A NUMERICAL METHOD FOR SUPersonic CONICAL FLOW 
W ithout AXial SYMMETRY*

By HIROMU SUGIYAMA

In this paper a numerical method for supersonic conical flow without axial symmetry was developed as follows: the governing equations for conical flow without axial symmetry were derived from a pair of stream functions of conical flow and a vorticity equation, and were transformed into Stocker & Mauger's coordinate system. By this method the numerical solutions were also presented for the supersonic flow past conical bodies which produced elliptic conical, nearly conical, and pseudo-triangular conical shock waves, and were compared with other investigator's solutions.

1. INTRODUCTION

A conical flow is found to occur around infinite conical bodies (or finite conical bodies near the vertices of conical bodies) in supersonic flow when viscosity and heat conductivity are neglected. The principal property of the conical flow is that the vector velocity and all the thermodynamic quantities are constant along every straight line starting from the vertices of conical bodies.

This paper treats a supersonic conical flow without axial symmetry which appears around elliptic conical bodies at zero angle of attack or circular cones at angles of attack. In this case, the conical flow is rotational, and a very thin layer of high vorticity (vortical layer) appears near the conical bodies.

The numerical methods for a conical flow are similar to the methods for a blunt-body flow with a detached shock wave, because the governing equations for these two flows are similar in mathematical properties (1). The inverse methods advancing the analysis for a prescribed shock shape and determining the body shape are, for instance, those of Briggs (2), Stocker & Mauger (3), Tamaki et al. (4), and Kurosaki (5). Briggs carried out his calculation in a physical plane, but he encountered the calculational difficulties in reaching the vortical layer. In the case of Stocker & Mauger, they used their own independent variables to perform calculations in a vortical layer, but they described no details of the vortical layer. Tamaki et al. and Kurosaki also analyzed the conical flow field by a truncation method. Their methods, however, do not give any of accurate solutions in the middle region away from symmetric planes.

Previously, Honda and Sugiyama (6) proposed a numerical inverse method for a three-dimensional blunt-body flow, adopting a pair of stream functions of three-dimensional flow and a physical coordinate as independent variables. In this paper, the same idea is applied to the analysis of a conical flow without axial symmetry, and a numerical method is developed for this conical flow as follows: The governing equations for the conical flow without axial symmetry are derived from a pair of stream functions of conical flow and a vorticity equation in a spherical polar coordinate system, and are transformed into a functional coordinate system composed of a stream function and a spherical polar coordinate, and, furthermore, the governing equations are transformed into Stocker & Mauger's coordinate system.

The distinction of this numerical method is: the pressure and the density are not adopted as dependent variables, so that numerical solutions (particularly pressure distributions) are considerably accurate near conical bodies.

Numerical computations were concretely carried out for the supersonic flow past conical bodies which produced elliptic conical, nearly conical, and pseudo-triangular conical shock waves, and the numerical results were compared with those of Stocker & Mauger (3) and Tamaki et al. (4).

NOMENCLATURE

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- : a constant related to shock shape, used in Eq. (57)
- : constants related to shock shape, used in Eq. (58)
- : speed of sound
- : pressure coefficient
- : expression for shock shape
- : entropy function, defined by Eq. (23)
- : function defined by Eq. (55)
- : unit vectors in r, θ, φ directions
2. Governing equations for conical flow without axial symmetry

2.1 A pair of stream functions

Let us consider here a three-dimensional, steady, viscid, compressible flow with a spherical polar coordinate system \((r, \theta, \phi)\) shown in Fig. 1. A pair of stream functions \(\sigma(\theta, \phi)\) and \(\psi(\theta, \phi)\) satisfying the continuity equation is introduced for a conical flow without axial symmetry as follows \((7)\):

\[
\rho \dot{\sigma} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \sigma}{\partial r} \right) \times \dot{\psi} \quad \cdots \quad (1)
\]

where \(\rho\) is the density, \(\mathbf{v}\) the velocity vector \((=u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\), and \(\mathbf{v}\) the differential operator defined as

\[
\mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \mathbf{i} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \mathbf{j} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \mathbf{k} \right)
\]

Equation (1) apparently yields

\[
\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) = 0 \quad \cdots \quad (2)
\]

\[

\mathbf{v} \cdot \mathbf{\nabla} = 0 \quad \cdots \quad (3)
\]

Under the conical flow condition \((\partial/\partial r = 0)\) Eq. (1) is reduced to

\[
\rho \dot{\sigma} = \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( r^2 \sigma \right) \times \dot{\psi} \quad \cdots \quad (4)
\]

where

\[
\dot{\psi} = \frac{1}{2} \mathbf{v} \times \sigma \times \mathbf{v} \quad \cdots \quad (5)
\]

\[
\mathbf{v} : \mathbf{v} \times \psi = 0 \quad \cdots \quad (6)
\]

Introducing the cross-flow velocity \(\sigma(\theta, \phi)\), Eq. (4) becomes

\[
\frac{\partial \sigma}{\partial \theta} = -\rho \dot{\sigma} \sin \theta \quad \cdots \quad (7)
\]

\[
\frac{\partial \psi}{\partial \phi} = \rho \dot{\psi} \quad \cdots \quad (8)
\]

Equations (2) and (3) become

\[
\frac{\partial \sigma}{\partial \theta} +\frac{w}{\sin \theta} \frac{\partial \psi}{\partial \theta} = 0 \quad \cdots \quad (9)
\]

\[
\frac{\partial \sigma}{\partial \phi} +\frac{w}{\sin \phi} \frac{\partial \psi}{\partial \phi} = 0 \quad \cdots \quad (10)
\]

The continuity equation is expressed as

\[
2\rho \dot{\sigma} \sin \theta + \frac{\partial}{\partial \theta} \left( \rho \dot{\sigma} \sin \theta \right) + \frac{\partial}{\partial \phi} \left( \rho \dot{\psi} \right) = 0 \quad \cdots \quad (11)
\]

2.2 Vorticity equation

Assume an isoviscous flow. Then a vorticity equation for the conical flow is as follows:

\[
\mathbf{v} \times \mathbf{\omega} = -\mathbf{T} \mathbf{S} \quad \cdots \quad (12)
\]

where \(\mathbf{S}\) is the entropy, \(\mathbf{T}\) the absolute temperature, and \(\mathbf{\omega}\) the vorticity vector described as

\[
\mathbf{\omega} = \omega_r \mathbf{i} + \omega_\theta \mathbf{j} + \omega_\phi \mathbf{k}
\]

\[
\omega_r = v \cot \theta + \frac{w}{\sin \theta} \quad \cdots \quad (13)
\]

\[
\omega_\theta = \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \phi} \quad \cdots \quad (14)
\]

\[
\omega_\phi = -\frac{w}{\sin \phi} \quad \cdots \quad (15)
\]

From the equation that the entropy is constant along the streamlines, i.e.

\[
\mathbf{v} \cdot \mathbf{\nabla} S = 0
\]

is yielded

\[
\mathbf{v} \cdot \mathbf{\nabla} S = 0 \quad \cdots \quad (16)
\]

From Eqs. (6) and (17) results

\[
\mathbf{v} \cdot \mathbf{\nabla} S = 0 \quad \cdots \quad (18)
\]

Now Eq. (3) is rewritten as

\[
\mathbf{v} \cdot \mathbf{\nabla} \psi = 0 \quad \cdots \quad (19)
\]

Therefore,
\[ S = S(\psi) \] .............................(20)

The meaning of this relation is that the entropy for the conical flow is constant on the \( \theta, \psi \) const. stream surface. Substituting the relation \( S = S(\psi) \) into Eq. (13), we obtain
\[ -c(\omega \cdot \nabla \psi) + \omega(\nu \cdot \nabla \psi) = 2 \rho u T \frac{dS}{d\psi} \] .............................(21)

For perfect gases with constant specific heats, the following holds
\[ \rho T = \frac{1}{c_s^2 \gamma - 1} \] .............................(22)

where \( \rho \) is the pressure, \( c_s \) the specific heat at constant volume, and \( \gamma \) the ratio of the specific heats. Introducing the entropy function
\[ F(\psi) = \frac{1}{\rho} \] .............................(23)

into Eq. (21), we obtain
\[ \frac{\partial \psi}{\partial t} + \frac{w}{\sin \theta} \frac{\partial \psi}{\partial \phi} = v^2 + w^2 \] .............................(24)

\[ (w_\nu - w_\omega) \frac{\partial \sigma}{\partial \theta} + \frac{1}{\sin \theta} (w_\nu - w_\omega) \frac{\partial \sigma}{\partial \phi} = 2u \frac{\rho}{\gamma - 1} \frac{dF}{d\psi} \] .............................(25)

The Bernoulli equation is expressed
\[ \frac{\partial \psi}{\partial \phi} = \frac{2u \sin \theta}{\gamma - 1} \] .............................(26)

where \( M_\infty \) is the free stream Mach number.

3. EQUATIONS OF MOTION IN A \((\Phi, \Psi)\) COORDINATE SYSTEM

We transform the governing equations in a \((\Theta, \Phi)\) coordinate system into those in a \((\Phi, \Psi)\) coordinate system \(3, 6\), where \( \Phi = \Phi \). Then the relations between the partial derivatives become
\[ \frac{\partial}{\partial \Phi} = \frac{\partial}{\partial \psi} - \frac{\rho \sin \theta}{\sigma} \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial \Psi} = \frac{\partial}{\partial \psi} + \frac{\rho \sin \theta}{\sigma} \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial \theta} = \frac{\omega}{\sin \theta} \frac{\partial}{\partial \phi} \] .............................(27) \(28\), (29)

Substitution of the above relations into Eqs. (10), (12), (24) and (25) results in
\[ \frac{1}{\sigma} \frac{\partial \sigma}{\partial \psi} = 2u \sin \theta \] .............................(30)

\[ \frac{\rho \sin \theta}{\sigma} \left( \frac{\partial}{\partial \psi} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \frac{\partial}{\partial \psi} \right) + \frac{\partial}{\partial \phi} = \frac{w}{c^2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta} \right) \] .............................(31)

\[ \frac{w}{c^2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta} \right) = \frac{2u \sin \theta + \cos \theta}{c^2} = 0 \] .............................(32)

\[ \frac{w}{c^2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta} \right) = \frac{w}{c^2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta} \right) = -\frac{\rho}{\gamma - 1} \frac{dF}{d\psi} \] .............................(33)

At this transformation, the following also holds
\[ \frac{\partial \theta}{\partial \phi} = -\frac{\omega}{\sin \theta}, \quad \frac{\partial \phi}{\partial \psi} = \frac{\sigma}{\rho \omega} \] .............................(34), (35)

4. EQUATIONS OF MOTION IN A \((\xi, \eta)\) COORDINATE SYSTEM

Following the transformation of Stocker & Mauger \(3\), let us transform the equations of motion in a \((\Phi, \Psi)\) coordinate system into those in a \((\xi, \eta)\) coordinate system, where
\[ \xi = \psi, \quad \eta = \log \frac{\gamma}{\sigma} \] .............................(36)

The function \( G(\xi) \), as described later, will be determined on a shock wave. In the \((\xi, \eta)\) coordinate system, a shock surface and a body surface correspond to \( \eta = 0 [ \sigma = G(\xi) ] \) and \( \eta = \infty \) (when \( \sigma = 0 \) lies in the body surface) respectively. From Eqs. (30) and (36) is yielded
\[ \frac{\partial \xi}{\partial \psi} = 2u \sin \theta \] .............................(37)

Using Eqs. (35) and (37), we obtain the following relations between the partial derivatives:
\[ \frac{\partial \xi}{\partial \phi} = \frac{2u}{\omega} \left( \frac{\sigma}{\rho \omega} \frac{\partial \phi}{\partial \psi} - \frac{\partial \phi}{\partial \xi} \right) \] .............................(38)

\[ \frac{\partial \xi}{\partial \phi} = \frac{2u}{\omega} \left( \frac{\sigma}{\rho \omega} \frac{\partial \phi}{\partial \psi} - \frac{\partial \phi}{\partial \xi} \right) \] .............................(39)

Substitution of these relations into Eqs. (31) - (34) yields
\[ \frac{u}{c^2} \frac{\partial \psi}{\partial \xi} = \frac{1}{c^2} \left( 1 - \frac{2u \sin \theta}{\gamma - 1} \right) \] .............................(40)

\[ \frac{\partial \psi}{\partial \psi} = \frac{2w}{\omega} \left( \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \psi} \right) \] .............................(41)

\[ \frac{\partial \phi}{\partial \phi} = \frac{2w}{\omega} \left( \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \psi} \right) \] .............................(42)

\[ \frac{\partial \phi}{\partial \phi} = \frac{2w}{\omega} \left( \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \psi} \right) \] .............................(43)

Equations (40) - (44) are the final ones to be numerically integrated. In this case, a remarkable point is that the number of equations to be intergrated is, as compared with Stocker & Mauger's equation \(4\), reduced to five from seven, and that the term \( 1/\omega \) of a singular point disappears at symmetric planes.

5. BOUNDARY CONDITIONS

The boundary conditions at a shock wave are given with Rankine-Hugoniot's relations. The shape of shock wave is described as
\[ f(\theta, \phi) = 0 \]  \hspace{1cm} \text{(45)}

then, the flow quantities just behind the shock wave become as follows (8):

\[ u = \cos \theta \]  \hspace{1cm} \text{(46)}

\[ v = -\sin \theta \left[ 1 - (1 - \epsilon) \frac{f_x^2}{N^2} \right] \]  \hspace{1cm} \text{(47)}

\[ w = (1 - \epsilon) \frac{f_x f_v}{N^2} \]  \hspace{1cm} \text{(48)}

\[ p = \frac{1}{\gamma M_w^2} + (1 - \epsilon) \frac{f_x^2 \sin^2 \theta}{N^2} \]  \hspace{1cm} \text{(49)}

\[ \nu = \frac{1}{\epsilon} \]  \hspace{1cm} \text{(50)}

where

\[ N^2 = f_x^2 + f_y^2 \sin^2 \theta \]  \hspace{1cm} \text{(51)}

\[ \epsilon = \frac{\gamma - 1}{\gamma + 1} \]  \hspace{1cm} \text{(52)}

subscripts \( \theta \) and \( \phi \) mean partial derivatives of \( f \).

One of the stream functions is chosen as

\[ \phi = \phi \]  \hspace{1cm} \text{(53)}

at a shock wave, then the other stream function yields

\[ \sigma = -p \epsilon \sin \theta \]  \hspace{1cm} \text{(54)}

The function \( G(\phi) \) therefore results in

\[ G(\phi) = - (p \epsilon \sin \theta) \gamma_0 \]  \hspace{1cm} \text{(55)}

6. COMPUTATIONAL PROCEDURE

To determine the values of unknown variables \( \theta, \phi, u, v, \) and \( w \), we numerically integrate Eqs. (40) – (44) with the forward integration method (8). Computational procedure is, at first, to calculate the unknown variables at the \( \gamma = \gamma_0 \) surface (shock surface) by the boundary conditions, and the second-derivatives of the unknown variables by a numerical differential formula, for instance \( \phi(\gamma_0) \) and \( \frac{\partial^2 \phi}{\partial \gamma^2} \).

Secondly it is to calculate the \( \gamma \) - derivatives of the unknown variables, for instance \( \frac{\partial \phi}{\partial \gamma} \), using Eqs. (40) – (44), and then to calculate the unknown variables, for instance \( \phi(\gamma_\alpha + \Delta \gamma) \), at the \( \gamma_\alpha + \Delta \gamma \) plane, by the relation

\[ \phi(\gamma_\alpha + \Delta \gamma) = \phi(\gamma_\alpha) + \Delta \gamma \frac{\partial \phi}{\partial \gamma} \]  \hspace{1cm} \text{(56)}

Repeating this procedure up to the body surface (theoretically \( \gamma = \infty \)), we obtain the numerical solution of the flow field between the given shock surface and the body surface.

Table 1 Numerical examples for the shock shapes described by Eq. (57)

<table>
<thead>
<tr>
<th>Case</th>
<th>( a_0 )</th>
<th>( a_0 )</th>
<th>( a_4 )</th>
<th>( a_8 )</th>
<th>( M_\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.166</td>
<td>-0.03</td>
<td>0</td>
<td>0</td>
<td>10^4</td>
</tr>
<tr>
<td>A2</td>
<td>0.128</td>
<td>-0.007</td>
<td>0.006</td>
<td>-0.0007</td>
<td>6</td>
</tr>
</tbody>
</table>

Now the velocities are obtained as mentioned above, and then the pressure and the density are calculated using Eqs. (23) and (26).

The numerical differential formula adopted is a 5-points Lagrange formula, and the mesh dimensions are taken as \( \Delta \gamma = \pi/108, \pi/72, \pi/36 \) and \( \Delta \gamma = 0.05, 0.1, 0.15 \).

Computations were carried out by making use of the electronic computer FACOM 230-75 at Hokkaido University.

7. NUMERICAL RESULTS AND DISCUSSIONS

To examine the adequacy of the present numerical results, we compared these results with those of Stocker & Mauger (3) and Tamaki et al. (4). First, numerical computations were carried out for the shapes of the shock wave adopted by Stocker & Mauger, that is

\[ \sin^2 \theta = a_0 + \sum a_i \cos i \phi \]  \hspace{1cm} \text{(57)}

where the constants \( a_i \) and the free stream Mach numbers \( M_\infty \) are given in Table 1. The ratio of specific heats of gas was \( \gamma = 1.4 \) for all calculations.
The cross-flow streamlines at the $r \cos \theta = \text{const.}$ cross sections are shown in Figs. 2(a) and 3(a) for the cases A1 and A2. In these figures, the body surface appears to be an envelope of the cross-flow streamlines. The dotted line indicates the body surface obtained by Stocker & Mauger (3). The shock layers calculated by this method were slightly thinner than those by the method of Stocker & Mauger. In other words, the present body shapes were slightly larger than those of Stocker & Mauger.

Figures 2(b) and 3(b) show the pressure distributions on the bodies (elliptic cones) for the cases A1 and A2. These pressure distributions agreed fairly well with the results of Stocker & Mauger (drawn by dotted lines) near the symmetric planes, but they were a little larger at the middle regions, i.e., $\phi = 15$ - 75 degrees and $\phi = 35$ - 85 degrees for the cases A1 and A2 respectively.

Next, the numerical computations were carried out for the shapes of the shock wave adopted by Tamaki et al. (4), that is,

$$\cot \theta = \cot \theta^* - e \cos b \phi$$  \hspace{1cm} (58)

where the constants $b$, $e$ and $\theta^*$, and $M_m$ are given in Table 2.

In Figs. 4(a) and 5(a), the cross-flow streamlines are shown for the cases B1 and B3. The body shapes calculated by the present method agreed with the results of Tamaki et al. (4) (the dotted lines) near the symmetric planes, and were a little larger at the intermediate regions.

The pressure distributions on the body surface (nearly circular cones) are shown in Figs. 4(b) and 5(b) for the cases B1 and B3. These pressure distributions also agreed with the results of Tamaki et al. (4) near the symmetric planes, but at the intermediate regions did not agree. Figures 4(b) and 5(b) also show the pressure distributions along the cross-flow streamlines of $\phi = 30$, 60, 90, 120 and 150 degrees. It is recognized from these

<table>
<thead>
<tr>
<th>Case</th>
<th>$b$</th>
<th>$e$</th>
<th>$\theta^*$ deg</th>
<th>$M_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>1</td>
<td>0.3</td>
<td>44</td>
<td>2.8</td>
</tr>
<tr>
<td>B2</td>
<td>1</td>
<td>0.3</td>
<td>44</td>
<td>5</td>
</tr>
<tr>
<td>B3</td>
<td>1</td>
<td>0.3</td>
<td>44</td>
<td>10</td>
</tr>
<tr>
<td>B4</td>
<td>3</td>
<td>0.05</td>
<td>44</td>
<td>4</td>
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<tr>
<td>B5</td>
<td>3</td>
<td>0.05</td>
<td>44</td>
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<td>B6</td>
<td>3</td>
<td>0.1</td>
<td>44</td>
<td>4</td>
</tr>
<tr>
<td>B7</td>
<td>3</td>
<td>0.1</td>
<td>44</td>
<td>10</td>
</tr>
</tbody>
</table>

![Flow pattern](image)

(a) Flow pattern

![Pressure profiles](image)

(b) Pressure profiles Fig.4 Case B1
figures that there were the thin layers in which the pressure abruptly changed along the cross-flow streamlines near the conical bodies. In this paper, the author reports only the fact that the thin layers mentioned above appeared near the conical body. Hereafter, the author will examine these thin layer structures.

Figures 6(a) and 7(a) show the cross-flow streamlines and the body shapes (nearly pseudo-triangular conical body shapes) for the cases B4 and B6. Figures 6(b) and 7(b) show the pressure distributions on the body surfaces and along the cross-flow streamlines for the cases B4 and B6. In these cases, we also recognized the appearance of the thin layers in which the pressure abruptly changed along the cross-flow streamlines near the conical bodies.

8. CONCLUSIONS

A numerical method for a supersonic conical flow without axial symmetry was developed like this: The governing equations for the conical flow without axial symmetry were derived from a pair of stream functions of conical flow and a vorticity equation in a spherical polar coordinate system, and were transformed into a functional coordinate system composed of a stream function and a spherical polar coordinate. The governing equations were, furthermore, transformed into Stocker & Mauger's coordinate system. The distinction of this numerical method is that the pressure and the density are not adopted as dependent variables and that numerical solutions — particularly pressure distributions — are considerably accurate near conical bodies.

By this method the numerical solutions were also presented for the supersonic flow past conical bodies which produced elliptic conical, nearly conical, and pseudo-triangular conical shock waves. And these solutions were compared with the solutions by Stocker & Mauger and Tanaki et al. It has been known from the present solutions that there existed the thin layers in which the pressure abruptly changed along the cross flow streamlines near the conical bodies.

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REFERENCES