$\times$	$\times$	$\times$	$\times$	Δ	$\bigcirc$

minimum of the positive definite function $h(\mathbf{x})$ in Eq.(6). As the development of the rapid analysis method is aimed at in this paper, the former is chosen and solved by the use of the Newton-Raphson method, whose convergence speed is inherently remarkable. The latter, however, is also solved by the optimization technique with respect to the initial position of the input crank and its solution is used as a good starting value for the former scheme. The Newton-Raphson method is applied as follows;

Suppose that the solution $\mathbf{x}_{n-1}$, for rotation θ_{n-1} of the input link, has been determined from the equation

$$f(\theta_{n-1}, \mathbf{x}_{n-1}) = 0 \quad (7).$$

Equation (7) no longer holds for the further rotation $d\theta$ of the input link, or

$$f(\theta_n, \mathbf{x}_{n-1}) \neq 0 \quad (8).$$

Here suppose that the solution $\mathbf{x}_n$ corresponding to θ_n is obtained by modification of the preceding solution $\mathbf{x}_{n-1}$ by $d\mathbf{x}$, or

$$f(\theta_n, \mathbf{x}_{n-1} + d\mathbf{x}) = 0 \quad (9).$$

Equation (9) can be expanded with respect to the increment $d\mathbf{x}$ in the following form

$$f(\theta_n, \mathbf{x}_{n-1} + d\mathbf{x}) = f(\theta_n, \mathbf{x}_{n-1}) + \frac{\partial f}{\partial \mathbf{x}} d\mathbf{x} + \dots = 0 \quad (10),$$

Fig.4 Flow chart of the displacement analysis of the linkage

Table 3 Input data for the analysis of the planar 4-bar linkage

	$\hat{r}$	$\hat{a}$	$\hat{u}$	$\hat{b}$	$\hat{v}$	$\hat{c}$
i	0.0	1.0	0.0	0.7071	0.0	0.0
j	0.0	0.0	0.0	0.7071	0.0	1.0
k	1.0	0.0	1.0	0.0	1.0	0.0

N	10	tolerable number of iterations
$\Delta\theta$	10°	increment of angular displacement of the input link
ϵ	10^{-6}	tolerable accuracy of
h	10^{-3}	increment for numerically evaluating the derivatives

$$a=2.0 \quad b=5.0 \quad c=4.0 \quad d=6.0$$

$$\phi=0.0 \quad \psi=0.0$$

where

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (11)$$

Neglecting all but the linear terms in Eq. (10), $d\mathbf{x}$ is solved as follows;

$$d\mathbf{x} = -\left(\frac{\partial f}{\partial \mathbf{x}}\right)^{-1} f(\theta_n, \mathbf{x}_{n-1}) \quad (12)$$

In practice, however, $\mathbf{x}_{n-1}$ in Eq.(12) need not be always taken as the preceding solution for $\theta = \theta_{n-1}$ but can be generally taken as an approximate estimate to the solution corresponding to $\theta = \theta_n$.

Each element of the matrix in Eq.(11) is evaluated by the use of the formula of numerical differentiation

$$\frac{\partial f_i}{\partial x_j} = \frac{f_i(\dots, x_j+h, \dots) - f_i(\dots, x_j-h, \dots)}{2h} \quad (13)$$

where h is an increment of the appropriate magnitude.

In the analysis of the linkage, usually Eq.(9) is desirably solved for rotations of the input link with a constant interval. As the solution $\mathbf{x}$ is normally continuous with respect to the rotation of the input link θ , a good approximate estimate $\mathbf{x}_i$ for $\theta = \theta_i$ is obtained by the parabolic extrapolation formula

$$\mathbf{x}_i = 3\mathbf{x}_{i-1} - 3\mathbf{x}_{i-2} + \mathbf{x}_{i-3} \quad (14),$$

where $\mathbf{x}_{i-3}$, $\mathbf{x}_{i-2}$, and $\mathbf{x}_{i-1}$ are solutions obtained for θ_{i-3} , $\theta_{i-2} = \theta_{i-3} + d\theta$, and $\theta_{i-1} = \theta_{i-3} + 2d\theta$, respectively.

It has been ascertained by examples that if this $\mathbf{x}_i$ is used as the starting value for θ_i the number of iterations of the Newton-Raphson method is reduced to half that attained by immediate use of the solution $\mathbf{x}_{i-1}$ corresponding to $\theta = \theta_{i-1}$.

4. APPLICATIONS

The examples of application of the present method to the kinematic linkages are illustrated in this section.

4.1 Planar 4-bar linkage

The numerical results shown in Table 4 are obtained when the present method is applied to the planar 4-bar linkage specified by Table 3.

Informations to be prepared before ex-

Table 4 Numerical results obtained by the analysis of the planar 4-bar linkage

No.	θ	$x(\Phi, \psi)$	start value*	start value**	error	No.***
			solution		f^2	No.***
0	0	0.0	0.0		0.43	3
		6.31	12.62		$< 10^{-6}$	12
1	10	6.31	12.62		1.2	4
		-8.75	8.23		$< 10^{-6}$	18
2	20	-8.75	8.23		1.3	4
		-23.71	5.21		$< 10^{-6}$	18
3	30	-23.71	5.21		1.5	2
		-38.55	3.585		5×10^{-4}	7
		-38.25	3.749		$< 10^{-6}$	7
4	40	-38.25	3.749		1.5	2
		-52.38	3.821		2×10^{-4}	7
		-52.24	3.774		$< 10^{-6}$	7
5	50	-52.24	3.774		1.5	2
		-65.67	5.295		2×10^{-4}	7
		-65.63	5.120		$< 10^{-6}$	7

* the preceding solution

** the parabolic extrapolation of the last three solutions

*** the number of iterations

**** the number of evaluations of function

ecution of analysis are parameters of the linkage and the position of the junction point at which a closed loop is separated into two open loops.

The latter can be chosen arbitrarily in this case and is given in Table 2 of the displacement sequence diagram. Particularly when the former informations are prepared, the following items should receive close attention.

- (a) The reference vectors, $\hat{a}_0, \hat{b}_0, \hat{c}_0$, and $\hat{d}_0$ which also serve as the respective initial vectors, can be given arbitrarily. But, for the speed-up and the stability of the convergence of the Newton-Raphson method it is desirable that the starting vector $\mathbf{x}_0$ be chosen as close as possible to the solution for the first position of the input link.
- (b) The data should be given accurately which supply implicitly the constraint conditions by the kinematic pairs, such as the initial position vector of the link and the revolute axis attached to it which jointly prescribe the relative orientation of the revolute axis.

One of the numerical results (Table 4) is verified by the result of graphical analysis for $\theta=30^\circ$ shown in Fig.5. It is noted in Table 4 that the numbers of iterations and the numbers of evaluations of the function in the cases of no's.0,1, and 2 are about twice those of the cases of no's.3,4, and 5, where the number of evaluations of the function means that of calling the subroutine by which updating of each unit link vector and junction point is done according to the displacement sequence diagram. This fact is caused by immediate use of the preced-

Fig.5 Configuration of the 4-bar linkage at $\theta=30^\circ$ for the starting value and the solution

Table 5 Input data for the analysis of the spatial 4-bar linkage RSCR

	$\hat{u}$	$\hat{a}$	$\hat{v}$	$\hat{d}$	$\hat{c}$	$\hat{b}$	
θ	Δ	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	rotation
ϕ	$\times$	$\times$	Δ	$\bigcirc$	$\bigcirc$	$\bigcirc$	rotation
ψ	$\times$	$\times$	$\times$	$\times$	Δ	$\bigcirc$	rotation
s	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\times$	translation

	$\hat{u}$	$\hat{a}$	$\hat{v}$	$\hat{d}$	$\hat{c}$	$\hat{b}$
i	0.7071	0.0	0.0	0.0	0.0	0.0
j	-0.7071	0.0	-1.0	0.0	0.8660	0.5
k	0.0	1.0	0.0	1.0	-0.5	0.8660

ing solution as the starting value of the cases of no's.0,1, and 2 due to lack of data sufficient for extrapolation of starting value.

4.2 Spatial 4-bar linkage RSCR

It will be shown in this section that the spatial 4-bar linkage can also be analyzed as easily as the planar 4-bar linkage.

Here, an RSCR type spatial 4-bar linkage shown in Fig.6 is taken as an example because of easiness of verification of the results obtained, where R, S, and C stand for revolute, spheric, and cylindric pairs, respectively. The input data to be provided beforehand are shown in Table 5 and Fig.7.

The matters which should be taken into consideration will be explained in some detail in what follows;

(a) Constitution of $\mathbf{f}$

Since there is a spheric pair S in the linkage, it is natural to take the center point of S as the junction point P which is named P_1 on the link 1 and P_2 on the link 2 respectively. Hence

$$\mathbf{f} = \overrightarrow{OP_1} - \overrightarrow{OP_2} \quad (15),$$

Fig.6 Spatial 4-bar linkage RSCR

Fig.7 Reference configuration for the linkage RSCR

$$a=2.0 \quad b=2.0 \quad d=1.5 \quad f=6.1962 \quad N=10$$

$$\Delta\theta=10^\circ \quad \epsilon=10^{-6} \quad h=10^{-3}$$

$$\phi=-1.0 \quad \psi=0.5 \quad s=5.0$$

where
 $\overrightarrow{OP_1} = a\hat{a} \quad \overrightarrow{OP_2} = f\hat{j} + d\hat{d} - s\hat{c} - b\hat{b} \quad (16).$

(b) Selection of elements of $\mathbf{x}$

Since in this spatial problem the equation $\mathbf{f}=\mathbf{0}$ is actually equivalent to three scalar equations, the number of kinematic variables should also be three so that the equation may be solved. Therefore it may be appropriate to assume three kinematic variables ϕ, ψ , and s as the elements of $\mathbf{x}$.

(c) Drawing of the displacement sequence diagram

This diagram concisely indicates the sequence of the procedure with which link vectors are computed for every new displacement $\mathbf{x}$. It can be drawn easily after determination of the junction point P and elements of $\mathbf{x}$.

(d) Computation of the displacement

Referring to the foregoing displacement sequence diagram, two positions of the junction point P are computed mechanically as follows: First, since Table 5 implies that the rotation θ of the input crank about the axis $\hat{a}$ has influence only upon the vector $\hat{a}$, the initial vector $\hat{a}_0$ is changed into $\hat{a} = R(\theta, \hat{a})\hat{a}_0$. Next, since the displacement of vectors $\hat{a}_0, \hat{c}_0$, and $\hat{b}_0$ is caused by the rotation ϕ about the axis of output link $\hat{v}$, they are updated by submitting them to the action of $R(\phi, \hat{v})$. For example, $\hat{b}_0$ is transformed into $\hat{b}'_0 = R(\phi, \hat{v})\hat{b}_0$. Further, since only displacement of vector $\hat{b}$ is caused by rotation ψ about the intermediate link $\hat{c}$ as is shown in Table 5, $\hat{b}'_0$ is finally transformed into $\hat{b} = R(\psi, \hat{c})\hat{b}'_0$.

As above illustrated, it is possible to translate mechanically the diagram into a subroutine program in which the displacements of link vectors and the positions of junction points are computed.

(e) Set up of the reference to the link vector

A set of reference vectors to link vectors may be given arbitrarily and it is not always necessary that it constitute a closed loop. However, use of that of almost closed loop similar to the initial configuration of the linkage as shown in Fig. 7 is more desirable because it also offers a good starting value for the solution in the Newton-Raphson iterations.

(f) Determination of the starting value for $\mathbf{x}$

The starting value for $\mathbf{x}$ is desirably as close as possible to the real solution. Therefore, the solution for the first position of the input link is utilized which is estimated roughly beforehand with the graphical method or the analytical one or else the optimization method. As an optimization method, the revised simplex method⁽⁶⁾ is used which searches for the minimum of a scalar objective function $w = f^2$.

Table 6 shows the numerical results obtained and Table 7 shows the results to be compared which are obtained by the use of the optimization method mentioned above.

From the comparison between them the following properties are obvious ;

(a) The Newton-Raphson method used together with an extrapolated starting value reduces the number of evaluations of the function to one tenth of that by the use of the optimization method.

(b) In application of the Newton-Raphson method, the use of an extrapolated starting value reduces the number of the function evaluations to half that by the immediate use of the solution for the preceding position of the input crank as a starting value.

By the way, the total number of the FORTRAN statements of the program for this problem is 220 and that of the subroutines included in it is 62. And only the latter undergoes a minimum change, depending on the linkage studied.

4.3 Spatial 4-bar linkage RFFF

The application of this method is illustrated with another spatial 4-bar linkage RFFF, where F stands for Hooke's joint and the analysis of the mechanism including it presents a difficult problem. Fig. 8 shows the configuration of the mechanism and Table 8 gives the displacement sequence diagram for it. It is natural that the rotations ϕ, ψ, χ , and ω , which are the relative rotations between adjacent links about the axes $\hat{a}, \hat{b}, \hat{c}$, and $\hat{d}$, respectively, are the elements of $\mathbf{x}$.

Table 6 Numerical results obtained by the analysis of the spatial 4-bar linkage RSCR

No.	θ	$\mathbf{x}(\phi, \psi, s)$			start value*	start value**	f^2	No.**
					solution			No.**
0	0	-1.0	0.5	5.0	0.366	0.2x10 ⁻⁶	4	4
		-1.3112	0.4548	4.873	0.12	0.8x10 ⁻⁶	23	
1	10	-1.3112	0.4548	4.873	0.12	0.8x10 ⁻⁶	3	
		-1.1251	0.4423	5.143	0.12	0.5x10 ⁻⁶	17	
2	20	-1.1251	0.4423	5.143	0.12	0.5x10 ⁻⁶	3	
		-0.9380	0.4218	5.395	0.12	0.3x10 ⁻⁶	3	
3	30	-0.9380	0.4218	5.395	0.12	0.3x10 ⁻⁶	2	
		-0.7499	0.3936	5.632	0.12	0.3x10 ⁻⁶	9	
		-0.7436	0.3949	5.626	0.12	0.3x10 ⁻⁶	9	
4	40	-0.7436	0.3949	5.626	0.12	0.3x10 ⁻⁶	2	
		-0.5419	0.3616	5.835	0.12	0.3x10 ⁻⁶	9	
		-0.5358	0.3634	5.829	0.12	0.4x10 ⁻⁶	9	
5	50	-0.5358	0.3634	5.829	0.12	0.3x10 ⁻⁶	2	
		-0.3145	0.3270	6.006	0.12	0.3x10 ⁻⁶	9	
		-0.3090	0.3293	6.002	0.12	0.4x10 ⁻⁶	9	

* the preceding solution

** the parabolic extrapolation of the last three solutions

*** the number of iterations

**** the number of function evaluations

Table 7 Numerical results obtained by the analysis of the spatial 4-bar linkage RSCR based on the optimization method

No.	θ	$\mathbf{x}(\phi, \psi, s)$			start value	solution	f^2	No.*
0	0	-1.0	0.0	5.0	3.73	< 10 ⁻⁶	94	
		-1.311	0.4549	4.873	0.12	< 10 ⁻⁶	78	
1	10	-1.311	0.4549	4.873	0.12	< 10 ⁻⁶	78	
		-1.125	0.4423	5.143	0.1	< 10 ⁻⁶	79	
2	20	-1.125	0.4423	5.143	0.1	< 10 ⁻⁶	73	
		-0.938	0.4218	5.396	0.12	< 10 ⁻⁶	73	
3	30	-0.938	0.4218	5.396	0.12	< 10 ⁻⁶	78	
		-0.744	0.3948	5.626	0.12	< 10 ⁻⁶	78	
4	40	-0.744	0.3948	5.626	0.12	< 10 ⁻⁶	73	
		-0.536	0.3634	5.829	0.12	< 10 ⁻⁶	73	
5	50	-0.536	0.3634	5.829	0.12	< 10 ⁻⁶	73	
		-0.309	0.3292	6.003	0.12	< 10 ⁻⁶	73	

* the number of function evaluations

$\hat{k}, \hat{\mu}$, and $\hat{\nu}$, respectively, as shown in Fig. 8, are taken as four unknown variables composing $\mathbf{x}$. Therefore the function $\mathbf{f}$ should be equivalent to four scalar equations which represent the following conditions:

(a) The coincidence of the junction points P's, which are taken as the double points representing the center of the cross of Hooke's joint on each open loop.

(b) Orthogonality of the axes $\hat{\xi}$ and $\hat{\pi}$, that is $\hat{\xi} \cdot \hat{\pi} = 0$.

Table 8 Displacement sequence diagram for the analysis of the spatial 4-bar linkage RFFF

	$\hat{f}$	$\hat{a}$	$\hat{\lambda}$	$\hat{k}$	$\hat{b}$	$\hat{c}$	$\hat{\xi}$	$\hat{\mu}$	$\hat{\nu}$	$\hat{d}$	$\hat{e}$	$\hat{\pi}$
θ	Δ	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$	$\times$
ϕ	$\times$	$\times$	Δ	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$	$\times$
ψ	$\times$	$\times$	$\times$	Δ	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$	$\times$
χ	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	Δ	$\circ$	$\circ$	$\circ$	$\circ$
ω	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	Δ	$\circ$	$\circ$	$\circ$

Table 9 Input data for the analysis of the spatial 4-bar linkage RFFF

	i	j	k	a	b	c	d	e	g	N	ϵ	h	ϕ	ψ	χ	ω
$\hat{f}$	0.5	-0.86603	0	2.0	1.0	2.8284	5.6568	0	7.0	10	10^{-8}	10^{-3}	0.0	0.0	0.0	0.0
$\hat{a}$	0	0	1													
$\hat{\lambda}$	0	-1	0													
$\hat{k}$	-1	0	0													
$\hat{b}$	0	1	0													
$\hat{c}$	0	0.7071	0.7071													
$\hat{\xi}$	0	-0.7071	0.7071													
$\hat{\mu}$	0	0	1													
$\hat{\nu}$	1	0	0													
$\hat{d}$	0	-0.7071	0.7071													
$\hat{e}$	.	.	.													
$\hat{\pi}$	0	0.7071	0.7071													

Table 10 Numerical results obtained by the analysis of the spatial 4-bar linkage RFFF

No.	θ	$\mathbf{x}(\phi, \psi, \chi, \omega)$	start value*	start value**	solution	f^2	No.***	No.****
0	0	0, 0, 0, 0	0	0	0	0.46×10^{-8}	2	2
1	10	0, 0, 0, 0	0	0	0	0.563	29	29
2	20	0.2726, 0.1118, -0.2385, 0.0610	0.2726, 0.1118, -0.2385, 0.0610	0.455	0.6	0.78×10^{-9}	4	30
3	30	0.2856, 0.2259, -0.3174, 0.1181	0.2856, 0.2259, -0.3174, 0.1181	0.396	0.205	0.6×10^{-9}	4	29
4	40	0.2560, 0.3385, -0.3633, 0.1725	0.2560, 0.3385, -0.3633, 0.1725	0.346	0.5	0.4×10^{-15}	3	20
5	50	0.1335, 0.5529, -0.4056, 0.2791	0.1335, 0.5529, -0.4056, 0.2791	0.302	0.1	0.1×10^{-2}	3	20

* the preceding solution
** the parabolic extrapolation of the last three solutions
*** the number of iterations
**** the number of function evaluations

The input data for the mechanism to be analyzed are shown in Table 9. Figure 9 shows the reference configuration which also serves as a starting configuration. Table 10 shows the numerical results obtained. Also, the numerical results obtained by the optimization method (the revised simplex method by Nelder and Mead) are shown in Table 11 for verification of results shown in Table 10.

4.4 General spatial 4-bar linkage

The applicability of the present method, particularly to the spatial 4-bar linkage, is summarized in Table 12. The linkages in-

Fig. 8 Spatial 4-bar linkage RFFF

Fig. 9 Reference configuration for the linkage RFFF

Table 11 Numerical results obtained by the analysis of the spatial 4-bar linkage RFFF based on the optimization method

No.	θ	$\mathbf{x}(\phi, \psi, \chi, \omega)$	start value	solution	f^2	No.*
0	0	0, 0, 0, 0	0	0	0.5×10^{-8}	5
1	10	0, 0, 0, 0	0	0	0.563	244
2	20	0.2725, 0.1118, -0.2385, 0.0610	0.2725, 0.1118, -0.2385, 0.0610	0.455	$< 10^{-8}$	203
3	30	0.2859, 0.2259, -0.3175, 0.1181	0.2859, 0.2259, -0.3175, 0.1181	0.395	$< 10^{-8}$	158
4	40	0.2559, 0.3385, -0.3633, 0.1726	0.2559, 0.3385, -0.3633, 0.1726	0.346	$< 10^{-8}$	158
5	50	0.2052, 0.4475, -0.3927, 0.2262	0.2052, 0.4475, -0.3927, 0.2262	0.302	$< 10^{-8}$	154

* the number of function evaluations

vestigated are composed of the classes of the kinematic pairs K_1, K_2 , and K_3 which stand for the kinematic pairs of 1, 2, and 3 degrees of freedom including R and P (prismatic pair), C and F, and S, respectively. Furthermore, the favourable selections of vectors $\mathbf{x}$ and $\mathbf{f}$ are suggested which are summarized as follows;

(a) The equation $\mathbf{f}$, in such cases of linkages including the spheric pair S as the cases of A, B, C, E, and F, holds upon the condition that the double points representing the center point of the pair S should coincide. Therefore three kinematic variables should be chosen as the elements of $\mathbf{x}$.

(b) For the linkage of the class D which has no spheric pair but Hooke's joint F, $\mathbf{f}$ is constituted upon the condition of orthogonality of the cross link of Hooke's joint and that of coincidence of the double point representing the center of the cross link. Therefore four kinematic variables should be chosen as the elements of $\mathbf{x}$.

Table 12 implies that the usual spatial 4-bar linkage jointed with the lower pairs can be analyzed by the method presented in this paper with no difficulty.

5. CONCLUSIONS

This paper outlines a method which can be used as a rapid computing tool for the displacement analysis of the kinematic linkage. Examples of the planar and spatial 4-bar linkages are used to demonstrate the validity of the method. Its advantages may be summarized as follows;

(1) The present method saves the user from the cumbersome manipulation of equations at the phase of formulation of the problem by the use of the new formulation method based on the compatibility equation expressing the condition of the coincidence of the double points at the assumed disjointed point to the closed loop.

(2) A set of equations obtained by the above procedure is solved by the Newton-Raphson method, which is noted for the performance of fast convergence of the solution process and the severe dependence of the stability upon closeness of the starting value to the final solution. Therefore the starting value for the position of the input link is estimated more precisely by the parabolic extrapolation among the last three solutions obtained previously for each position of the input link instead of by the immediate use of the last solution obtained for the preceding position of the input link. Consequently the convergence rate of the iterations is about twice that of the immediate use of the previous final solution.

APPENDIX

Revolute matrix⁽⁷⁾

Let $\hat{\mathbf{t}}$ be the unit directional vector of the axis of the revolute pair, which joints the moving link with the frame, and θ be the angular displacement of the moving link from the reference position as shown in Fig. 10. Then the revolute matrix is defined by

$$R(\theta, \hat{\mathbf{t}}) = \begin{bmatrix} t_x^2 V \theta + C \theta & t_x t_y V \theta - t_x S \theta & t_x t_z V \theta + t_y S \theta \\ t_x t_y V \theta + t_x S \theta & t_y^2 V \theta + C \theta & t_y t_z V \theta - t_x S \theta \\ t_x t_z V \theta - t_y S \theta & t_y t_z V \theta + t_x S \theta & t_z^2 V \theta + C \theta \end{bmatrix}$$

where

$\hat{\mathbf{t}} = (t_x, t_y, t_z)$ $S\theta = \sin \theta$ $C\theta = \cos \theta$ $V\theta = 1 - C\theta$
The expression $\mathbf{a} = R(\theta, \hat{\mathbf{t}}) \mathbf{a}_0$ means that the rotation of the moving link θ about the axis $\hat{\mathbf{t}}$ in a positive sense with respect to the right-hand rule changes the orientation of the constant vector $\mathbf{a}_0$ fixed in the moving link into $\mathbf{a}$.

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Fig. 10 Rotation of a rigid body about the axis fixed to the frame

Table 12 Survey of the applicability to the spatial 4-bar linkages

class	model	$\mathbf{f}, \mathbf{x}$
A	RSCR	* (ϕ, ψ, S)
K_1, K_3, K_2, K_1	RSFR	* (ϕ, ψ, χ)
8 versions		
B	RSRC	* (S, ϕ, ψ)
K_1, K_3, K_1, K_2	PSRF	* (ϕ, ψ, χ)
8 versions		
C	RCSR	* (ϕ, ψ, r)
K_1, K_2, K_3, K_1	RFSR	* (ϕ, ψ, χ)
8 versions		
D	RCCC	** $(\phi, \psi) \rightarrow (r, S, t)$
K_1, K_2, K_2, K_2	RFFF	*** $(\phi, \psi, \chi, \omega)$
16 versions		
E	RCRS	* (ϕ, ψ, S)
K_1, K_2, K_1, K_3		8 versions
F	PPSC	* (S, t, ϕ)
K_1, K_1, K_3, K_2		8 versions
G	RRCS	* (S, ϕ, ψ)
K_1, K_1, K_2, K_3		8 versions

total: 64 versions r, S : translation

ϕ, ψ, χ : rotation

* coincidence of center of S

** coincidence of direction of the axis of C

*** coincidence of center and direction of F