A Method for Estimating the Condition That a Rotor Can Pass through Resonance *

By Katsumasa MATSUURA **

Synopsis

About a rotor accelerated across a resonance, which possesses a linear oscillating system with a single degree of freedom, excited by an unbalanced rotating mass, it is said that by investigating the non-stationary transitions of motion of the rotor under the critical condition that the rotor can pass barely through resonance or not, it is possible to formulate the condition; and an estimating expression for passage through resonance is introduced as follows;

$$K = \pi \mu_I \mu_M q \rho_{\max} < 1.0$$

In this case, if the rotor has a circular oscillating system moving in a plane, the estimating expression is given as

$$K = 2 \pi \mu_I \mu_M q \rho_{\max} < 1.0$$

where $\rho_{\max}$ is estimated by Eq. (9).

Furthermore, about a rotor having an uncoupled oscillating system with two degrees of freedom, 2 kinds of estimating expressions are proposed, which are divided into 2 parts depending on the frequency difference between 2 resonances.

1. Introduction

It is well-known since old times that there arise phenomena^{1),2)} where a rotor can not pass through resonance or where it takes much time for a rotor to pass through resonance when the rotor with an insufficient energy source is started exceeding the resonance. It is necessary to discuss the phenomena from a view point of the interaction between the oscillating system and the energy source. J.G. Baker³⁾, C.B. Biezeno and R. Grammel⁴⁾, I. Fernlund⁴⁾, and others made researches for a method of estimating the condition for a rotor to pass through resonance. Nevertheless, these methods obtained were not able to estimate the condition for a rotor to pass through a resonance.

Since 1960⁵⁾, many papers studied from a view point of the interaction have been published by V.O. Kononenko⁵⁾, R. Kawai, T. Iwatsubo⁶⁾, and others. These papers describe chiefly the stationary solutions of motion or the unstable criteria of motion in the resonance region, and a few non-stationary transitions of motion during the resonance, obtained by Vogolyubov's perturbation method, in which the problem of whether a rotor can pass through a resonance is hardly studied.

In the present paper by investigating the non-stationary transitions of motion at resonance, obtained by numerical analysis, it is found possible to formulate the transient values at resonance under the critical condition whether the rotor can pass barely

through a resonance or not, and an estimating expression for passage through a resonance is introduced in case of a rotor with a single-degree-of-freedom oscillating system.

Furthermore, the estimating expressions are proposed for passage through resonances in case of a rotor moving in a plane and possessing a linear and uncoupled oscillating system with two degrees of freedom by combinations of the result in case of one degree of freedom and the stationary dynamic load torque.

2. Nomenclature

m = rotating unbalanced mass
 M = mass of rotor excepting m
 e = radius of gyration of unbalanced mass
 c = damping coefficient
 k = spring constant
 θ = angular displacement of rotation
 ω = angular velocity of rotation
 I = polar moment of inertia of rotor
 k_O = reduction gear ratio
 T = driving torque of an energy source
 R = non-dimensional displacement of motion
 ζ = damping factor of oscillation
 $\omega_n, \omega_x, \omega_y$ = natural angular frequencies
 f_x, f_y, f_n = natural frequencies (Hz)
 h = angular acceleration, $k_O T / (2\pi(I + me^2))$
 $\mu_I, \mu_M = me^2 / (I + me^2), m / (M + m)$
 ϕ = phase lag of displacement of oscillation to the centrifugal force
 ρ = amplification factor of oscillating displacement
 ρ_{ns}, ϕ_{ns} = non-stationary values of ρ, ϕ
 $\rho_x, \rho_y, \phi_x, \phi_y$ = stationary values of ρ, ϕ
 TDL = stationary dynamic load torque on motor axis of rotation
 TD_{ns} = non-stationary value of TDL
 $h_{Dns} = k_O T D_{ns} / (2\pi(I + me^2))$
 q = coefficient for acceleration, f_n^2 / h
 TD_{max} = maximum value of TD_{ns}
 K = estimating value for passage through

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** Senior Researcher,
Hitachi Research Laboratory,
Hitachi, Ltd., Hitachisi.

resonance

3. Equations of motion

Fig. 1 shows a schematic diagram of the system treated in the paper. The differential equations of motion for a rotor started, excited by a rotating unbalanced mass and moving in a plane, as shown in Fig. 1, are given as follows;

$$\left. \begin{aligned} M\ddot{x} + C_x \dot{x} + k_x x &= -m \frac{d^2}{dt^2} (x + e \cos \theta) \\ M\ddot{y} + C_y \dot{y} + k_y y &= -m \frac{d^2}{dt^2} (y + e \sin \theta) \\ (I + me^2) \ddot{\theta} &= k_0 T + m\ddot{x} e \sin \theta - m\ddot{y} e \cos \theta \end{aligned} \right\} (1)$$

, where it is assumed that two-degree-of-freedom system is linear and uncoupled. Using the nomenclature, Eq. (1) are rewritten as follows;

$$\left. \begin{aligned} \ddot{R}_x + 2\zeta_x \omega_x \dot{R}_x + \omega_x^2 R_x &= \ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta \\ \ddot{R}_y + 2\zeta_y \omega_y \dot{R}_y + \omega_y^2 R_y &= -\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta \\ \ddot{\theta} &= 2\pi h + \mu_I \mu_M (\ddot{R}_x \sin \theta - \ddot{R}_y \cos \theta) \end{aligned} \right\} (2)$$

4. A method for estimating the condition that a rotor can pass through resonance in case of a single-degree-of-freedom oscillating system ($R_y = 0.0$)

4.1 Numerical analysis

4.1.1 Input data

Non-stationary transitions of motion of a rotor, given by Eqs. (1), (2) are calculated by a digital computer using Runge-Kutta-Gill method, and the initial values are given as follows;

$$t = 0 \quad ; \quad \theta, \dot{\theta}, R_x, \dot{R}_x = 0$$

Fig. 1 Schematic diagram of the system treated in the paper

Furthermore, the main input data used in the numerical calculation are as follows;

$$M = 0.06 \text{ kg s}^2/\text{cm}, I = 2.88 \text{ kg cm s}^2, e = 24 \text{ cm}, k_0 = 4., f_x = 2.5 \text{ Hz}$$

4.1.2 Angular velocity characteristics of rotation

Fig. 2 shows the angular velocity characteristics of rotation obtained by numerical analysis, where we plot real time t (sec.) in ordinate and frequency ratio $\frac{\dot{\theta}}{\omega_x}$ in abscissa. In Fig. 2 the following phenomena can be observed concerning the transient characteristics of angular velocity of rotation; ① in a case that $\zeta_x = 0.05$, $m = 0.001 \text{ kg s}^2/\text{cm}$, $h = 0.5 \text{ c/s}^2$, the rotational speed of the rotor increases as time passes and we obtain that $\frac{\dot{\theta}}{\omega_x} \approx 1.045$ at $t = 6.2 \text{ sec.}$, after that it slows down gradually as time passes, which shows the phenomenon where the rotor can't exceed the critical speed. ② when $\zeta_x = 0.0$, $m = 0.002 \text{ kg s}^2/\text{cm}$, $h = 1.0 \text{ c/s}^2$, the rotational speed of the rotor shows the maximum value $\frac{\dot{\theta}}{\omega_x} = 0.952$ at 3.2 sec. after the start, and then it is stagnant at resonance while swelling. When ③ $\zeta_x = 0.05$, $m = 0.001 \text{ kg s}^2/\text{cm}$, $h = 0.7 \text{ c/s}^2$, ④ $\zeta_x = 0.05$, $m = 0.001 \text{ kg s}^2/\text{cm}$, $h = 1.0 \text{ c/s}^2$, ⑤ $\zeta_x = 0.0$, $m = 0.001 \text{ kg s}^2/\text{cm}$, $h = 1.0$, ⑥ $\zeta_x = 0.0$, $m = 0.002$, $h = 2.0$, the rotational speeds increase as time passes and the critical speeds can be exceeded in these cases.

Fig. 2 Non-stationary transitions of angular velocity of rotation in case of a rotor with a single-degree-of-freedom oscillating system

4.1.3 Non-stationary transitions of motion of a rotor across a resonance

Fig. 3 shows the non-stationary transitions of motion corresponding to the velocity characteristics shown in Fig. 2. In case of ①, the displacement amplitude of the rotor increases as time passes and approaches the stationary amplitude $X_{\max}$ at resonance, given as $X_{\max} = \frac{1}{2\zeta_x} = 10$. In case of ②, as the rotor can not exceed the critical speed as shown in Fig. 2, the amplitude of displace-

ment increases with time passing. But it tends to approach a limited value though the damping factor is 0.0, because the average value of the rotational speed with swelling is smaller than the resonant speed. In cases of ③ and ⑤, the maximum amplitudes of displacement occur at 4.9 sec. and 3.52 sec. from the start respectively, afterwards the amplitudes decrease as time passes. In case of ⑤, we take notice of the phenomena where the amplitude of displacement, after resonance, does not decrease and it keeps slightly smaller than the maximum value.

Fig. 3 Non-stationary transitions of motion of a rotor with a single-degree-of-freedom oscillating system excited by a rotating unbalanced mass

4.1.4 Non-stationary transition of angular phase difference between the centrifugal force and the motion of a rotor

Fig. 4 shows non-stationary transitions of the phase angles corresponding to cases shown in Fig. 2, and we can find the following phenomena concerning the transient phase angles.

Fig. 4 Non-stationary transitions of the phase angle between exciting force and motion of oscillation in case of a single-degree-of-freedom system

In case of ①, the phase angle φ_{ns} becomes $\frac{\pi}{2}$ at 6 sec. from the start and at $t=7.0$ sec. occurs the maximum $\varphi_{max} \approx 0.6\pi$, after that, with time passing, it decreases as given by $\varphi_{ns}=0.57\pi$ at 8 sec. In case of ②, after the maximum phase angle $\varphi_{ns}=0.31\pi$ occurs at 3.2 sec., the phase angle decreases as time passes while swelling. Afterwards, the phase angle repeats between 0.0 and 0.16π . In cases of ③ and ⑤, the phase angles increase as time passes after the phase angles show 0.5π at 4.1 sec. and 2.9 sec. respectively.

4.2 Comparison of non-stationary solutions of motion with stationary ones

Fig. 5 shows relations between the non-stationary solutions of motion and the stationary ones in case of ③ described above.

In Fig. 5 we plot frequency ratio $\frac{\dot{\theta}}{\omega_x}$ in ordinate and amplification factor of displacement ρ_{ns} or phase angle φ_{ns} in abscissa. As shown in Fig. 5, the maximum amplification factor of displacement in the non-stationary state equals to approximately 75% of the stationary value at resonance, and the critical speed at which the amplitude becomes maximum is approx. 15% higher than the natural frequency, and at the frequency ratio $\frac{\dot{\theta}}{\omega_x} \approx 1.07$, the phase angle $\varphi_{ns} \approx \frac{\pi}{2}$ occurs. These phenomena depend on the angular acceleration of rotation at resonance.

Fig. 5 Comparison of non-stationary characteristics of ρ_{ns} , φ_{ns} , $k_0 T_{Dns}$ with stationary ones of ρ_x , φ_x , $k_0 T_{DL}$

4.3 Dynamic load torque in the non-stationary state

As presented in the preceding paper⁷⁾, we know the dynamic load torque in the stationary state is given by the following expressions:

$$T_{DL} = \frac{m e^2}{2 k_0} \mu_M \omega^2 (\rho_x \sin \varphi_x + \rho_y \sin \varphi_y) \quad (3)$$

Thus, it can be assumed that the dynamic load torque T_{Dns} in the non-stationary state, generated by motion of a rotor with a single-degree-of-freedom oscillating system is given as follows:

$$T_{Dns} = \frac{me^2}{2k_0} \mu_M \dot{\theta}^2 \rho_{ns} \sin \varphi_{ns} \quad (4)$$

by putting $\rho_y = 0$ because of the system being single-degree-of-freedom, and by replacing respectively ρ_x , φ_x in Eq. (3) with ρ_{ns} , φ_{ns} which show the amplitude of displacement and the phase angle in the non-stationary state.

Meanwhile, the third expression of Eq. (1) shows torque balance in the driving system, and the second term in the righthand side of the equation shows the non-stationary dynamic load torque generated by oscillations.

Hereupon, we suppose that $k_0 T_{Dns}$ given by Eq. (4) becomes equal to the average value of the dynamic load torque obtained by numerical analysis of Eq. (1) for one cycle of oscillation. Thus, we rewrite the equation given by the third expression of Eq. (1) as follows:

$$(I + me^2) \ddot{\theta} = k_0 (T - T_{Dns}) \quad (5)$$

Dividing both sides of Eq. (5) by $I + me^2$, we obtain

$$\ddot{\theta} = 2\pi (h - h_{Dns}) \quad (6)$$

$$h_{Dns} = \frac{1}{4\pi} \mu_I \mu_M \dot{\theta}^2 \rho_{ns} \sin \varphi_{ns}$$

As seen from Eqs. (5) and (6), a rotor can pass through resonance when the following condition holds at resonance:

$$T > T_{Dns} \quad \text{or} \quad h > h_{Dns} \quad (7)$$

In Table 1, the values of h_{Dns} and T_{Dns} in the resonance region are shown, which are calculated by substituting the values of ρ_{ns} , φ_{ns} , $\dot{\theta}$ obtained from Figs. 2, 3, 4 into Eqs. (4), (5). Furthermore, the driving torque T and the driving acceleration of rota-

tion of rotation h are written in Table 1 to compare with T_{Dns} and h_{Dns} .

In case of ① in Table 1, we can find that $h > h_{Dns}$, namely $T > T_{Dns}$ occurs at $t = 6.2$ sec.. Thus, the angular velocity of rotation $\dot{\theta}$ decreases as time becomes longer than 6.2 sec. and then in this case, we can find that the rotor can't exceed the critical speed. If the angular velocity of rotation is stagnant at resonance, the amplitude of displacement increases as time passes, so that it becomes more difficult for a rotor to exceed resonance because the dynamic load torque increases as time passes.

In case of ② in Table 1, $T_{Dns} > T$ occurs at $t = 3.2$ sec. when the angular velocity shows the maximum value given as $\frac{\dot{\theta}}{\omega_x} \approx 0.9515$.

When $t > 3.2$ sec., the angular velocity decreases with time passing. Meanwhile, in this case, the phase difference φ_{ns} also decreases as the velocity decreases, and at $t = 4.0$ sec., the minimum $\varphi_{ns} = 0.03$ occurs. Under the condition, the rotor is again accelerated because $T > T_{Dns}$, and then, the phase difference and the dynamic load torque T_{Dns} again increase with time passing, and the second condition $T < T_{Dns}$ occurs, so that the second decrease of the velocity begins. As a result, the phase angle φ_{ns} tends to repeat between 0. and 0.16π , and the angular velocity ratio of rotation is reduced to swelling around the average value given as $\frac{\dot{\theta}}{\omega_x} \approx 0.965$.

In cases of ③ and ⑤ as shown in Fig. 2, $k_0 T_{Dns} = 8.8$ and 13.4 kgcm occur at 4.5 and 3.25 sec. respectively, which show the maxima of dynamic load torques in non-stationary states. Meanwhile the driving torques are given as $k_0 T = 12.66$ and 18.1 kgcm which are respectively larger than the maxima of T_{Dns} . As a result, we can find in these cases that the rotor can exceed the critical speed. Furthermore, we can find from ③ of Table 1 that the velocity at which T_{Dns}

Table 1 Transitional values in the region of resonance

No.	Input data	t	$\frac{\dot{\theta}}{\omega_x}$	φ_{ns}	ρ_{ns}	P^{*1}	h_{Dns}	$k_0 T_{Dns}$	$k_0 T$	ρ_{max}	$k_0 T_{Dmax}$	K	Note
①	$\zeta_x = 0.05$ $m = 0.001$ $h = 0.5$	5.4	1.0	0.37π	5.0	4.58	0.233	4.21	9.04	7.31	8.6 (9.56) ^{*2}	0.957	*1 $P = (\frac{\dot{\theta}}{\omega_x})^2 \rho_{xns}$ $\times \sin \varphi_{xns}$
		6.0	1.04	0.5π	6.9	7.46	0.485	7.58					
		6.2	1.045	0.54π	7.4	7.99	0.52	9.41					
		8.0	1.0	0.57π	9.1	8.86	0.576	10.42					
②	$\zeta_x = 0.0$ $m = 0.002$ $h = 1.0$	3.2	0.952	0.31π	6.64	4.96	1.28	23.17	18.1	9.45	44.4 (43.5)	2.474	*2 $9.56 = 8.6/\beta$ β is approximately obtained by Fig. 6
		4.0	0.93	0.03π	9.3	0.73	0.19	3.44					
		4.75	1.0	0.2π	11.1	6.43	1.67	30.22					
③	$\zeta_x = 0.05$ $m = 0.001$ $h = 0.7$	2.9	0.8	0.17π	1.4	0.456	0.0292	0.532	12.66	6.64	7.97 (8.74)	0.621	Input data $M = 0.06 \text{ kg s}^2/\text{cm}$ $I = 2.88 \text{ kgcm s}^2$ $e = 2.4 \text{ cm}$ $f_x = 2.5 \text{ Hz}$
		3.25	0.9	0.25π	2.2	1.26	0.082	1.48					
		3.75	1.0	0.36π	3.6	3.24	0.21	3.8					
		4.1	1.07	0.5π	5.5	6.27	0.408	7.38					
		4.5	1.1	0.6π	6.5	7.48	0.486	8.8					
		4.9	1.15	0.78π	7.5	6.36	0.413	7.48					
		5.15	1.2	0.91π	7.0	2.81	0.18	3.26					
		5.5	1.3	0.95π	4.5	1.19	0.078	1.4					

becomes maximum is higher than the velocity where $\varphi_{ns} \doteq \frac{\pi}{2}$ occurs, and is smaller than the velocity where ρ_{ns} becomes maximum. In Fig. 5, the values of kOT_{DL} are plotted to compare with kOT_{Dns} . Table 2 shows an example of numerical calculation results shown in Fig. 5. As seen in Table 2, the dynamic load torque obtained by numerical analysis of Eq. (1) changes from $-2.44 \text{ kg}^{\text{cm}}$ upto $16.87 \text{ kg}^{\text{cm}}$ during a cycle, but the

average value of kOT_{Dns} in one cycle of oscillation is given as $kOT_{Dns} = 7.18 \text{ kg}^{\text{cm}}$, and the average value of $\frac{\dot{\theta}}{\omega_x}$ in that cycle becomes 1.15. The point $\textcircled{A}$ in Fig. 5 characterizes these average values, which overlap the curve of kOT_{Dns} obtained by Eq. (4). Thus, it is reasonable to use Eq. (4) instead of numerical analysis, as an approximate expression for estimating the non-stationary dynamic load torque.

Table 2 An example of numerical calculation of dynamic load torque in non-stationary state

t (s)	R_x	$\dot{\theta}/\omega_x$	θ/π	kOT_{Dns}
4.7244	4.7391	1.1310	14.9938	0.2918
4.7294	4.2535	1.1325	15.0221	-0.8027
4.7344	3.7377	1.1340	15.0504	-1.6188
4.7394	3.1952	1.1356	15.0787	-2.1718
4.7444	2.6295	1.1373	15.1071	-2.4444
4.7494	2.0447	1.1389	15.1356	-2.4271
4.7544	1.4445	1.1406	15.1641	-2.1193
4.7594	0.8332	1.1422	15.1926	-1.5285
4.7644	0.2149	1.1437	15.2212	-0.6712
4.7694	-0.4062	1.1451	15.2408	0.4282
4.7744	-1.0256	1.1464	15.2784	1.7374
4.7794	-1.6392	1.1475	15.3071	3.2172
4.7844	-2.2425	1.1485	15.3358	4.8229
4.7894	-2.8314	1.1498	15.3645	6.5054
4.7944	-3.4018	1.1498	15.3932	8.2125
4.7994	-3.9494	1.1502	15.4220	9.8909
4.8044	-4.4706	1.1504	15.4507	11.4877
4.8094	-4.9614	1.1505	15.4795	12.9522
4.8144	-5.4185	1.1504	15.5082	14.2379
4.8194	-5.8385	1.1502	15.5370	15.3032
4.8244	-6.2184	1.1498	15.5657	16.1138
4.8294	-6.5555	1.1494	15.5945	16.6432
4.8344	-6.8472	1.1489	15.6232	16.8734
4.8394	-7.0915	1.1485	15.6519	16.7961
4.8444	-7.2867	1.1480	15.6806	16.4124
4.8494	-7.4311	1.1477	15.7093	15.7327
4.8544	-7.5239	1.1474	15.7380	14.7764
4.8594	-7.5642	1.1472	15.7666	13.5714
4.8644	-7.5517	1.1472	15.7953	12.1528
4.8693	-7.4865	1.1473	15.8240	10.5623
4.8743	-7.3689	1.1476	15.8526	8.8465
4.8793	-7.1990	1.1482	15.8813	7.0556
4.8843	-6.9803	1.1489	15.9100	5.2420
4.8893	-6.7119	1.1498	15.9837	3.4589
4.8943	-6.3964	1.1509	15.9675	1.7584
4.8993	-6.0360	1.1522	15.9936	0.1911
		A.V* 1.15		A.V 7.18

* A.V is short for the value averaged in a cycle.

4.4 Expression for estimating passage through resonance

4.4.1 Approximate formulation of variables

ρ_{ns} , φ_{ns} , $\frac{\dot{\theta}}{\omega_x}$ when the maximum of T_{Dns} occurs at resonance

For obtaining the expression for estimating conditions that the rotor can pass through resonance, it is necessary to formulate non-stationary values of variables ρ_{ns} , φ_{ns} , $\frac{\dot{\theta}}{\omega_x}$ corresponding to the maximum of the non-stationary dynamic load torque. Hereupon, we investigate Table 1 where the

numerical results are calculated under the critical condition whether the rotor can pass barely through resonance or not.

Firstly, concerning φ_{ns} , we find that in case of $\textcircled{1}$, T_{Dns} becomes maximum when $\varphi_{ns} = 0.6\pi$. Though these values are slightly

larger than $\frac{\pi}{2}$ in the critical condition, magnitudes of $\sin \varphi_{ns}$ are not so different from the value of $\sin \frac{\pi}{2}$.

Thus, we approximate the value of φ_{ns} as follows:

$$\varphi_{ns} \approx \frac{\pi}{2} \quad \text{..... (8)}$$

, where T_{Dns} becomes maximum. Secondly, we know already that the maximum amplification factor ρ_{max} of displacement is approximately given as⁸⁾

$$\rho_{max} \approx 3.78\sqrt{q}e^{-a}, a=1.16q^{0.379}(2\zeta)^{0.7} \dots (9)$$

, provided a rotor has an ideal energy source.

If we compare the values of ρ_{ns} given in Table 1 with those of ρ_{max} calculated by Eq. (9), it is found that in case of ①, the value $\rho_{max}=7.31$ exists between $\rho_{ns}=6.9$ at $t=6.0$ sec. and $\rho_{ns}=7.4$ at $t=6.2$ sec., which show the critical conditions for the rotor to be unable to pass through resonance; and in case of ③, the value of $\rho_{max}=6.64$ approximately equals to the value of $\rho_{ns}=6.5$ which occurs at the velocity where T_{Dns} becomes the maximum, i.e. 8.8 in this particular case, when the rotor can pass through resonance. Thus, it is assumed that the value of ρ_{ns} corresponding to the maximum value of T_{Dns} is approximately given by the following expression under the critical condition :

$$\rho_{ns} \approx \rho_{max} \dots (10)$$

Thirdly, it is found from Table 1 that in case of ① the frequency ratio $\frac{\dot{\theta}}{\omega_x}$ corresponding to the maximum of T_{Dns} is about 1.045, and in case of ③ it is about 1.1. We know already that in case of a rotor with an ideal energy source, the critical speed where the maximum amplitude of displacement occurs varies with the angular acceleration at resonance. But, as obviously seen in above results, in case of a rotor with limited energy source not so large, the velocity corresponding to the maximum of T_{Dns} under the critical condition is not so different from the natural frequency because the dynamic load torque becomes very large and the difference between the driving torque and the dynamic load torque becomes very small, and then the angular acceleration at resonance becomes very small under the critical condition. As a result, it is assumed that under the critical condition the approximate value of $\frac{\dot{\theta}}{\omega_x}$ where the maximum of T_{Dns} occurs is given as follows :

$$\frac{\dot{\theta}}{\omega_x} \approx 1.0 \dots (11)$$

Substituting Eqs. (8), (10), (11) into Eq. (4), we obtain

$$T_{Dmax} = \frac{me^2}{2k_0} \mu_M \omega_x^2 \rho_{max} \dots (12)$$

Thus, the rotor can pass over the critical speed when $T > T_{Dmax}$, which is rewritten as

$$K = \pi \mu_I \mu_M q \rho_{max} < 1.0 \dots (13)$$

where $q = f_x^2/h$. The non-dimensional value K given in the estimating expression Eq. (13) for passage through resonance is called the estimating value.

We show some examples of estimation for passage through resonance as follows : (A) in case of ④ in Fig. 2, $\rho_{max} \approx 5.94$ is obtained from Eq. (9) because $\zeta_x = 0.05$, $h = 1.0 \text{ c/s}^2$, $m = 0.001 \text{ kgs}^2/\text{cm}$, thus from Eq. (13) we

obtain $K = 0.389$. (B) in case of ⑥ in Fig. 2, $\rho_{max} = 6.68$ because $\zeta_x = 0.0$, $h = 2.0 \text{ c/s}^2$, $m = 0.002 \text{ kgs}^2/\text{cm}$, so that $K = 0.875$. (C) in case of ⑤, $\rho_{max} = 9.45$, so that $K = 0.619$.

In above 3 cases the rotor can be driven exceeding the critical speed because $K < 1.0$. The values K and $k_0 T_{Dmax}$ calculated by Eqs. (12), (13), in cases of ①, ②, ③ are shown in Table. 1. The above results exclusive of the case ① agree well with the results of the numerical analysis as shown in Fig. 2.

4.4.2 Correction of the estimating value

It is found from Fig. 2 that though the value of K is about 0.957, and smaller than 1.0 in case of ①, the rotor can not pass through resonance. This result shows that the estimation of maximum dynamic load torque by Eq. (12) contains a little error. So we introduce a correcting factor β for estimation of passage through resonance, and give the estimating expression as follows :

$$K = \pi \mu_I \mu_M q \rho_{max} < \beta \dots (14)$$

Where, we calculate the value of β from the numerical analysis, and investigate the magnitude of an accidental error of estimating value K . Fig. 6 describes the variation of β with ζ_x when $q = 6.25$ and β with q when $\zeta_x = 0.05$. As seen in Fig. 6, β shows approximately the value between 0.9 and 1.0.

Fig. 6 Examples of correcting factor introduced from results of numerical analysis

Thus, the maximum magnitude of an error of K or T_{Dmax} is at most 10%. Consequently, if we take $\beta = 0.9$ when the value of β is not known, we can find the condition where the rotor can pass certainly through resonance. Hence, we can introduce the maximum dynamic load torque T_{Dmax} from Eq. (14) as follows:

$$T_{Dmax} = \frac{me^2}{2k_0\beta} \mu_M \omega_x^2 \rho_{max} \quad (15)$$

If we consider the value of β in case of ③, we obtain $k_0 T_{Dmax} = 8.74 \text{ kgcm}$, and so the region of m when the rotor can pass through resonance is given as follows:

$$k_0 T = 12.66 > k_0 T_{Dmax} \approx 8.74 \times 10^6 \text{ m}^2$$

$$\therefore m < 1.203 \times 10^{-3} \text{ kg s}^2/\text{cm} \quad (\because \rho_{max} \approx 6.64)$$

Fig. 7 shows the velocity characteristics of a rotor under the critical condition decided by considering the value of β . These results show good coincidence with the above results.

Fig. 7 Angular velocity characteristics of a rotor under the critical condition decided considering the value of β

4.4.3 Non-stationary characteristics after resonance

Fig. 8 shows mainly the non-stationary characteristics after resonance of ρ_{ns} , φ_{ns} , T_{Dns} when $m = 1.2 \times 10^{-3} \text{ kg s}^2/\text{cm}$ in case of Fig. 7. In Fig. 8 the values of $k_0 T_{Dns}$ are given by averaging numerical calculation values at intervals of each $1/2$ cycle. As shown in Fig. 8, the non-stationary characteristics of ρ_{ns} , φ_{ns} , T_{Dns} approach the stationary characteristics after resonance while swelling as time passes. T_{Dns} shows negative values in the region given as

$1.2 \leq \frac{\dot{\theta}}{\omega_x} < 1.4$, and then the energy generated

by oscillations is supplied to the driving system in this region, so that the velocity is accelerated by oscillations and leaves rapidly the resonance in the region.

Fig. 8 Comparison of non-stationary solutions with stationary ones in the region after resonance

5. Estimation for a rotor to pass through resonance when it has a two-degree-of-freedom oscillating system

Our study shown in preceding chapters has been limited to discussion only when the rotor possesses a one-degree-of-freedom oscillating system. In this chapter, we discuss on a rotor in plane motion oscillating with two degrees of freedom as shown in Fig. 1.

5.1 A case of circular oscillating system ($\omega_x = \omega_y$, $\zeta_x = \zeta_y$)

In this case, the dynamic load torque is given by twice the value in case of a single degree of freedom, and so from Eqs. (14), (15) we obtain the following expressions to estimate the condition of a rotor for passage through resonance:

$$K = 2\pi\mu_I\mu_M q \rho_{max} < \beta \approx 1.0 \quad (16)$$

$$T_{Dmax} = \frac{1}{k_0\beta} me^2 \mu_M \omega_n^2 \rho_{max} \quad (17)$$

5.2 A case that one of the natural frequencies is apart from the other.

In this case it is assumed that the dynamic load torque in each resonance is given by the sum of the non-stationary dynamic load torque at one resonance and the stationary load torque at the speed of rotation in a system having another resonance, because the non-stationary transitions of motion in case of a single degree of freedom become close to the stationary ones in the outside region of resonance as shown in Fig. 7. Thus, the estimating expression for passage through resonance and the maximum value of the dynamic load torque at one resonance, for example ω_y , are given as follows:

$$K = \pi \mu_I \mu_M q_y \{ (\rho_x \sin \varphi_x) \dot{\theta} = \omega_y + \rho_{y\max} / \beta \} \dots (18)$$

$$T_{D\max} = \frac{1}{2k_0} m e^2 \mu_M \omega_y^2 \{ (\rho_x \sin \varphi_x) \dot{\theta} = \omega_y + \rho_{y\max} / \beta \} \dots (19)$$

where symbols having subscript y show an oscillating system of y axial direction, subscript x indicates x axial direction. Furthermore, the value of β may be approximately given by 1.0 when we don't know the correct value of β .

Now, we illustrate some examples estimated by Eqs. (18), (19). (A) if $f_x = 2.5$ Hz, $f_y = 5.0$ Hz, $\zeta_x = \zeta_y = 0.05$, $T = 4.0$ kg·cm, and other input data are the same as in the cases of Table 1, we obtain: $q_y = 28.25$, $\rho_{y\max} = 8.83$, $\rho_x = 1.333$, $\varphi_x = 176^\circ 10'$ for the second resonance ω_y , and then substituting these values in Eq. (19), we obtain

$$m < 0.597 \times 10^{-3} \text{ kg s}^2/\text{cm}$$

which shows the region of m for passage through the second resonance ω_y , where $\beta = 0.94$. Furthermore, (B) the region of m for passage through the first resonance ω_x is given as $m < 1.402 \times 10^{-3} \text{ kg s}^2/\text{cm}$ by Eqs. (18), (19). (C) if $f_x = 2.0$ Hz, $f_y = 2.5$ Hz, and other input data are the same as in the case of (A), we obtain: $q_y = 7.06$, $\rho_{y\max} = 6.18$, $1/2 m e^2 \mu_M = 4.8 \times 10^3 \text{ m}^2$, $\beta = 0.9$, $\rho_x = 4.44$, $\varphi_x = 167^\circ 30'$, accordingly we obtain $k_0 T_{D\max} = 9.21 \text{ kg cm}$ from Eq. (19), so that the region of m for passage through the second resonance ω_y is given as $m < 1.32 \times 10^{-3} \text{ kg s}^2/\text{cm}$. Fig. 9 shows the numerical results calculated for the above three cases (A), (B), (C). It is proved from Fig. 9 that the numerical calculation results coincide well with the results of estimation for passage through resonances.

Fig. 9 Angular velocity characteristics in cases that one of the natural frequencies is away from the other

Fig. 10 shows the non-stationary characteristics of ρ_{ns} , φ_{ns} depending on the angular velocity of rotation when $m = 0.0013 \text{ kg s}^2/\text{cm}$ in case of (C) of Fig. 9. As shown in Fig. 10, $\rho_{y\text{ns}}$, $\varphi_{y\text{ns}}$ approach rapidly each stationary solution at the second

resonance ω_y , and $\rho_{x\text{ns}}$, $\varphi_{x\text{ns}}$ also approach the stationary state at the frequency ratio $\frac{\dot{\theta}}{\omega_x} = 1.06 \frac{\omega_y}{\omega_x} = 1.325$.

Furthermore, it is found from Fig. 10 that both of $\varphi_{y\text{ns}}$ and $\varphi_{x\text{ns}}$ make a loop-line near $\frac{\dot{\theta}}{\omega_y} = 1.0$ and $\frac{\dot{\theta}}{\omega_x} = 1.325$ where the rotor is stagnant at rotational speed.

Fig. 10 Frequency characteristics of ρ_{ns} , φ_{ns} in case of a rotor with a two-degree-of-freedom uncoupled oscillation

5.3 A case that one of the natural frequencies is close to the other.

In this case, it is assumed that $T_{D\max}$ is given by the summation of the non-stationary dynamic load torques in each resonance, so we obtain:

$$T_{D\max} = \frac{1}{2k_0} m e^2 \mu_M \{ \omega_x^2 \rho_{x\max} / \beta_x + \omega_y^2 \rho_{y\max} / \beta_y \} \dots (20)$$

$$K = \pi \mu_I \mu_M \{ q_x \rho_{x\max} / \beta_x + q_y \rho_{y\max} / \beta_y \} < 1.0 \dots (21)$$

If we put $\omega_x = \omega_y$ and $\rho_{x\max} = \rho_{y\max}$ in Eqs. (20), (21), these equations agree with Eqs. (17), (16) in case of a circular oscillating system. To give some examples, (A) if $f_x = 2.4$ Hz, $k_0 T = 16 \text{ kg cm}$, $\zeta_x = \zeta_y = 0.05$, we obtain: $q_x = 6.51$, $\rho_{x\max} = 6.02$, $q_y = 7.06$, $\rho_{y\max} = 6.18$, $\beta_x = \beta_y = 0.9 \pi \mu_I \mu_M = 10.456 \times 10^3 \text{ m}^2$, and then we obtain the region of m for passage through resonance from Eq. (21) as $m < 1.025 \times 10^{-3} \text{ kg s}^2/\text{cm}$. Furthermore, if we estimate this case by Eq. (18), we obtain $m < 1.015 \times 10^{-3} \text{ kg s}^2/\text{cm}$, which is not so different from the result estimated by Eq. (21). If we apply Eq. (21) to the case of (C) in the section 5.2, we obtain $m < 1.063 \times 10^{-3} \text{ kg s}^2/\text{cm}$, which is fairly different from the result estimated by Eq. (18). As a result, it is supposed that if the difference between two natural frequencies is smaller than 10%, it will be necessary for the estimation to use Eqs. (20) or (21).

(B) if $f_x = 2.5 \text{ Hz}$, $f_y = 2.6 \text{ Hz}$ and other input data are the same as in the case of (A), we obtain: $q_y = 7.06$, $\rho_{x\max} = 6.18$, $q_y = 7.65$, $\rho_{y\max} = 6.34$, and then in the case the region for passage through resonance is given as $m < 0.97 \times 10^{-3} \text{ kg s}^2/\text{cm}$. If we estimate the case using Eq. (18), it is given as $m < 0.95 \times 10^{-3} \text{ kg s}^2/\text{cm}$, which agrees approximately with the result by Eq. (21).

Fig. 11 shows the velocity characteristics obtained by the numerical analysis, corresponding to cases (A) and (B). As shown in Fig. 11, the numerical calculation results coincide well with the results estimated by Eqs. (20), (21).

Fig. 11 Angular velocity characteristics in cases that one of the natural frequencies is close to the other

5.4 A case where an oscillating system has a damping factor larger than 0.1^9

If the damping factor ζ is larger than 0.1 , the values of $\rho_{\max}$ given by Eq. (9) are approximated as follows:

$$\rho_{\max} \approx \frac{1}{2\zeta} \quad (22)$$

, which shows the stationary amplitude at resonance, so that in case of $\zeta > 0.1$, we can obtain the condition for passage through resonance from TDL characteristics given by Eq. (3).

Fig. 12 shows a driving torque together with dynamic load torque characteristics in cases that $f_x = 2.0 \text{ Hz}$, $f_y = 2.5 \text{ Hz}$, $m = 0.002 \text{ kg s}^2/\text{cm}$, $\zeta_x = \zeta_y = 0.15$ and $\zeta_x = \zeta_y = 0.3$. From Fig. 12, we see that the maximum value of TDL is about 5.5 kg cm in case of $\zeta_x = \zeta_y = 0.15$ at f_y , and that the driving torque at f_y is 8.2 kg cm . In this case, as the value of TDL is approximately proportional to m^2 , we can obtain the magnitude of m under the critical condition which decides whether the rotor can pass through resonance or not as follows:

$$m \approx 0.002 \sqrt{\frac{8.2}{5.5}} \approx 0.00244 \text{ kg s}^2/\text{cm}$$

Fig. 12 Torque-speed characteristics of the driving torque and the stationary dynamic load torque

Fig. 13 shows non-stationary characteristics of oscillations when $m = 0.0025 \text{ kg s}^2/\text{cm}$. It is understood from Fig. 13 that the rotor can not pass through resonance in this case.

Fig. 13 Non-stationary characteristics of oscillations and angular velocity of rotation when the rotor having the damping factors larger than 0.1 is excited by a rotating mass $m = 0.0025 \text{ kg s}^2/\text{cm}$ corresponding to the critical condition under which it can't pass through resonance

In Fig. 14, we show the frequency characteristics of ρ_{ns} , ϕ_{ns} when $m = 0.0024 \text{ kg s}^2/\text{cm}$.

As shown in Fig. 14, the values of $\rho_{y\text{ns}}$ as well as $\phi_{y\text{ns}}$ come close to the stationary

values near the frequency ratio $\frac{\omega}{\omega_y} = 1.0$, and ρ_{xns} , φ_{xns} rapidly converge to each stationary value near $\frac{\omega}{\omega_y} = \frac{\omega_y}{\omega_x} = 1.25$.

The results shown in Figs. 13 and 14 prove that it is possible to estimate the condition for a rotor to pass through resonance using the maximum value of stationary dynamic load torque when the damping factors of the system in x and y axial directions are larger than 0.1.

Fig. 14 Frequency characteristics of ρ_{ns} , φ_{ns} in case of a 2-degree-of-freedom oscillating system having the damping factors larger than 0.1, which is excited by a rotating unbalanced mass $m = 0.0024 \text{ kg s}^2/\text{cm}$ corresponding to the critical condition under which the rotor can exceed barely resonance

6. Conclusions

By investigating theoretically the transient characteristics of a rotor under the critical condition deciding whether it can pass through resonance or not, the following results are introduced :

(1) It is found that the transient values of the amplitude of displacement, the phase lag of the displacement of oscillation to the centrifugal force, and of the angular velocity of rotation, are approximately given by the values capable of estimating when the dynamic load torque becomes maximum.

(2) Thus the maximum value of the transient dynamic load torque at resonance, generated by oscillations under the critical condition when a rotor has a single-degree-of-freedom oscillating system is given by the following expression :

$$T_{Dmax} = \frac{m e^2}{2 k_0} \mu_M \omega_x^2 \rho_{max}$$

Furthermore, using the value of T_{Dmax} , the estimating expression for passage through resonance is given as

$$K = \pi \mu_I \mu_M q \rho_{max} < 1.0$$

In this case, if the rotor has a circular oscillating system moving in a plane, the twofold values of K and T_{Dmax} are used for estimating passage through resonance.

(3) Combining the result introduced in case of single degree of freedom with the stationary dynamic load torque, estimating expressions for passage through resonances in case of a rotor moving in a plane, possessing a two-degree-of-freedom uncoupled oscillation are proposed, which are divided into 2 parts owing to the frequency difference between two resonances.

If the above results are applied to a practical machine, it will be necessary for a rotor to consider other load torque, e.g. the static load torque due to a mechanical friction.

In this case, however, the results obtained in this paper shall be applied to the machine by using the value of difference between the driving torque and the static load torque instead of the driving torque.

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