Transient Response of Second Order Relay

Servomechanism*

By Shusaku OGINO**

An exact solution in mathematical sense for the transient response of a second order relay servomechanism is obtained. By applying the succession function method, the condition for the system to have a cycle is given and the proof that this cycle is a limit cycle is established. Based on this proof an algorithm and a chart to calculate the period and the amplitude of the cycle are constructed.

Moreover, the approximate periodic solution obtained by the describing function method is also studied.

1. Introduction

The problem to find the transient response of a second order relay servo is well known in automatic control theory and it has been taken up as an example or an exercise in most textbooks on automatic control. However, it seems that its treatments have been limited to depicting the trajectory in the phase plane by specifying numerical values to the system parameters, and the limit cycle which we often meet in these problems has been explained based on such numerical results⁽¹⁾.

On the other hand, the describing function method leads to a satisfactory approximation for the frequency and the amplitude of the limit cycle in most cases. However, the method disregards the proof or the explanation that the cycle is a limit cycle. And also the well known Hamel's and Tsypkin's methods in on-off control system are of the same kind in this respect⁽²⁾.

In this paper, we shall treat the case in which the characteristic of the nonlinear element is symmetric with respect to the origin of the phase plane as usual. For this case, we shall expect that the limit cycle assuming its existence must be symmetric with respect to the origin. G.Kovatch⁽³⁾ has tried to give a proof of the symmetric property of the cycle for that case but it is questionable whether he has succeeded or not.

Now, in this paper, at first we shall take parameters of the control system as arbitrary constants and present an algorithm to give a general solution for a transient response of the system. Then, in some cases, there is such a case that a procedure to get a trajectory is given by an endless repetition of the procedure to get its partial arc. In this case the behavior of the trajectory as time t tends to infinity is studied by using the well

known succession function method⁽⁴⁾. Thus the existence and symmetric property of a closed cycle are made clear and a proof that the closed cycle is a limit cycle is given in easily comprehensible manner.

The cycle is also studied by the describing function method. It is pointed out that in some cases, in spite of the true existence of a cycle, this method fails to catch the cycle. And the condition for these cases to occur is made clear.

2. Procedure for computation of phase trajectory

A block diagram of the system studied in this paper is shown in Fig.1. The nonlinear element is a three-position relay with hysteresis and dead space. For this element, we assume that $d \geq 0$ and $-1 \leq \delta \leq 1$. Therefore if we take (a) $d=0$, (b) $d>0$ and $\delta=1$ or (c) $d>0$ and $\delta=-1$, then the nonlinear element becomes (a) a three-position ideal relay, (b) a three-position relay with dead space, or (c) a two-position relay with hysteresis respectively. Accordingly, the following study may involve these as special cases.

From Fig.1, we have

$$\varepsilon = -C, \quad T\ddot{\varepsilon} + \dot{\varepsilon} = km, \quad m = Mn(\varepsilon)$$

where

$$n(\varepsilon) = \begin{cases} -1, & \varepsilon < -d \\ 0, & -\delta d < \varepsilon < \delta d \\ 1, & d < \varepsilon \end{cases} \quad \text{or} \quad \begin{cases} -d < \varepsilon < -\delta d, & \varepsilon > 0 \\ \delta d < \varepsilon < d, & \varepsilon > 0 \\ -d < \varepsilon < -\delta d, & \varepsilon < 0 \\ \delta d < \varepsilon < d, & \varepsilon < 0 \end{cases} \quad (1)$$

Accordingly we have $T\ddot{\varepsilon} + \dot{\varepsilon} + kMn(\varepsilon) = 0$.

Replacing t with t/T , we have

$$\ddot{\varepsilon} + \dot{\varepsilon} + Kn(\varepsilon) = 0 \quad (2)$$

where $K = kMT$. Therefore we can study the system in Fig.2 instead of one in Fig.1

Fig.1 Relay servo

Fig.2 Normalized Relay servo

* Received 9th October, 1974.

** Professor, Faculty of Engineering, Yamagata University, Yonezawa.

without loss of generality.

Now, before we attempt to arrive at a general solution of the differential equation (2) with arbitrary initial conditions, it is appropriate to solve the following problem for a positive number ζ .

Problem. Find the solution of Eq.(2) with initial conditions $\varepsilon(0)=-d$, $\dot{\varepsilon}(0)=-\zeta$.

And construct its phase trajectory.

In the following, we shall express this trajectory as $\Gamma(P)$, where P denotes the point $P(-d, -\zeta)$ in the phase plane.

At first, as we have $n(\varepsilon)=-1$ on and after the time $t=0$, Eq.(2) reduces to

$$\ddot{\varepsilon} + \dot{\varepsilon} - K = 0 \quad \dots\dots\dots(3)$$

Let the solution of Eq.(3) with $\varepsilon(0)=-d$, $\dot{\varepsilon}(0)=-\zeta$ be $x(t)$, and we have

$$\begin{aligned} x(t) &= -(K+d+\zeta) + (K+\zeta)e^{-t} + Kt \\ \dot{x}(t) &= -(K+\zeta)e^{-t} + K \end{aligned} \quad \dots\dots\dots(4)$$

Solving the equation $\dot{x}(t)=0$ ($t>0$), we have $t=\tau_0$ where $e^{-\tau_0}=K/(K+\zeta)$.

The minimum value of $x(t)$ for $t \geq 0$ is $x(\tau_0) = K\tau_0 - (d+\zeta)$, and we have $x(\tau_0) < x(0) = -d \leq -\delta d$. As $x(t)$ is a monotonously increasing function for $t > \tau_0$ and $x(\tau_0) < -\delta d$, $x(\infty) = \infty$, we have one and only one value of $t > \tau_0$ satisfying $x(t) = -\delta d$. Putting it as t_1 , we have

$$-(K+d+\zeta) + (K+\zeta)e^{-t_1} + Kt_1 = -\delta d \quad \dots\dots\dots(5)$$

and $\dot{x}(t_1) > 0$. Here we take four half lines

$$\text{DY: } \varepsilon = -d, \dot{\varepsilon} \leq 0, \quad \text{EZ: } \varepsilon = -\delta d, \dot{\varepsilon} \geq 0$$

$$\text{D'Y': } \varepsilon = d, \dot{\varepsilon} \geq 0, \quad \text{E'Z': } \varepsilon = \delta d, \dot{\varepsilon} \leq 0$$

in the phase plane (see Fig.3). Then the above discussion shows that a trajectory $\Gamma(P)$ of Eq.(2) whose initial point $P(-d, -\zeta)$ is on the half line DY meets the half line EZ at a point $Q(-\delta d, \dot{\varepsilon}(t_1))$. The arc PQ of $\Gamma(P)$ is given analytically by Eq.(4) with $0 \leq t \leq t_1$.

Secondly, in order to find the trajectory of Eq.(2) on and after $t=t_1$, we proceed to solve Eq.(2) with the initial point Q. Then we have $n(\varepsilon)=0$ on and after $t=0$ and Eq.(2) reduces to

$$\ddot{\varepsilon} + \dot{\varepsilon} = 0 \quad \dots\dots\dots(6)$$

Let the solution of Eq.(6) with the initial point Q be $y(t)$, and we have

$$\begin{aligned} y(t) &= -\delta d + \dot{\varepsilon}(t_1)(1-e^{-t}) \\ \dot{y}(t) &= \dot{\varepsilon}(t_1)e^{-t} \end{aligned} \quad \dots\dots\dots(7)$$

From Eqs.(7), we have $y + \dot{y} = -\delta d + \dot{\varepsilon}(t_1)$. This is a straight line passing through the point Q whose gradient is -1. Let the coordinate of the intersection of this line and the line $\varepsilon = d$ be (d, ζ) , and we have

$$\zeta = \dot{\varepsilon}(t_1) - (1+\delta)d \quad \dots\dots\dots(8)$$

Fig.3 Explanation of trajectory

Here we consider the following cases.

Case 1. $\zeta \leq 0$. Let the intersection of the straight line and ε -axis be P' , and we have $P'[\dot{\varepsilon}(t_1) - \delta d, 0]$. On the other hand, from Eqs.(7), we have $y(\infty) = \dot{\varepsilon}(t_1) - \delta d$, $\dot{y}(\infty) = 0$. Therefore, the representative point on the trajectory(7) tends to the point P' as $t \rightarrow \infty$. Thus the trajectory $\Gamma(P)$ of Eq.(2) in this case is the arc PQP' . The partial arc QP' of $\Gamma(P)$ is given analytically by Eqs.(7) with $0 \leq t$.

Case 2. $\zeta > 0$. From Eqs.(7), we have

$$y(\infty) - d = \dot{\varepsilon}(t_1) - (1+\delta)d = \zeta$$

and accordingly $y(\infty) > d$. As $y(t)$ is a monotonously increasing function for $t (> 0)$ and $y(0) = -\delta d$, $y(\infty) > d$, we obtain one and only one value of $t > 0$ satisfying $y(t) = d$. Put it as t_2 and we have $y(t_2) = d$ or

$$\dot{\varepsilon}(t_1)(1-e^{-t_2}) = (1+\delta)d \quad \dots\dots\dots(9)$$

From Eq.(9), we have

$$\dot{y}(t_2) = \dot{\varepsilon}(t_1)e^{-t_2} = \dot{\varepsilon}(t_1) - (1+\delta)d = \zeta.$$

Thus, in this case, the trajectory $\Gamma(P)$ of Eq.(2) meets the half line $D'Y'$ at a point $\tilde{P}(d, \zeta)$. The partial arc QP of $\Gamma(P)$ is given analytically by Eqs.(7) with $0 \leq t \leq t_2$.

In the following, let us consider the case $\zeta > 0$.

In order to find the trajectory $\Gamma(P)$ of Eq.(2) on and after $t=t_1+t_2$, we proceed to solve Eq.(2) with initial point $\tilde{P}(d, \zeta)$. Then we have $n(\varepsilon)=1$ on and after $t=0$ and Eq.(2) reduces to

$$\ddot{\varepsilon} + \dot{\varepsilon} + K = 0 \quad \dots\dots\dots(10)$$

Let the solution of Eq.(10) with $\varepsilon(0)=d$, $\dot{\varepsilon}(0)=\zeta(>0)$ be $\eta(t)$ and also the solution of Eq.(3) with $\varepsilon(0)=-d$, $\dot{\varepsilon}(0)=-\zeta$ be $\phi(t)$. Then, we have $\eta(t) = -\phi(t)$ because Eq.(10) becomes Eq.(3) by substituting $-\varepsilon$ for ε in the former.

Thus a calculating procedure to get the trajectory $\Gamma(P)$ of Eq.(2) whose initial point P is on the half line DY is given by the flow chart in Fig.4. If it happens that $\zeta \leq 0$, the calculating procedure will be finished with a finite number of repetitions and the representative point of the trajectory will converge to a point on the segment ED' or DE' in Fig.3. On the contra-

Fig.4 Calculating procedure of trajectory

ry, if we have $\zeta > 0$ for each repetition, e.g. the case in which $(1+\delta)d=0$, the procedure to get the first partial arc PQP will continue endlessly.

What one of conditions among the given constants d, δ, K can establish the condition $\zeta > 0$ for each repetition? Moreover for the case in which $\zeta > 0$, how does the trajectory $\Gamma(P)$ of Eq.(2) behave as $t \rightarrow \infty$? We shall discuss these matters in the following chapter.

3. Behavior of trajectory

$\Gamma(P)$ of Eq.(2) as $t \rightarrow \infty$

Solving Eq.(5) with respect to ζ , we have

$$\zeta = [Kt_1 - (1-\delta)d]/(1-e^{-t_1}) - K \quad (11)$$

From Eq.(11), we have

$$d\zeta/dt_1 = Ke^{-t_1}(e^{t_1}-t_1-1+(1-\delta)d/K)/(1-e^{-t_1})^2.$$

As $e^{t_1}-t_1-1 > 0$ for $t_1 > 0$ and $(1-\delta)d \geq 0$, we have $d\zeta/dt_1 > 0$ for $t_1 > 0$. Therefore $\zeta(t_1)$ is a monotonously increasing function of t_1 (> 0).

If $(1-\delta)d=0$, we have $\zeta(+0)=0$ and $\zeta(\infty)=\infty$. On the other hand, if $(1-\delta)d > 0$, we have $\zeta(+0)=-\infty$ and $\zeta(\infty)=\infty$, so there is one and only one value of t_1 satisfying $\zeta(t_1)=0$ and $t_1 > 0$. Let it be t_{10} , and we have

$$t_{10} + e^{-t_{10}} - 1 = (1-\delta)d/K \quad (12)$$

Thus we can depict the outline of the graph $\zeta(t_1)$ for these cases (see Fig.6).

Next, we consider $\bar{\zeta}(t_1)$. Substituting Eq.(11) into Eq.(8), we have

$$\bar{\zeta} = K - (1+\delta)d - [Kt_1 - (1-\delta)d]/(e^{t_1}-1) \quad (13)$$

and $d\bar{\zeta}/dt_1 = \zeta/(e^{t_1}-1)$.

If $(1-\delta)d=0$, we have $\zeta(t_1) > 0$ for $t_1 > 0$ and $d\bar{\zeta}/dt_1 > 0$ for $t_1 > 0$. In addition we have $\bar{\zeta}(+0) = -(1+\delta)d$, $\bar{\zeta}(\infty) = K - (1+\delta)d$. Thus we can depict the outline of the graph $\bar{\zeta}(t_1)$ for this case (Fig.6). If $(1-\delta)d > 0$, we have $\bar{\zeta}(+0) = \infty$, $\bar{\zeta}(\infty) = K - (1+\delta)d$ and $d\bar{\zeta}/dt_1 \leq 0$ according to $(0 <) t_1 \leq t_{10}$. Therefore $\bar{\zeta}(t_1)$ takes the minimum value $\bar{\zeta}(t_{10})$ at $t_1 = t_{10}$ and by using Eq.(12), we have $\bar{\zeta}(t_{10}) = Kt_{10} - 2d$. If $d > 0$ and $\delta = -1$, we have $Kt_{10} - 2d = K(1 - e^{-t_{10}})$ from Eq.(12), so we have $\bar{\zeta}(t_{10}) > 0$. For the other case in which $d > 0$ and $-1 < \delta < 1$, we shall investigate the sign of the minimum value $\bar{\zeta}(t_{10})$. For that purpose, in Eq.(12), we regard d, δ as given constants and t_{10} as a function of K : $t_{10} = t_{10}(K)$. Then we have

$$dt_{10}/dK = (e^{t_{10}} - t_{10}e^{t_{10}} - 1)/K(e^{t_{10}} - 1)$$

and $d\bar{\zeta}(t_{10}(K))/dK = (e^{t_{10}} - t_{10} - 1)/(e^{t_{10}} - 1) > 0$.

Therefore, $\bar{\zeta}[t_{10}(K)]$ is a monotonously increasing function of K . Moreover we have

Fig.5 Relation between K_0/d and δ

$\bar{\zeta}[t_{10}(+0)] = -(1+\delta)d < 0$ and $\bar{\zeta}[t_{10}(\infty)] = \infty$, so there is one and only one value of K satisfying $\bar{\zeta}[t_{10}(K)] = 0$ and $K > 0$. Let it be K_0 , and we have $\bar{\zeta}(t_{10}) \leq 0$ according to $K \leq K_0$. Thus we can depict the outline of the graph of $\bar{\zeta}(t_1)$ for these cases (see Fig.(6)).

A value of K_0 is obtained from $K_0 = 2d/t_{10}$ where $1 - e^{-t_{10}} = (1+\delta)t_{10}/2$, because from these two equations, we have $t_{10} + e^{-t_{10}} - 1 = (1-\delta)d/K_0$ (see Eq.(12)).

The relationship between K_0/d and δ is given in Fig.5. The graph K_0^*/d in Fig.5 will be explained in §5.

Moreover, we shall compare the value of $\zeta(t_1)$ with the value of $\bar{\zeta}(t_1)$ for the same value of t_1 . For that purpose, we put $\varphi(t_1) = \zeta(t_1) - \bar{\zeta}(t_1)$, then we have

$$d\varphi/dt_1 = Ke^{t_1}\phi(t_1)/(e^{t_1}-1)^2$$

where

$$\phi(t_1) = e^{t_1} - e^{-t_1} - 2t_1 + 2(1-\delta)d/K.$$

As $d\phi/dt_1 = e^{t_1} + e^{-t_1} - 2 > 0$ for $t_1 > 0$ and $\phi(0) = 2(1-\delta)d/K \geq 0$, we have $\phi(t_1) > 0$ for $t_1 > 0$. This leads to $d\varphi/dt_1 > 0$ for $t_1 > 0$. Therefore $\varphi(t_1)$ is a monotonously increasing function of t_1 (> 0) and we have the following results.

For the case in which $d=0$, we have

$$\zeta(t_1) > \bar{\zeta}(t_1) \text{ for } t_1 > 0, \quad \zeta(0)=0 \text{ and } \bar{\zeta}(0)=0.$$

For the case in which $d > 0$ and $\delta = 1$, we have

$$\zeta(t_1) > \bar{\zeta}(t_1) \text{ for } t_1 > 0, \quad \zeta(0)=0 \text{ and } \bar{\zeta}(0)=-2d.$$

Besides, we shall examine the case in which $d > 0$, $-1 \leq \delta < 1$.

We begin with the case in which $d > 0$ and $\delta = -1$. For that case, we have

$$\varphi(t_{10}) = \zeta(t_{10}) - \bar{\zeta}(t_{10}) = -\bar{\zeta}(t_{10}) < 0.$$

and $\varphi(\infty) = \infty$, so there exists one and only one $t_1 (> t_{10})$ satisfying $\varphi(t_1) = 0$ that is $\zeta(t_1) = \bar{\zeta}(t_1)$. Let it be t_{20} , and we have

$$\zeta(t_1) \leq \bar{\zeta}(t_1) \text{ according to } (t_{10} \leq) t_1 \leq t_{20}.$$

We put $\zeta_0 = \zeta(t_{20}) = \bar{\zeta}(t_{20})$.

Next, for the case in which $d > 0$, $-1 < \delta < 1$, we have

$$\varphi(t_{10}) = -\bar{\zeta}(t_{10}) \leq 0 \text{ according to } K \leq K_0$$

and $\varphi(\infty) = \infty$. Thus we have the following results.

For the case in which $d > 0$, $-1 < \delta < 1$ and $K < K_0$, we have

$$\zeta(t_1) > \bar{\zeta}(t_1) \text{ for } t_1 > t_{10}, \quad \zeta(t_{10})=0, \quad \bar{\zeta}(t_{10}) < 0.$$

For the case in which $d > 0$, $-1 < \delta < 1$ and $K = K_0$, we have

$$\zeta(t_1) > \bar{\zeta}(t_1) \text{ for } t_1 > t_{10}, \quad \zeta(t_{10})=0, \quad \bar{\zeta}(t_{10})=0.$$

For the case in which $d > 0$, $-1 < \delta < 1$ and $K > K_0$, there exists one and only one value of $t_1 (> t_{10})$ satisfying $\varphi(t_1) = 0$, that is $\zeta(t_1) = \bar{\zeta}(t_1)$. Let it be t_{20} again, and we have

$$\zeta(t_1) \leq \bar{\zeta}(t_1) \text{ according to } (t_{10} \leq) t_1 \leq t_{20}$$

We put $\zeta_0 = \zeta(t_{20}) = \bar{\zeta}(t_{20})$ again.

Thus we can obtain Fig.6 which clarifies the relationship between $\zeta(t_1)$ and $\bar{\zeta}(t_1)$.

Now, if we take a positive ζ , we can determine one and only one value $t_1 > 0$ for the case in which $(1-\delta)d = 0$, and one and only one value of $t_1 > t_{10}$ for the case in

Fig.6 Relation between $\zeta(t_1)$ and $\bar{\zeta}(t_1)$

which $(1-\delta)d > 0$. And also from Eq.(13) we can find one and only one value of ζ for that t_1 . Thus ζ is a one valued function of ζ (≥ 0). Let this function be $\chi(\zeta)$, and we have the following lemma.

Lemma 1. $\chi(\zeta)$ is a strict monotonously increasing continuous function of ζ (≥ 0).

This follows from

$$d\zeta/d\zeta = (d\zeta/dt_1)/(d\zeta/dt_1) > 0 \quad \text{for } \zeta > 0.$$

Moreover, using $\chi(\zeta)$, the relationship between $\zeta(t_1)$ and $\zeta(t_2)$ clarified in the above is expressed as follows (Fig.7^(a)).

Lemma 2.1. For the case in which $d > 0$, $\delta = -1$ or $d > 0$, $-1 < \delta < 1$, $K > K_0$, there exists one and only one value of $\zeta > 0$ satisfying $\zeta = \chi(\zeta)$. Let it be ζ_0 , and we have

$$\zeta \leq \chi(\zeta) \quad \text{according to } \zeta \leq \zeta_0.$$

Lemma 2.2. For the case in which $d = 0$ or $d > 0$, $-1 < \delta < 1$, $K = K_0$, we have

$$\zeta > \chi(\zeta) \quad \text{for } \zeta > 0 \quad \text{and} \quad \chi(0) = 0.$$

Lemma 2.3. For the case in which $d > 0$, $\delta = 1$ or $d > 0$, $-1 < \delta < 1$, $K < K_0$, we have

$$\zeta > \chi(\zeta) \quad \text{for } \zeta > 0 \quad \text{and} \quad \chi(0) < 0.$$

Now, we take a positive number ζ . If $\chi(\zeta) > 0$, we put $\chi^2(\zeta) = \chi[\chi(\zeta)]$ and generally, if $\chi(\zeta), \chi^2(\zeta), \dots, \chi^n(\zeta) > 0$, we put $\chi^{n+1}(\zeta) = \chi[\chi^n(\zeta)]$. If $\chi^n(\zeta) > 0$, $\chi^n(\zeta)$ denotes the orthogonal distance from the intersection of the trajectory $\Gamma(P)$ and the half line $D'Y'$ (for the case in which n is odd) or the half line DY (for the case in which n is even) to ε -axis. If $\chi^{n+1}(\zeta) > 0$, $\chi^n(\zeta) \leq 0$, we may conclude that the representative point on the trajectory tends to a point on the segment DD' as $t \rightarrow \infty$. Thus by investigating the behavior of $\chi^n(\zeta)$ as $n \rightarrow \infty$, we can clarify the behavior of the trajectory $\Gamma(P)$ as $t \rightarrow \infty$. In the following, this will be studied in detail.

1. Case in which $d > 0, \delta = -1$ or $d > 0, -1 < \delta < 1, K > K_0$.

1.1. If $\zeta = \zeta_0$, we have $\chi(\zeta_0) = \zeta_0$ from Lemma 2.1 and so we have generally $\chi^i(\zeta_0) = \zeta_0$ for $i = 1, 2, \dots$. It follows that in this case the trajectory $\Gamma(P)$ becomes a closed cycle Γ_0 passing through the two points $(-d, -\zeta_0)$, (d, ζ_0) on the phase plane. Therefore, the thus obtained closed trajectory Γ_0 is a symmetric cycle with respect to the origin of the phase plane.

1.2. If $0 \leq \zeta < \zeta_0$, we have $\zeta < \chi(\zeta)$ from Lemma 2.1 and $\chi(\zeta) < \chi^2(\zeta) = \zeta_0$ from Lemma 1. Therefore we have $\zeta < \chi(\zeta) < \chi^2(\zeta)$. Employing Lemma 1 again, we have $\chi(\zeta) < \chi^2(\zeta) < \chi^3(\zeta)$, so we have $\zeta < \chi(\zeta) < \chi^2(\zeta) < \chi^3(\zeta) < \dots < \chi^n(\zeta) < \dots < \zeta_0$. Therefore we have $\lim_{n \rightarrow \infty} \chi^n(\zeta) = \zeta_0$. Moreover we can prove that $\lim_{n \rightarrow \infty} \chi^n(\zeta) = \zeta_0$ (see Appendix 1).

If $\zeta > \zeta_0$, we have $\zeta > \chi(\zeta) > \chi^2(\zeta) > \dots > \chi^n(\zeta) > \dots > \zeta_0$, and we obtain again $\lim_{n \rightarrow \infty} \chi^n(\zeta) = \zeta_0$ as well as the above.

Fig.7 Relation between $\chi(\zeta)$ and ζ

Thus the trajectory $\Gamma(P)$ of Eq.(2) for such ζ that $\zeta \neq \zeta_0$ and $\zeta \geq 0$ spirals around the cycle Γ_0 obtained in the case 1.1 from inside or outside of Γ_0 (ref. Appendix 2). Therefore, the cycle Γ_0 is a stable limit cycle.

Now, we can calculate the period and the amplitude of a limit cycle by following the flow chart in Fig.8. At this juncture we note that the calculating procedure is based on the above proof of the existence and the uniqueness of a limit cycle.

For the case in which $d = 1$ we calculate the period and the amplitude of the limit cycle for each value of K . The results are depicted in Fig.9. The amplitude and the period of the limit cycle for the system in Fig.1 may be obtained as follows:

At first we seek for the amplitude (or period) of the limit cycle for $K = kMT/d$, then multiply it by d (or T).

2.1. Case in which $d = 0$.

2.1.1. If $\zeta = 0$, we have $\chi(\zeta) = 0$. This shows that the system is in an equilibrium state.

2.1.2. If $\zeta > 0$, we have $\zeta > \chi(\zeta) > \chi^2(\zeta) > \dots > \chi^n(\zeta) > \dots > 0$ which leads to $\lim_{n \rightarrow \infty} \chi^n(\zeta) = 0$, and moreover we can obtain $\lim_{n \rightarrow \infty} \chi^n(\zeta) = 0$ also. Thus the trajectory $\Gamma(P)$ for such ζ that $\zeta > 0$ spirals around the origin from outside and the representative point on the trajectory tends to the origin as $t \rightarrow \infty$. Therefore the origin is asymptotically stable.

2.2. Case in which $d > 0, -1 < \delta < 1$ and $K = K_0$.

2.2.1. If $\zeta = 0$, we have $\chi(\zeta) = 0$. The trajectory $\Gamma(P)$ is the arc DFD' in Fig.10, where $F[-\delta d, (1+\delta)d]$. And the representative point on the trajectory tends to the point $D'(d, 0)$ as $t \rightarrow \infty$.

2.2.2. If $\zeta > 0$, we have $\zeta > \chi(\zeta) > \dots > \chi^n(\zeta) > \dots > 0$ which leads to $\lim_{n \rightarrow \infty} \chi^n(\zeta) = 0$. Let the arc $D'F'D$ be a symmetric figure of the arc DFD' with respect to the origin (Fig.10). And the trajectory $\Gamma(P)$ spirals around the closed curve $DFD'F'D$ from outside as $t \rightarrow \infty$.

3. Case in which $d > 0, \delta = 1$ or $d > 0, -1 <$

Fig.8 Calculating procedure of period and amplitude of limit cycle for given K, d, δ

Fig.9 Chart for period and amplitude of limit cycle, where $d=1$ and δ, K are given

$\delta < 1, K < K_0$.

For an arbitrary $\zeta (\geq 0)$ we have a positive integer n such that $\chi^n(\zeta) \leq 0$ so the representative point on the trajectory $\Gamma(P)$ tends to a point on the segment DD' .

By the above discussion, the behavior of the trajectory $\Gamma(P)$ as $t \rightarrow \infty$ is made clear.

In the same way, we may account for the case of a trajectory whose initial point belongs to the half line $D'Y'$. However, some additional considerations are necessary to seek a phase trajectory whose initial point is any point on the phase plane. We shall discuss it in next chapter.

4. Transient response of second order relay servo

At first, we begin with the following easily obtained results.

Lemma 3. Let $(1+\delta)d > 0$. The trajectory $\Gamma(F)$ of Eq.(2) with the initial point $F[-\delta d, (1+\delta)d]$ is the segment FD' , and the representative point on the trajectory tends to the point D' as $t \rightarrow \infty$. The representative point on the trajectory of Eq.(2) whose initial point is a point on the segment EF tends to a point on the segment ED' as $t \rightarrow \infty$. On the other hand, the trajectory whose initial point is a point on the half line FZ intersects at a point with the half line $D'Y'$.

The trajectory for the last case in Lemma 3 may be obtained as a prolongation (towards $t < 0$) of the trajectory $\Gamma(P)$ where

P belongs to the half line $D'Y'$. Therefore the behavior of the representative point on this trajectory may be deduced from the results of §3.

Similar results may be obtained for the trajectory $\Gamma(P)$ where P belongs to the half line $E'Z'$.

Next, we shall consider the movement of the trajectory $\Gamma(P)$, when its initial point $P(-d, -\zeta)$ moves on the half line DY , that is ζ varies from 0 to ∞ . The trajectory $\Gamma(P)$ intersects at $Q[-\delta d, x(t_1)]$ with the half line EZ , so we shall start with the consideration of the movement of Q as ζ varies from 0 to ∞ .

From Eq.(8) we have $x(t_1) = \zeta + (1+\delta)d$. As ζ is a monotonously increasing function of $\zeta (\geq 0)$, $x(t_1)$ is also a monotonously increasing function of $\zeta (\geq 0)$. And we know that $\zeta \rightarrow K - (1+\delta)d$ as $\zeta \rightarrow \infty$. Therefore we have $x(t_1) \rightarrow K$ as $\zeta \rightarrow \infty$. Next, we shall let $\zeta \rightarrow +0$, then we have the following results.

For the case in which $(1-\delta)d = 0$, we have $\zeta \rightarrow -(1+\delta)d$ and $x(t_1) \rightarrow 0$. For the case in which $(1-\delta)d > 0$, we have $x(t_1) \rightarrow x(t_{10})$, where

$$x(t_{10}) = K(1 - e^{-t_{10}}) = Kt_{10} - (1-\delta)d \quad (14)$$

We take a point $Q_\infty(-\delta d, K)$ on the half line EZ and moreover a point $Q_0[-\delta d, x(t_{10})]$ if $(1-\delta)d > 0$. The point Q_0 is an intersection of the trajectory $\Gamma(D)$ and the half line EZ (see Fig.11). Now if we assign the initial point $P(-d, -\zeta)$ to all the points on the half line DY , then the point Q moves all the points on the segment EQ_∞ when $(1-\delta)d = 0$ (Fig.11 (a), (b)) or the points on

Fig.10 Limiting curve of trajectory for $K=K_0$

Fig.11 Explanation of domains to which an initial point of Eq.(2) belongs

the segment Q_0Q_∞ when $(1-\delta)d > 0$ (Fig.11 (c) $\sim$ (f)) and the partial arc PQ of the trajectory $\Gamma(P)$ fills out the entire domain B designated in Fig.11. Therefore, each of the trajectories of Eq.(2) whose initial points belong to the domain B can be obtained as a prolongation (towards $t < 0$) of the trajectory $\Gamma(P)$ where P belongs to the half line DY.

Let B' be a symmetric domain of B with respect to the origin, and in a similar way we may study the trajectory of Eq.(2) whose initial point belongs to the domain B'.

Next, as the trajectory of Eq.(2), whose initial point belongs to the domain $A: \varepsilon \leq -\delta d, \varepsilon \geq K$, intersects at a point with the half line $Q_\infty Z$, it is obtained as a prolongation (towards $t < 0$) of the trajectory whose initial point is on the half line $Q_\infty Z$. And in a similar way we may study the trajectory of Eq.(2) whose initial point belongs to the domain $A': \varepsilon \geq \delta d, \varepsilon \leq -K$.

Assume the initial point P belongs to the domain A or B or A' or B'. Then from the above and the results of §3, we can make clear the behavior of the trajectory $\Gamma(P)$ of Eq.(2) as $t \rightarrow \infty$. Therefore, for the case in which $(1-\delta)d = 0$ (Fig.11(a), (b)), the discussion is already finished. However, for the case in which $(1-\delta)d > 0$ (Fig.11(c) $\sim$ (f)), we must continue to investigate the trajectory $\Gamma(P)$ of Eq.(2) whose initial point P belongs to the domain C or C' (see Fig.11 (c) $\sim$ (f)).

First, we consider the case in which $d > 0, \delta = -1$ (Fig.11 (c)). A trajectory $\Gamma(P)$ whose initial point P belongs to the domain C always meets at a point with the segment D'Q₀, so the trajectory $\Gamma(P)$ may be obtained as a prolongation (towards $t < 0$) of the trajectory $\Gamma(Q)$ whose initial point Q belongs to the half line D'Y'. Thus the behavior of the trajectory $\Gamma(P)$, where P belongs to C, as $t \rightarrow \infty$ may be known by applying the results of §3. In a similar manner, the behavior of the trajectory $\Gamma(P)$ where P belongs to C', as $t \rightarrow \infty$ may be discussed as well.

Finally, for the case in which $d > 0, -1 < \delta < 1$, we shall consider the trajectory $\Gamma(P)$ of Eq.(2) where P belongs to C.

Now, it is noted that the equation $\dot{x}(t_0) - (1+\delta)d = Kt_0 - 2d$ (see Eq.(14)) yields the relations

$$\dot{x}(t_0) - (1+\delta)d \approx 0 \quad \text{according to } K \approx K_0.$$

For the case in which $K < K_0$, the point $Q_0[-\delta d, \dot{x}(t_0)]$ lies on the segment EF (Fig. 11(d)), so the trajectory $\Gamma(P)$ where P belongs to the domain C, always meets at a point with the segment EQ₀. Thus the representative point on the trajectory $\Gamma(P)$, where P belongs to C, tends to a point on the segment ED' as $t \rightarrow \infty$. For the case in which we take C' instead of C in the above, we may deduce similar results.

For the case in which $K = K_0$, the point Q₀ is coincident with the point F (Fig.11 (e)), so the representative point of the trajectory $\Gamma(P)$, where P belongs to C or C', tends to a point on the segment DD' as $t \rightarrow \infty$.

For the case in which $K > K_0$, the point Q₀ lies on the segment FQ₀ (Fig.11 (f)), so

the prolongation (towards $t < 0$) of the trajectory of Eq.(3), whose initial point is the point F, meets at $P_0[K \ln(1 - (1+\delta)d/K) + d, 0]$ with the segment DE. We divide the domain C into two small domains C₁ and C₂ by the arc FP₀ (Fig.11(f)). The trajectory $\Gamma(P)$ of Eq.(2), where P belongs to C₁ meets at a point with the segment FQ₀. And the trajectory of Eq.(6) starting from the intersection meets at a point with the half line D'Y'. Therefore any trajectory $\Gamma(P)$ of Eq.(2), whose initial point P belongs to C₁, is obtained as a prolongation (towards $t < 0$) of the trajectory $\Gamma(P')$, whose initial point P' belongs to the half line D'Y'. On the other hand, it is easily understood that the representative point of each trajectory $\Gamma(P)$, whose initial point P belongs to C₂ tends to a point on the segment ED' as $t \rightarrow \infty$.

For the case in which we take C' instead of C in the above, we may deduce similar results also.

Thus we can explain the behavior of the trajectory $\Gamma(P)$ of Eq.(2), whose initial point P is any point on the phase plane, as $t \rightarrow \infty$, and the results will be arranged in §6.

5. Approximate solution obtained by describing function method

We put $G(s) = K/s(1+s)$ in Fig.2. Let the describing function for the relay element in Fig.2 be $N(E)$, and we have

$$N(E) = (2/\pi E) \{ \sqrt{1 - (\delta d/E)^2} + \sqrt{1 - (d/E)^2} - j(1-\delta)d/E \}, \quad E \geq d.$$

Now we assume the existence of a periodic solution $\varepsilon = E \sin \omega t$ in the system of Fig.2, then we have

$$1 + G(j\omega)N(E) = 0 \quad (15)$$

We put $\tilde{E} = E/d, \tilde{K} = K/d$. From Eq.(15), we have

$$\omega^2 = (2\tilde{K}/\pi\tilde{E}) (\sqrt{1 - \delta^2\tilde{E}^2} + \sqrt{1 - \tilde{E}^2}),$$

$$\omega = 2\tilde{K}(1-\delta)/(\pi\tilde{E}^2).$$

Eliminating ω from the two, we have

$$\tilde{E}^2 (\sqrt{\tilde{E}^2 - \delta^2} + \sqrt{\tilde{E}^2 - 1}) = 2(1-\delta)^2 \tilde{K} \pi.$$

We put $x = \tilde{E}^2$ and

$$f(x) = x(\sqrt{x - \delta^2} + \sqrt{x - 1}) - 2(1-\delta)^2 \tilde{K} \pi,$$

then we have

$$f'(x) = (\sqrt{x - \delta^2} + \sqrt{x - 1}) (1 + x/2\sqrt{x - \delta^2}\sqrt{x - 1}) > 0.$$

Thus $f(x)$ is a monotonously increasing function of x and we have $f(\infty) = \infty$, so the condition that there exists x satisfying $f(x) = 0$ and $x \geq 1$ is $f(1) \leq 0$, that is

$$\tilde{K} \geq (\pi/2) \sqrt{(1+\delta)/(1-\delta)^3}$$

or $K/d \geq (\pi/2) \sqrt{(1+\delta)/(1-\delta)^3}$.

Here we put

$$K_0^*/d = (\pi/2) \sqrt{(1+\delta)/(1-\delta)^3}.$$

The graph of K_0^*/d for δ is shown in Fig.5.

In Fig.5, let the domain enclosed by the graph K_0^*/d and K_0^*/d be S, and there exists a limit cycle of the system in Fig.2 for such d, δ, K that the point $(\delta, K/d)$ belongs to S, but the cycle cannot be found by using the describing function method.

6. Conclusion

The results of the investigation are summarized as follows.

1. A flow chart to get the transient response of a second order relay servo is given (see Fig.4).

2. For given $d > 0$, $-1 < \delta < 1$ we take t_0 and K_0 such that $1 - e^{-t_0} = (1 + \delta)t_0/2$, $t_0 > 0$ and $K_0 = 2d/t_0$. Then the necessary and sufficient condition that an isolated periodic solution (cycle) exists is $K > K_0$. Put the cycle be Γ_0 .

2.1. Case in which $K > K_0$.

If we take an initial point P belonging to the outside of the closed curve $DD_0FD_0P_0F'D$ (see Fig.11(f)), the trajectory $\Gamma(P)$ of Eq.(2) spirals around the above cycle Γ_0 as $t \rightarrow \infty$. Therefore the cycle is a stable limit cycle. If we take an initial point P belonging to the inside of the closed curve (including the circumference), the representative point of the trajectory $\Gamma(P)$ tends to a point on the segment DD' as $t \rightarrow \infty$.

2.2. Case in which $K = K_0$.

If we take an initial point P belonging to the outside of the closed curve $DD_0FD_0F'D$ (see Fig.11(e)), the trajectory $\Gamma(P)$ of Eq.(2) spirals around the closed curve as $t \rightarrow \infty$. If we take an initial point P belonging to the inside of the closed curve (including the circumference), the representative point of the trajectory $\Gamma(P)$ tends to a point on the segment DD' as $t \rightarrow \infty$.

3.1. Case in which $d > 0$, $\delta = -1$ (case of two-position relay with hysteresis).

There exists an isolated periodic solution (cycle), and it is a stable limit cycle.

3.2. Case in which $d = 0$ (case of three-position ideal relay).

For any initial point P excluding the origin, the trajectory $\Gamma(P)$ of Eq.(2) spirals around the origin as $t \rightarrow \infty$. Thus the origin is a stable equilibrium point.

3.3. Case in which $d > 0$, $\delta = 1$ (case of three-position relay with dead space).

For any initial point P , the representative point of the trajectory $\Gamma(P)$ of Eq.(2) tends to a point on the segment DD' as $t \rightarrow \infty$.

4. A flow chart to calculate the period and the amplitude of a periodic solution

of Eq.(2) is given (see Fig.8 or Fig.4).

5. A chart to obtain the period and the amplitude of a periodic solution (for given k, T, d, δ) of the system in Fig.1 is given (see Fig.9).

Appendix 1. Proof of $\lim_{n \rightarrow \infty} \chi^n(\zeta) = \zeta_0$

We put $\lim_{n \rightarrow \infty} \chi^n(\zeta) = \xi_0$. Then we have

$$\chi^n(\zeta) \rightarrow \xi_0, \chi^{n+1}(\zeta) \rightarrow \xi_0 \text{ as } n \rightarrow \infty.$$

Therefore, letting $n \rightarrow \infty$ in $\chi[\chi^n(\zeta)] = \chi^{n+1}(\zeta)$, we have

$$\chi(\xi_0) = \xi_0$$

Thus we can conclude $\xi_0 = \zeta_0$ from Lemma 2.1.

Appendix 2

In this paper, only piecewise-linear differential equations are treated.

As the phase trajectories corresponding to their solutions are obtained by the so-called piecewise-linear approach⁽⁶⁾, the existence and uniqueness theorem⁽⁷⁾ for the differential equation also holds good. Thus the trajectories corresponding to two solutions either do not intersect or else they coincide.

References

- (1) e.g. Seifert, W.W. and Steeg, C.W., Control Systems Engineering, (1960), p.717, McGraw-Hill.
- (2) Thaler, G.J. and Pastel, M.P., Analysis and Design of Nonlinear Feedback Control System, (1962), p.276, McGraw-Hill.
- (3) Kovatch, G., IEEE Trans. Autom. Contr, AC-8(1963), p.538.
- (4) Pontryagin, L.S., Ordinary Differential Equation (Translated by Kacinskis, L. and Counts, W.B.), (1962), p.223, Addison-Wesley Publishing Company, Inc.
- (5) Pontryagin, L.S., op.cit. p.226.
- (6) Shinnars, S.M., Modern Control System Theory and Application (1972), p.400, Addison-Wesley Publishing Company, Inc.
- (7) Pontryagin, L.S., op.cit. p.20.