On the Relation between the Probability of
Failure and the Factor of Safety in the Design
of Machine Parts under Repeated Load*

Yuichi KAWADA**

On the assumption that the values of the fatigue limit of machine parts disperse following the law of normal distribution, though the working stress amplitude on those parts is maintained constant, the relation between the allowable stress or the factor of safety of a machine part, subjected to a repeated stress, and the probability of failure of the machine part was investigated. In order to clarify the relation, the data of the fatigue limit of notched specimens of any size collected by a committee in the Japan Society of Mechanical Engineers were utilized. Formulas to get the arithmetic mean of fatigue limit of machine parts were derived. The standard deviations and the coefficients of variance of fatigue limit of machine parts were also calculated. As the results, the factor of safety for any probability of failure was obtained just from the tensile strength of the material and the dimensions of the machine part.

1. Introduction

The fatigue limit of a machine part is known to disperse even if it is made of the same material, and the fatigue limit should be considered as a statistic quantity. As the result, it should not be talked about whether the machine part fails or not, but the probability of failure should be estimated when the design stress and the allowable fatigue limit of a machine part are given.

The present study works out the method to obtain such a factor of safety as the probability of failure becomes a desirable value, when a machine part, made of structural alloy steel, and subjected to a constant amplitude repeated load, has to be designed against fatigue failure.

2. Relation between the probability of failure and the factor of safety

In the present study, stresses are all expressed by nominal values, and the design stress which is a constant stress amplitude applied on a machine part is assumed not to disperse. The fatigue limit of a machine part, on the contrary, is assumed to disperse following the law of normal distribution. Then the probability density function y of the standard unit t is expressed as follows,

$$y = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad (1)$$

where, $t = (\sigma_w - X_p)/\sigma$, σ_w = allowable fatigue limit of a machine part, X_p = arithmetic mean of fatigue limit of the machine part, σ = the standard deviation of fatigue limit of the machine part. The relation of Eq. (1) is expressed by the curve in Fig.1, as is well known. Assuming that the design stress σ_d is taken equal to the fatigue limit σ_w , which corresponds to the value t on the abscissa in Fig.1, then the probability of failure P_F is equal to the area under the probability density curve and leftward of the value t , i.e., the hatched area in Fig.1, and is expressed by the next equation.

Fig.1 Failure probability

$$P_F = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt. \quad (2)$$

As to the relation between the values P_F and t , a table is already published which is available for us, and a part of this table is shown in Table 1.

Table 1. Failure probability P_F and t

t	-2.00	-3.00	-3.09	-3.72
P_F	0.02275	0.00135	0.001	0.0001

Thus, the fatigue limit of a machine part having a dispersion should be expressed by two parameters, one the arithmetic mean X_p , and the other the standard

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** President, Metropolitan College of Technology, Tokyo, Hino-shi.

deviation σ or the coefficient of variance $v = \sigma/X_p$, and the design stress σ_d is expressed as follows,

$$\sigma_d = \sigma_w = X_p + t\sigma = X_p(1+tv). \quad (3)$$

The factor of safety S , which is defined in this paper as the ratio of the arithmetic mean of fatigue limit of a machine part X_p to the design stress σ_d , is also expressed as follows.

$$S = \frac{X_p}{\sigma_d} = \frac{X_p}{X_p + t\sigma} = \frac{1}{1+tv} \quad (4)$$

At present, the values of the arithmetic mean X_p and the coefficient of variance v of fatigue limit of a machine part are not available, and the machine design considering the probability of failure is impossible. In the present study, therefore, it was made clear how to estimate the values of the arithmetic mean and the coefficient of variance of fatigue limit of a machine part. The assumptions made in the present study were also discussed.

The machine parts mentioned in this study are round bars with a fillet, a circumferential Vee-notch and a hole in diametral direction of any size, made of structural alloy steels, and their forms and materials were chosen, because in the first place the data of fatigue limit for such forms and materials are available from the data book⁽¹⁾ published by the Japan Society of Mechanical Engineers (JSME) and also from the results of experiments by members of a committee in the Japan Society for the Promotion of Science (JSPS)⁽²⁾, and in the second place these three forms are considered to be the typical forms existing in all machine elements.

3. On the arithmetic mean of fatigue limit of a machine part

The fatigue limit of a machine part worked from a material is, in general, estimated by correcting the fatigue limit of the standard specimen for size effect and notch effect, if the surface of the machine part is finished smoothly. The fatigue limit of the standard specimen of a material is known to increase in proportion to the tensile strength σ_B of the material, in many materials, such as structural alloy steels, and the arithmetic mean of fatigue limit of a machine part X_p is obtained from the next equation.

$$X_p = (\sigma_{wb,cal}, \sigma_{wz,cal}, \tau_{w,cal}) = c\sigma_B \zeta k \quad (5)$$

where, $\sigma_{wb,cal}, \sigma_{wz,cal}, \tau_{w,cal}$ = the arithmetic mean of fatigue limit calculated by Eq.(5) when the loading is rotating bending, reversed tension and compression, and reversed torsion respectively, c = a constant determined by the form of a machine part and the manner of loading, σ_B = tensile strength of the material,

ζ = coefficient of size effect which is calculated by the empirical formulas⁽³⁾ published by the Committee for Compiling Data of Notch Effect, Scale Effect and Surface Effect in JSME, $k = 1/\beta$, β = fatigue notch factor which is calculated by the empirical formulas^{(4)*} published by the same Committee. Assuming that the roughness of the surface of machine parts is nearly equal to that of the standard specimen, the correction term for surface roughness was omitted. If the roughness of machine parts is different from that of standard specimens, a correction term should be added in Eq.(5).

The value of the constant c was determined utilizing the fatigue data on notched specimens, published by JSME⁽⁵⁾ and also by a committee in JSPS such that the ratio of the fatigue limit by experiment to the fatigue limit by Eq.(5) becomes unity on the average. The values of c thus calculated for various forms and loadings are shown in Table 2, together with the quantity of data.

Table 2 . Value of c in Eq (5)

Round bar with	Load	c	Number of data
Fillet	Bending	0.500	58
	Torsion	0.327	31
Vee-notch	Bending	0.460	146
	Ten-comp	0.423	71
Hole	Torsion	0.301	38
	Bending	0.515	74
	Ten-comp	0.506	47
	Torsion	0.284	41

Thus, the arithmetic mean X_p of fatigue limit of a machine part is obtained by the formula (5) utilizing the value of c in Table 2. The relations between the experimental values and the calculated values of fatigue limit for various forms and loadings are shown in Figs. 2 to 9.

4. On the coefficient of variance of fatigue limit of a machine part

Utilizing the 58 data on the rotary bending fatigue limit of round bars with a fillet, the standard deviations of the ratios $\sigma_{wb,exp}/\sigma_{wb,cal}$ were investigated, on the assumption that these ratios disperse under the law of normal distribution. Thus, the value of the standard deviation σ as well as the coefficient of variance v was found to be 0.159. Utilizing this result, the design stress lines which correspond to -3σ and -3.72σ for the arithmetic mean of fatigue limit were drawn as broken lines passing through the origin in

* Empirical formula of β for round bars with a circumferential Vee-notch under reversed tension and compression published by the committee in JSME is found to have been revised in accordance with the new data published by JSPS. The revised empirical formula is used in this study (cf. Appendix).

Fig. 2 Comparison of rotary bending fatigue limits of round bars with fillet

Fig. 3 Comparison of reversed torsion fatigue limits of round bars with fillet

Fig. 4 Comparison of rotary bending fatigue limits of round bars with circumferential Vee-notch

Fig. 5 Comparison of reversed tension and compression fatigue limits of round bars with circumferential Vee-notch

Fig. 6 Comparison of reversed torsion fatigue limits of round bars with circumferential Vee-notch

Fig. 7 Comparison of rotary bending fatigue limits of round bars with hole in diametral direction

Fig. 8 Comparison of reversed tension and compression fatigue limits of round bars with hole in diametral direction

Fig.9 Comparison of reversed torsion fatigue limits of round bars with hole in diametral direction

Fig.2. The assumption that the standard deviation of the ratio $\sigma_{wb,exp}/\sigma_{wb,cal}$ is constant independently of the value of $\sigma_{wb,cal}$, means that the value of standard deviation of $\sigma_{wb,exp}$ increases in proportion to the value of $\sigma_{wb,cal}$. The dispersion of the data in Fig.2, on the contrary, shows that the standard deviation of $\sigma_{wb,exp}$ seems to be nearly constant, independently of the value of $\sigma_{wb,cal}$. As the result, the first method seems not to be appropriate for the estimation of the design stress of any failure probability. As the second method, then, the standard deviation of the difference ($\sigma_{wb,exp} - \sigma_{wb,cal}$) was investigated for round bars with a fillet under the assumption that the dispersion of this difference follows the law of normal distribution, and the value of $\sigma = 4.25 \text{ kg/mm}^2$ was obtained. Then the design stress line which corresponds to -3σ is expressed as a dotted line inclined 45° to the abscissa, and not passing through the origin in Fig.2. This dotted line seems fit the data better than the broken line by the first method, but there is an inconvenience. The ordinate of the dotted line, that is, the design stress corresponding to the failure probability of 0.00135 becomes zero, or the factor of safety becomes infinity when the value of abscissa $\sigma_{wb,cal}$ takes a value of about 13 kg/mm^2 , and such a case is not considered to exist. Then, the author proposes a third method for the evaluation of the design stress. Divide all the data in Fig.2 into two groups in such a manner that each group contains nearly the same quantity of

data or the range of $\sigma_{wb,cal}$ of each group becomes nearly equal. In Fig.2, the first group contains 25 data, the range of $\sigma_{wb,cal}$ being 10.0 to 19.8 kg/mm^2 , and the second group contains 26 data, the range of which is 20.0 to 29.6 kg/mm^2 . For each group, the standard deviation of the ratio $\sigma_{wb,exp}/\sigma_{wb,cal}$ was calculated, resulting in 0.171 and 0.143 respectively. Dividing these values of standard deviation by the means of the ratio 0.98 and 0.98 respectively, the values of coefficient of variance are obtained as 0.174 and 0.146 respectively (cf. Table 3). Calculating the mean of $\sigma_{wb,cal}$ for each group, the values of 16.5 kg/mm^2 and 25.1 kg/mm^2 are obtained (cf. Table 3). Taking the mean values of $\sigma_{wb,cal}$ on abscissa, and the values of design stress which correspond to -3σ and -3.72σ for the mean values of $\sigma_{wb,cal}$ for each group on ordinate, the points of square mark are obtained on Fig.2. Connecting the origin and the point of square mark of the first group with straight lines, and connecting the points of square mark of the first and the second group with straight lines, these bent lines were considered to be the design stress lines of -3σ and -3.72σ , that is, the failure probability of 0.00135 and 0.0001 . The author proposes this third method for estimation of the design stress and the factor of safety.

The design stress lines obtained by this third method for different loads and forms are shown in Figs.3 to 9. The numerical values necessary to draw the design stress lines are shown in Table 3, and the meaning of symbols in Fig. 10. For the case of round bars with a circumferential

Fig.10 Relation between design stress σ_d and arithmetic mean of fatigue limit X_p

Table 3. Coefficient of variance and mean design stress of each group

Round bar with	load	1st group						2nd group					
		$\sigma_{w,cal}$		n^*	ν_1	σ_{d1}		$\sigma_{w,cal}$		n^*	ν_2	σ_{d2}	
		range	X_{p1}			-3σ	-3.72σ	range	X_{p2}			-3σ	-3.72σ
Fillet	Bending	10.0-19.8	16.5	25	0.174	8.2	6.2	20.0-29.6	25.1	26	0.146	14.1	11.5
	Torsion	8.4-19.9	14.8	19	0.166	7.5	5.8	21.1-34.9	27.6	12	0.113	18.4	16.2
Notch	Bending	6.7-16.9	12.8	75	0.216	4.5	2.5	17.3-39.0	22.7	71	0.155	12.0	9.5
	Ten-comp	10.2-20.4	15.5	71	0.133	9.3	7.8	-	-	-	-	-	-
	Torsion	8.3-14.8	11.7	19	0.188	5.0	3.4	14.9-24.0	18.4	19	0.115	12.3	10.8
Hole	Bending	13.2-24.8	18.5	45	0.167	9.0	6.7	25.8-38.4	30.7	29	0.115	20.7	18.3
	Ten-comp	10.6-19.3	15.0	24	0.172	6.8	4.9	20.3-31.0	25.1	23	0.112	17.1	15.1
	Torsion	7.6-16.9	13.2	20	0.144	7.5	6.1	17.2-22.5	19.3	21	0.114	13.0	11.5

n^* ; number of data.

Vee-notch, subjected to a reversed tensile and compressive load, the range of the values of calculated fatigue limit $\sigma_{wz,cal}$ is narrow such as 10.2 to 20.4 kg/mm², and all the data are treated as a group. The design stress lines for different failure probabilities can also be drawn in the same manner in Figs. 2 to 9.

Let the coordinates of the arithmetic means of design stresses and the coefficients of variance of the first and second group be denoted by X_{P1} , σ_{d1} , v_1 and X_{P2} , σ_{d2} , v_2 as shown in Fig. 10, then the coefficient of variance v and the design stress σ_d are calculated from the following formulas respectively,

for $X_P < X_{P1}$

$$v = v_1 \quad (6)$$

$$\sigma_d = X_P(1+tv_1), \quad (7)$$

for $X_P > X_{P1}$

$$v = v_1 + \frac{v_2 - v_1}{X_{P2} - X_{P1}}(X_P - X_{P1}) \quad (8)$$

$$\sigma_d = X_{P1}(1+tv_1) + \frac{X_{P2}(1+tv_2) - X_{P1}(1+tv_1)}{X_{P2} - X_{P1}}(X_P - X_{P1}) \quad (9)$$

5. Factor of safety considering the failure probability for fatigue limit of a machine part

Since the factor of safety S is defined as the ratio of the arithmetic mean of fatigue limit of a machine part to the design stress, as is shown in Eq. (4), the ratio of the ordinate of the line inclined 45° to the abscissa, passing through the origin to the ordinate of the bent line in Figs. 2 to 9, becomes a factor of safety S , and the factor is expressed as follows,

$$\text{for } X_P < X_{P1}, \quad S = \frac{1}{1+tv_1}, \quad (10)$$

for $X_P > X_{P1}$,

$$S = \frac{X_P}{X_{P1}(1+tv_1) + \frac{X_{P2}(1+tv_2) - X_{P1}(1+tv_1)}{X_{P2} - X_{P1}}(X_P - X_{P1})} \quad (11)$$

Applying the numerical values in Table 3 for Eqs. (10) and (11), the factor of safety which corresponds to -3σ , or the failure probability of $P_F = 0.00135$, is obtained as shown in Fig. 11. From this figure it is clear that the factor of safety which corresponds to a failure probability decreases with an increase of calculated fatigue limit above about 12 kg/mm², even if the form of a machine part and the manner of loading

Fig. 11 Factor of safety corresponding to failure probability $P_F = 0.00135 (-3\sigma)$

are the same. The factor of safety which corresponds to the failure probability of 0.00135 is nearly 2 or less for a bar with a fillet or a hole drilled in diametrical direction, irrespective of the manner of loading, but the factor of safety becomes larger than 2 for a round bar with a circumferential Vee-notch subjected to rotary bending or reversed torsion.

6. discussions

On the assumptions and results in this paper, some discussions will be made.

6-1. On the population of data

Though the population parameters of the fatigue limit of a machine part were investigated statistically in the present study, there is a problem of whether the data utilized are a random sample from a population or not. The data as a sample are collected from domestic and foreign literatures which include data obtained by experiments at various laboratories on test pieces of various materials made at various steel makers. The actual machine parts, however, are not considered to be only those machined by a specified machine tool on the materials produced by a specified material maker, though the dimensions and the material specifications are the same; and in actual machine design, it is rather preferable to use the population parameter obtained in the present study, considering the collected data as a random sample from a population. It is, however, necessary to discuss whether the scatter of the data follows the law of normal distribution or not.

6-2. On the distribution function

It is assumed in this study that the ratio of the experimental fatigue limit to its arithmetic mean disperses following the law of normal distribution for each form and manner of loading. In order to ascertain the validity of this law, the relation between the ratio of the experimental fatigue limit to its arithmetic mean (X_{exp}/X_{cal}) and the probability of failure was plotted on a probability paper for each form and manner of loading. An example of this plotting is shown in Fig. 12. The probability of failure P_F is calculated by the next formula,

$$P_F = \frac{i}{n+1}, \quad (12)$$

where, i = number in the order when the ratios of the experimental fatigue limit

Fig. 12 $P_F-(\sigma_{wb,exp}/\sigma_{wb,cal})$ diagram of round bars with fillet in rotary bending

to its arithmetic mean are placed in an increasing order, n = the total number of these ratios. As the arithmetic means of the fatigue limit X_p were divided into two groups in Fig. 2, and the data were plotted on the probability paper for each group. From Fig. 12, it is clear that the data do not lie on a straight line strictly, and the ratio of the fatigue limit does not distribute strictly in accordance with the normal law. This fact shows that the data used were not a random sample from a population. On this account, the probability of failure thus obtained should be considered not to be a strict value. But the probability of failure thus obtained gives a relative value, and designers should practice their design utilizing this value of failure probability as a relative value.

6-3. On the coefficient c in the equation of the arithmetic mean of fatigue limit of a machine part

The value of coefficient c in equation (5), which is shown in Table 2, is considered to be a proper value, because this value is about 0.5 in the cases of rotating bending and reversed tension and compression, and about 0.3 in the case of reversed torsion, and is nearly the same as the ratio of the fatigue limit of plain specimens to the tensile strength. Nevertheless, the value of coefficient c , e.g., for rotating bending was such a value as 0.500, 0.460, 0.515 for bars with a fillet, a circumferential groove and a perforated hole respectively and differed from each other. This inconsistency is considered to be caused by an insufficient number of samples. But, at present, the author wishes the designers to utilize the values in Table 2.

6-4. On the coefficient of variance of fatigue limit of a machine part

The value of the coefficient of variance of the ratio of experimental to arithmetic mean of the fatigue limit dispersed in such a wide range as 0.112 to 0.216, as is shown in Table 3, for each group of the form and the manner of loading. The values of the coefficient of variance for round bars with a circumferential groove under rotary bending and reversed torsion were especially large such as 0.216 and 0.188, but in these cases, next reason is considered. In machining a Vee-notch, it has been known that work hardened layers of various grades are made at the bottom of the Vee-notch by different workers, and these layers affect the fatigue strength in various ways. Moreover, it has been also known that the strengthening effect of the work hardened layers is especially large in the case that a stress gradient exists in the diametral direction just as in bending and torsion.

6-5. On the diagram of factor of safety

As the design stress was expressed by a bent line as is shown in Figs. 2 to 10, the factor of safety was also expressed by a straight line and a hyperbola as is shown in Fig. 11, changing its inclination discontinuously at the crossing. Though it might be reasonable if the inclination could be changed continuously, it is difficult to find such a method from the present data. There might be another opinion that a curve should be drawn as a mean of the factor of safety for all the forms and the manners of loading. But, as was discussed in 6-4, there was a reason for the increase of factor of safety for round bars with a circumferential groove under rotary bending and reversed torsion, and it might be reasonable to consider that the probability of failure becomes 0.00135 if the factor of safety is taken 2 except in the above two cases. Nevertheless, the author wishes the designer to utilize, at present, the value of factor of safety shown in Fig. 11.

6-6. On the value of the probability of failure

Now a problem of how to choose the value of the probability of failure will be discussed. As is stated already, the coefficient of variance was obtained for a sample from a population which does not follow the law of normal distribution strictly. Therefore, even if the value of probability of failure is obtained for a factor of safety, this value can not be called strictly correct.

There is another impropriety in estimating such a small probability of failure as 0.001 from a small number of data, for instance, less than 75 as is shown in Table 3. Therefore the value of the probability of failure obtained by this method is to be considered a relative value, and designers are forced to choose the proper value for the probability of failure, utilizing the experience on the reliability of parts in the past.

7. Conclusions

Under the assumption that the fatigue limit distributes following the normal law, a method to obtain the factor of safety which corresponds to any probability of failure is proposed for the case that machine parts are to be designed considering the fatigue limit. Examples of calculation of the factor of safety are shown. Though there are some problems in choosing the value of the probability of failure, in practice it is possible to get a more reasonable factor of safety by this study.

Appendix

Correction of the empirical formula of fatigue notch factor

The empirical formula of a fatigue notch factor for round bars with a circumferential groove under reversed tension and compression is published by the committee for Compiling Data of Notch Effect, Scale Effect and Surface Effect in JSME, as follows:

$$\beta = 1 + \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \quad (13)$$

$$\xi_1 = c_1 + c_2 \sigma_B \quad (14)$$

$$\xi_2 = 1 - e^{-c_3 d} \quad (15)$$

$$\xi_3 = 1 - e^{-c_4 d/\rho} \quad (16)$$

$$\xi_4 = 1 - e^{-c_5 \{1 - (d/D)\}} \quad (17)$$

$$\xi_5 = 1 - e^{-c_6 (\pi - \theta)} \quad (18)$$

where, d = shaft diameter at the bottom of the groove in mm, D = shaft diameter, ρ = radius of curvature at the bottom of the groove, θ = angle of the groove in radian, c_1 to c_6 = constants shown in upper line in Table 4.

Table 4. Value of constants in Eqs. 14~18

	c_1	c_2	c_3	c_4	c_5	c_6
In use	3.9	0.016	0.070	0.082	12	1.7
Corrected	0.15	0.0195	0.40	0.082	12	1.7

The values of constants in this formula are found to be improper if we include the new data by JSPS which have been published after the publication of the formula. Fitness of the values of constants is inspected as follows. The data on the fatigue notch factor collected by JSME are those for different values of σ_B , d , d/ρ , d/D and θ . In order to investigate the relation e.g., between the value of $\beta-1$ and d , utilizing all data collected, the values of $\beta-1$ are corrected for definite values of σ_B , d/ρ , $1-(d/D)$, and θ , assuming that the present empirical formula is correct. In practice, the values of $\beta-1$ are corrected such that the values of ξ_1 , ξ_3 , ξ_4 and ξ_5 become unity respectively, then the corrected values are obtained from Eq.(19).

$$\xi_2 = \frac{\beta-1}{\xi_1 \xi_3 \xi_4 \xi_5} \quad (19)$$

where, β is the fatigue notch factor obtained by experiment, ξ_1 , ξ_3 , ξ_4 and ξ_5 are the values calculated by Eqs.(14), (16), (17) and (18). Then the corrected data ξ_2 by Eq.(19) are plotted against the shaft diameter d with the curves by Eq.(15), in Fig.13.

Fig.13 Comparison of values of ξ_2 by empirical formula and experiment

The curve by empirical formula passes the middle of data collected by JSME, but passes far from the points of data obtained by JSPS. Now the constants in empirical formula are revised as shown in the lower line in Table 4, and the curve by empirical formula passes the middle of all data collected, as shown in Fig.14. It has of course been ascertained that, with use of corrected constants, the empirical formulas also pass the middle of the data in other relations between, such as ξ_1 and σ_B , ξ_3 and d/ρ , ξ_4 and $1-(d/D)$, ξ_5 and $\pi-\theta$.

Fig 14 Comparison of values of ξ_2 by corrected empirical formula and experiment

References

- (1) The design data of the fatigue strength of metals, part 1, 1961, published by JSME (in Japanese).
 - (2) Preprint of the symposium on fatigue strength and size effect of steels, 1967, Committee 129, JSPS (in Japanese).
 - (3) Ref.(1), p.147 to 148.
 - (4) Ref.(1), p.113 to 116.
 - (5) Ref.(1), p.122 to 141.
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