539.383

Some Elastic Contact Problems of an Elliptical Punch*

By Toshikazu SHIBUYA**

The present paper deals with elastic contact problems of a half-space loaded by a rigid cylindrical punch of elliptical cross-section in the cases of (a) indentation, (b) bending and (c) torsion. In each case, the contact region of punch is mapped onto the unit circle. In the image plane, it is assumed that the distribution of contact stress on the stamp is of the same form as that for a rigid circular stamp, and the surface displacements of the half-space are obtained. It is shown that the results are the solutions of the elastic contact problems of the half-space loaded by a rigid elliptical punch.

1. Introduction

Contact problems concerning an elastic half-space loaded by a rigid circular punch, have been investigated by some authors in the cases of indentation⁽¹⁾, bending⁽²⁾ and torsion⁽³⁾⁽⁴⁾. On the other hand, there are few investigations about the contact problem for a rigid elliptical punch. Galin⁽⁵⁾ and Green and Sneddon⁽⁶⁾ analyzed the indentation problem using elliptical co-ordinates. Lur'e⁽⁷⁾ treated the same problem by assuming the contact pressure. Recently, we⁽⁸⁾ measured experimentally the surface displacements of the elastic body, when the body was indented by a rectangular and by a circular punch. It was found that the iso-displacement curves produced on the free surface by a rectangular punch approximately agree with those by a circular punch when the free surfaces are related through a conformal mapping. So, we speculate that a similar result holds for the distribution of contact pressure on the punch.

In the present paper, the same method is applied to contact problems for a half-space loaded by the elliptical punch, in which the conformal mapping function is represented in the simplest form. In each case of (a) indentation, (b) bending and (c) torsion, we assume that the distribution of contact stress on the elliptical punch is given in the same form as that for the circular punch in the image plane, and obtain the surface displacements. It is shown that the results are the solutions of the elastic contact problems of the half-space loaded by the elliptical punch.

2. Stress functions

If (u, v, w) denotes the displacement vector in the Cartesian co-ordinates (x, y, z) , the general solutions of the elastic equations without body forces can be

represented using Boussinesq's stress functions φ_0 , φ_3 and λ_3 as follows:

$$\left. \begin{aligned} u &= \frac{\partial \varphi_0}{\partial x} + z \frac{\partial \varphi_3}{\partial x} + \frac{\partial \lambda_3}{\partial y} \\ v &= \frac{\partial \varphi_0}{\partial y} + z \frac{\partial \varphi_3}{\partial y} - \frac{\partial \lambda_3}{\partial x} \\ w &= \frac{\partial \varphi_0}{\partial z} + z \frac{\partial \varphi_3}{\partial z} - (3-4\nu)\varphi_3 \end{aligned} \right\} \dots\dots\dots (1)$$

where

$$\begin{aligned} \nabla^2 \varphi_0 &= \nabla^2 \varphi_3 = \nabla^2 \lambda_3 = 0 \\ \nabla^2 &= \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \end{aligned}$$

and ν is Poisson's ratio.

The stresses corresponding to Eq.(1) are

$$\left. \begin{aligned} \frac{\sigma_x}{2G} &= \frac{\partial^2 \varphi_0}{\partial x^2} + z \frac{\partial^2 \varphi_3}{\partial x^2} - 2\nu \frac{\partial \varphi_3}{\partial x} + \frac{\partial^2 \lambda_3}{\partial x \partial y} \\ \frac{\sigma_y}{2G} &= \frac{\partial^2 \varphi_0}{\partial y^2} + z \frac{\partial^2 \varphi_3}{\partial y^2} - 2\nu \frac{\partial \varphi_3}{\partial y} - \frac{\partial^2 \lambda_3}{\partial x \partial y} \\ \frac{\sigma_z}{2G} &= \frac{\partial^2 \varphi_0}{\partial z^2} + z \frac{\partial^2 \varphi_3}{\partial z^2} - 2(1-\nu) \frac{\partial \varphi_3}{\partial z} \\ \frac{\tau_{xy}}{2G} &= \frac{\partial^2 \varphi_0}{\partial x \partial y} + z \frac{\partial^2 \varphi_3}{\partial x \partial y} + \frac{1}{2} \left(\frac{\partial^2 \lambda_3}{\partial y^2} - \frac{\partial^2 \lambda_3}{\partial x^2} \right) \\ \frac{\tau_{xz}}{2G} &= \frac{\partial^2 \varphi_0}{\partial x \partial z} + z \frac{\partial^2 \varphi_3}{\partial x \partial z} - (1-2\nu) \frac{\partial \varphi_3}{\partial x} + \frac{1}{2} \frac{\partial^2 \lambda_3}{\partial y \partial z} \\ \frac{\tau_{yz}}{2G} &= \frac{\partial^2 \varphi_0}{\partial y \partial z} + z \frac{\partial^2 \varphi_3}{\partial y \partial z} - (1-2\nu) \frac{\partial \varphi_3}{\partial y} - \frac{1}{2} \frac{\partial^2 \lambda_3}{\partial x \partial z} \end{aligned} \right\} \dots\dots\dots (2)$$

where G is the shear modulus.

3. Indentation by the elliptical punch

We assume that a rigid cylindrical punch of elliptical cross-section is pressed to a depth ε_0 on the surface of the half-space ($z \geq 0$) and the interface between the punch and the half-space is frictionless. Then, the boundary conditions are

$$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \\ (b) \quad (w)_{z=0} &= \varepsilon_0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ (c) \quad (\tau_{xz})_{z=0} &= (\tau_{yz})_{z=0} = 0 \\ (d) \quad \text{All components of displacement} & \text{ and stress vanish at infinity.} \end{aligned} \right\} \dots\dots\dots (3)$$

Considering that $(w)_{z=0}$ is even in x and y , the stress functions may be expressed as follows:

* Received 10th April, 1975.

** Associate Professor, Department of Engineering, Tokyo Institute of Technology, Tokyo.

$$\left. \begin{aligned} 2G\varphi_0 &= -(1-2\nu) \int_0^\infty \int_0^\infty \frac{f(\lambda, \xi)}{\sqrt{\lambda^2 + \xi^2}} \cos \lambda x \\ &\quad \times \cos \xi y \exp(-x\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \\ 2G\varphi_3 &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \cos \xi y \\ &\quad \times \exp(-x\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \end{aligned} \right\} \dots (4)$$

where $f(\lambda, \xi)$ is an arbitrary function of λ and ξ . Eq.(4) satisfies the boundary conditions (c) and (d) of Eq.(3). Using the boundary conditions (a) and (b) of Eq.(3), we find that the function $f(\lambda, \xi)$ satisfies the dual integral equations

$$\left. \begin{aligned} -\frac{G(u)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \cos \xi y d\lambda d\xi \\ &= -\frac{G\varepsilon_0}{1-\nu}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \cos \lambda x \\ &\quad \times \cos \xi y d\lambda d\xi = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \end{aligned} \right\} \dots (5)$$

Now, we represent the mapping function $W(\zeta)$ by which the contact region of the elliptical punch is mapped conformally onto the unit circle of ζ -plane by

$$x+iy=W(\zeta)=R(\zeta+m\bar{\zeta}), \quad (0 \leq m < 1) \dots (6)$$

where $\bar{\zeta}$ is a conjugate value of ζ and

$$\left. \begin{aligned} x &= R\rho(1+m) \cos \theta, \quad y = R\rho(1-m) \sin \theta \\ a &= R(1+m), \quad b = R(1-m) \end{aligned} \right\} \dots (7)$$

We can extend the mapping function to the exterior of the unit circle. The cases of $\rho < 1$, $\rho = 1$ and $\rho > 1$ denote respectively the interior, the boundary and the exterior regions of the elliptical punch.

The present problem is analogous to the indentation problem of a rigid circular punch of radius 1. Then, we consider the case that $(\sigma_z)_{z=0}$ is given, on the mapping plane, by

$$(\sigma_z)_{z=0} = \frac{A}{\sqrt{1-\rho^2}} H(1-\rho) \dots (8)$$

where A is a constant $H(\rho)$ is a step function.

Performing inverse-Fourier transform of the second equation of Eq.(5) and Eq.(8), we obtain

$$\sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{A}{\sqrt{1-\rho^2}} \cos \lambda x \cos \xi y dx dy \dots (9)$$

Changing the variables x and y to ρ and θ , and putting

$$R(1+m)\lambda = p \sin \phi, \quad R(1-m)\xi = p \cos \phi \dots (10)$$

we get

$$\begin{aligned} \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) &= \frac{A}{\pi^2} \int_0^1 \int_0^{2\pi} \frac{1}{\sqrt{1-\rho^2}} \cos \{p\rho \cos \theta \sin \phi\} \cos \{p\rho \sin \theta \cos \phi\} \left\{ \frac{\partial x}{\partial \rho} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \rho} \right\} d\rho d\theta \\ &= \frac{A}{2\pi^2} R^2 (1-m^2) \int_0^1 \frac{\rho d\rho}{\sqrt{1-\rho^2}} \int_0^{2\pi} [\cos \{p\rho \sin(\theta+\phi)\} + \cos \{p\rho \sin(\theta-\phi)\}] d\theta \dots (11) \end{aligned}$$

Using formulae⁽⁹⁾

$$\int_0^{2\pi} \cos \{p\rho \sin(\theta+\phi)\} d\theta = 2\pi J_0(p\rho), \quad \int_0^1 \frac{\rho J_0(p\rho)}{\sqrt{1-\rho^2}} d\rho = \frac{\sin p}{p} \dots (12) \quad (13)$$

(a) The elliptical punch on the half-space.

(b) The surface plane of the half-space.

(c) The mapping plane.

Fig.1. Geometry of the problems.

we get $f(\lambda, \xi)$ as

$$f(\lambda, \xi) = \frac{2A}{\pi} R^3(1-m^2)(1-m) \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p}{p^2} \dots (14)$$

where $k^2 = 1 - \{(1-m)^2/(1+m)^2\}$, $p = R\sqrt{(1+m)^2 \lambda^2 + (1-m)^2 \xi^2}$, $\phi = \tan^{-1}\{(1+m)\lambda/(1-m)\xi\}$.

Substituting Eq.(14) into $(w)_{z=0}$, we obtain

$$\frac{G(w)_{z=0}}{1-\nu} = -\frac{2}{\pi} AR^3(1-m^2)(1-m) \int_0^\infty \int_0^\infty \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p}{p^2} \cos \lambda x \cos \xi y d\lambda d\xi \dots (15)$$

Changing the variables λ and ξ to p and ϕ , we obtain

$$\begin{aligned} \frac{G(w)_{z=0}}{1-\nu} &= \frac{2}{\pi} AR^3(1-m^2)(1-m) \int_0^{\pi/2} \int_0^\infty \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p}{p^2} \cos \{p\rho \sin \phi \cos \theta\} \\ &\times \cos \{p\rho \cos \phi \sin \theta\} \left\{ \frac{\partial \lambda}{\partial p} \frac{\partial \xi}{\partial \phi} - \frac{\partial \lambda}{\partial \phi} \frac{\partial \xi}{\partial p} \right\} d\phi dp = -\frac{AR(1-m)}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} \\ &\times \int_0^\infty \frac{\sin p}{p} [\cos \{p\rho \sin(\phi+\theta)\} + \cos \{p\rho \sin(\phi-\theta)\}] dp \\ &= -\frac{AR(1-m)}{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} [H(1-|\rho \sin(\phi+\theta)|) + H(1-|\rho \sin(\phi-\theta)|)] \dots (16) \end{aligned}$$

On the contact region of the punch ($\rho \leq 1$), each step function in Eq.(16) is always 1 for arbitrary values of ρ and θ , and $(w)_{z=0}$ is

$$(w)_{z=0} = -\frac{1-\nu}{G} AR(1-m)K(k) \quad (\rho \leq 1) \dots (17)$$

Then, $f(\lambda, \xi)$ given by Eq.(14) satisfies completely all boundary conditions, where $K(k)$ is a complete elliptic integral of first kind and

$$A = -\frac{1}{R(1-m)K(k)} \frac{G\varepsilon_0}{1-\nu}$$

The resultant force P is

$$\begin{aligned} P &= -\int_{\rho < 1} (\sigma_z)_{z=0} dx dy \\ &= \frac{2\pi}{K(k)} \frac{(1+m)RG\varepsilon_0}{1-\nu} \dots (18) \end{aligned}$$

As the result, it is known that the solution of the dual integral equations(5) is

$$f(\lambda, \xi) = -\frac{2}{\pi} R^2(1-m^2) \frac{1}{K(k)\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p}{p^2} \frac{G\varepsilon_0}{1-\nu} \dots (19)$$

In the special case of the circular punch ($m=0$), the results are

$$\left. \begin{aligned} a=b=R, \quad k=0, \quad K(k)=\pi/2, \quad A &= -\frac{2G\varepsilon_0}{\pi R(1-\nu)} \\ f(\lambda, \xi) &= -\frac{4}{\pi^2} \frac{R^2 G\varepsilon_0}{1-\nu} \frac{\sin p}{p^2}, \quad P = \frac{4RG\varepsilon_0}{1-\nu} \\ (\sigma_z)_{z=0} &= -\frac{2G\varepsilon_0}{\pi R(1-\nu)} \frac{1}{\sqrt{1-\rho^2}} H(1-\rho) \\ (w)_{z=0} &= \begin{cases} \varepsilon_0, & (\rho \leq 1) \\ \frac{2}{\pi} \varepsilon_0 F\left(\sin^{-1} \frac{1}{\rho}, 0\right) = \frac{2\varepsilon_0}{\pi} \sin^{-1} \frac{1}{\rho}, & (\rho > 1) \end{cases} \end{aligned} \right\} \dots (20)$$

and agree with those in Ref.[1].

4. Bending by the elliptical punch

We now assume that the elliptical punch on the surface of the half-space is subjected to bending around the x -axis, and is inclined to a small angle α . Then, the boundary conditions are

$$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \\ (b) \quad (w)_{z=0} &= \alpha x, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ (c) \quad (\tau_{xz})_{z=0} &= (\tau_{yz})_{z=0} = 0 \\ (d) \quad \text{All components of displacement} & \text{ and stress vanish at infinity.} \end{aligned} \right\} \dots (21)$$

Considering $(w)_{z=0}$ is even in x and odd in y , the stress functions may be expressed

$$\left. \begin{aligned} -\frac{G(w)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \sin \xi y d\lambda d\xi = -\frac{G\alpha}{1-\nu} y, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \cos \lambda x \sin \xi y d\lambda d\xi = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \end{aligned} \right\} \dots (23)$$

Similarly to the previous section, let us assume that the problem is one of a circular punch on the mapping plane. Then, we consider the case that $(\sigma_z)_{z=0}$ on the contact region is given by

$$(\sigma_z)_{z=0} = \frac{A\rho \sin \theta}{\sqrt{1-\rho^2}} H(1-\rho) \dots (24)$$

Performing inverse-Fourier transform of the second equation of Eq.(23) and Eq.(24), we get

$$\sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) = \frac{AR^2(1-m^2)}{2\pi^2} \int_0^1 \frac{\rho^2 d\rho}{\sqrt{1-\rho^2}} \int_0^{2\pi} \sin \theta [\sin \{p\rho \sin(\theta+\phi)\} + \sin \{p\rho \sin(\theta-\phi)\}] d\theta \dots (25)$$

Using the formulae⁽⁹⁾

$$\int_0^{2\pi} \sin \theta \sin \{p \rho \sin (\theta + \phi)\} d\theta = 2\pi \cos \phi J_1(p\rho) \quad \int_0^1 \frac{\rho^2 J_1(p\rho)}{\sqrt{1-\rho^2}} d\rho = \frac{\sin p - p \cos p}{p^2} \quad (26) \quad (27)$$

we obtain

$$f(\lambda, \xi) = \frac{2A}{\pi} R^3(1-m^2)(1-m) \frac{\cos \phi}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p - p \cos p}{p^2} \quad (28)$$

Substituting Eq.(28) into $(w)_{z=0}$, we get

$$\begin{aligned} \frac{G(w)_{z=0}}{1-\nu} = & -\frac{AR(1-m)}{\pi} \int_0^{\pi/2} \frac{\cos \phi d\phi}{\sqrt{1-k^2 \sin^2 \phi}} \int_0^\infty \frac{\sin p - p \cos p}{p^2} [\sin \{p \rho \sin (\phi + \theta)\} \\ & - \sin \{p \rho \sin (\phi - \theta)\}] dp = -\frac{AR(1-m)}{2} \int_0^{\pi/2} \frac{\cos \phi}{\sqrt{1-k^2 \sin^2 \phi}} \\ & \times [\rho \sin (\phi + \theta) H(1-|\rho \sin (\phi + \theta)|) - \rho \sin (\phi - \theta) H(1-|\rho \sin (\phi - \theta)|)] d\phi \quad (29) \end{aligned}$$

On the contact region of the punch ($\rho \leq 1$), each step function in Eq.(29) is always 1 for arbitrary values of ρ and θ , and $(w)_{z=0}$ is

$$(w)_{z=0} = -\frac{1-\nu}{G} AR(1-m) \rho \sin \theta \int_0^{\pi/2} \frac{\cos^2 \phi}{\sqrt{1-k^2 \sin^2 \phi}} d\phi = -\frac{1-\nu}{G} Ay \frac{1}{k^2} [E(k) - (1-k^2)K(k)] \propto y \quad (\rho \leq 1) \quad (30)$$

Then, $f(\lambda, \xi)$ given by Eq.(28) satisfies completely the boundary conditions (21), where $E(k)$ is a complete elliptic integral of second kind, and

$$A = -\frac{k^2}{E(k) - (1-k^2)K(k)} \frac{G\alpha_z}{1-\nu}$$

The resultant moment M_z is

$$M_z = -\int_{\rho < 1} (\sigma_z)_{z=0} y dx dy = \frac{2\pi}{3} R^3(1-m^2)(1-m) \frac{k^2}{E(k) - (1-k^2)K(k)} \frac{G\alpha_z}{1-\nu} \quad (31)$$

As the result, it is known that the solution of the dual integral equation (23) is

$$f(\lambda, \xi) = -\frac{2R^3(1-m^2)(1-m)}{\pi} \frac{k^2}{E(k) - (1-k^2)K(k)} \frac{\cos \phi}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p - p \cos p}{p^2} \frac{G\alpha_z}{1-\nu} \quad (32)$$

In the special case of the circular punch, we get

$$\left. \begin{aligned} a=b=R, \quad k=0, \quad E(k)=K(k)=\frac{\pi}{2}, \quad A &= -\frac{4}{\pi} \frac{G\alpha_z}{1-\nu}, \quad M_z = \frac{8}{3} R^3 \frac{G\alpha_z}{1-\nu} \\ f(\lambda, \xi) &= -\frac{8}{\pi^2} R^3 \cos \phi \frac{\sin p - p \cos p}{p^2} \frac{G\alpha_z}{1-\nu}, \quad (\sigma_z)_{z=0} = -\frac{4}{\pi} \frac{G\alpha_z}{1-\nu} \frac{\rho \sin \theta}{\sqrt{1-\rho^2}} H(1-\rho) \\ (w)_{z=0} &= \begin{cases} \alpha_z y & (\rho \leq 1) \\ \alpha_z y \left[\frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{1}{\pi} \sin \left\{ 2 \sin^{-1} \left(\frac{1}{\rho} \right) \right\} \right] & (\rho > 1) \end{cases} \end{aligned} \right\} \quad (33)$$

Next, we consider the case that the elliptical punch is subjected to bending around y-axis, and is inclined to a small angle α_y . The boundary conditions are

$$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, \quad (b) \quad (w)_{z=0} = \alpha_y x, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \quad (c) \quad (\tau_{xz})_{z=0} = (\tau_{yz})_{z=0} = 0 \\ (d) \quad & \text{All components of displacement and stress vanish at infinity.} \end{aligned} \right\} \quad (34)$$

Considering that $(w)_{z=0}$ is odd in x and even in y, we express the stress functions as follows:

$$\left. \begin{aligned} 2G\varphi_0 &= -(1-2\nu) \int_0^\infty \int_0^\infty \frac{f(\lambda, \xi)}{\sqrt{\lambda^2 + \xi^2}} \sin \lambda x \cos \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \\ 2G\varphi_3 &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \sin \lambda x \cos \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \end{aligned} \right\} \quad (35)$$

Eq.(35) satisfies the boundary conditions (c) and (d) of Eq.(34) and the problem is reduced to finding the solution of dual integral equations

$$\left. \begin{aligned} -\frac{G(w)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \sin \lambda x \cos \xi y d\lambda d\xi = -\frac{G\alpha_y}{1-\nu} x, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \sin \lambda x \cos \xi y d\lambda d\xi = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \end{aligned} \right\} \quad (36)$$

Similarly, we consider the case that $(\sigma_z)_{z=0}$ is given by

$$(\sigma_z)_{z=0} = \frac{A\rho \cos \theta}{\sqrt{1-\rho^2}} H(1-\rho) \quad (37)$$

Performing inverse-Fourier transform of the second equation of Eq.(36) and Eq.(37), and using the formula

$$\int_0^{2\pi} \cos \theta \sin(p\rho \sin(\theta + \phi)) d\theta = 2\pi \sin \phi J_1(p\rho) \quad (38)$$

we obtain

$$f(\lambda, \xi) = \frac{2A}{\pi} R^3(1-m^2)(1-m) \frac{\sin \phi}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p - p \cos p}{p^3} \quad (39)$$

Substituting Eq.(39) into $(w)_{z=0}$, we get

$$\frac{G(w)_{z=0}}{1-\nu} = -\frac{AR(1-m)}{2} \int_0^{\pi/2} \frac{\sin \phi}{\sqrt{1-k^2 \sin^2 \phi}} [\rho \sin(\phi + \theta) H(1 - |\rho \sin(\phi + \theta)|) + \rho \sin(\phi - \theta) H(1 - |\rho \sin(\phi - \theta)|)] d\phi \quad (40)$$

On the contact region, we get

$$(w)_{z=0} = -\frac{1-\nu}{G} AR(1-m) \rho \cos \theta \int_0^{\pi/2} \frac{\sin^2 \phi}{\sqrt{1-k^2 \sin^2 \phi}} d\phi = -\frac{1-\nu}{G} \frac{1-m}{1+m} Ax \frac{1}{k^2} [K(k) - E(k)] \propto x \quad (\rho \leq 1) \quad (41)$$

Then, $f(\lambda, \xi)$ given by Eq.(39) satisfies completely the boundary conditions, where

$$A = -\frac{1+m}{1-m} \frac{k^2}{K(k) - E(k)} \frac{G\alpha_y}{1-\nu}$$

The resultant moment M_y is

$$M_y = -\int_{\rho < 1} (\sigma_z)_{z=0} x dx dy = \frac{2\pi}{3} R^3(1+m)^3 \frac{k^2}{K(k) - E(k)} \frac{G\alpha_y}{1-\nu} \quad (42)$$

As the result, the solution of the dual integral equations (36) is given by

$$f(\lambda, \xi) = -\frac{2}{\pi} R^3(1-m^2)(1+m) \frac{k^2}{K(k) - E(k)} \frac{\sin \phi}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p - p \cos p}{p^3} \frac{G\alpha_y}{1-\nu} \quad (43)$$

In the special case of the circular punch, we get

$$\left. \begin{aligned} a=b=R, \quad A &= -\frac{4}{\pi} \frac{G\alpha_y}{1-\nu}, \quad M_y = \frac{8}{3} R^3 \frac{G\alpha_y}{1-\nu}, \quad f(\lambda, \xi) = -\frac{8}{\pi^2} R^3 \sin \phi \frac{\sin p - p \cos p}{p^3} \frac{G\alpha_y}{1-\nu} \\ (\sigma_z)_{z=0} &= -\frac{4}{\pi} \frac{G\alpha_y}{1-\nu} \frac{\rho \cos \theta}{\sqrt{1-\rho^2}} H(1-\rho) \\ (w)_{z=0} &= \begin{cases} \alpha_y x & (1 \geq \rho) \\ \alpha_y x \left[\frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{1}{\pi} \sin \left(2 \sin^{-1} \left(\frac{1}{\rho} \right) \right) \right] & (1 < \rho) \end{cases} \end{aligned} \right\} \quad (44)$$

5. Torsion by the elliptical punch

We consider the case that the elliptical region on the surface of the half-space is subjected to torsion and the rotation around z-axis, $(\omega_z)_{z=0}$, has a constant value α_z . The boundary conditions are

$$\left. \begin{aligned} (a) \quad (\tau_{xz})_{z=0} &= (\tau_{yz})_{z=0} = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \\ (b) \quad (\omega_z)_{z=0} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_{z=0} = \alpha_z, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \\ (c) \quad (\sigma_z)_{z=0} &= 0 \\ (d) \quad \text{All components of displacement and stress vanish at infinity.} \end{aligned} \right\} \quad (45)$$

Considering that $(u)_{z=0}$ is odd in y and $(v)_{z=0}$ odd in x, we express the stress functions as follows:

$$2G\lambda_3 = \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \cos \xi y \exp(-x\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \quad (46)$$

Eq.(46) satisfies the boundary conditions (c) and (d) of Eq.(45). From Eq.(46), we get

$$f(\lambda, \xi) = -\frac{4A}{\pi} R^4(1-m^2)^2 \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} \frac{\sin p - p \cos p}{p^4} \quad (49)$$

Substituting Eq.(49) into $(\tau_{xz})_{z=0}$, we obtain

$$(\tau_{xz})_{z=0} = -\frac{A}{4\pi} \frac{1+m}{1-m} \int_0^\infty \frac{\sin p - p \cos p}{p} dp \int_0^{2\pi} \cos \phi [\sin(p\rho \sin(\phi + \theta)) - \sin(p\rho \sin(\phi - \theta))] d\phi \quad (50)$$

Using Eq.(38) and the formula

$$\left. \begin{aligned} 2G(u)_{z=0} &= -\int_0^\infty \int_0^\infty f(\lambda, \xi) \xi \cos \lambda x \sin \xi y d\lambda d\xi \\ 2G(v)_{z=0} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \lambda \sin \lambda x \cos \xi y d\lambda d\xi \\ (\tau_{xz})_{z=0} &= \frac{1}{2} \int_0^\infty \int_0^\infty f(\lambda, \xi) \xi \sqrt{\lambda^2 + \xi^2} \cos \lambda x \sin \xi y d\lambda d\xi \\ (\tau_{yz})_{z=0} &= -\frac{1}{2} \int_0^\infty \int_0^\infty f(\lambda, \xi) \lambda \sqrt{\lambda^2 + \xi^2} \sin \lambda x \cos \xi y d\lambda d\xi \\ 2G(\omega_z)_{z=0} &= \frac{1}{2} \int_0^\infty \int_0^\infty f(\lambda, \xi) (\lambda^2 + \xi^2) \cos \lambda x \cos \xi y d\lambda d\xi \end{aligned} \right\} \quad (47)$$

On the mapping plane, the problem is one of the torsion by the circular punch of radius 1. Then, we consider the case that $(\tau_{yz})_{z=0}$ is given by

$$(\tau_{yz})_{z=0} = \frac{A\rho \cos \theta}{\sqrt{1-\rho^2}} H(1-\rho) \quad (48)$$

Performing inverse-Fourier transform of the fourth equation of Eq.(47) and Eq.(48), we get

$$\int_0^\infty \frac{\sin p - p \cos p}{p} J_1(p\rho) dp = \frac{\rho}{\sqrt{1-\rho^2}} H(1-\rho) \quad (51)$$

we obtain $(\tau_{xz})_{z=0}$ as

$$(\tau_{xz})_{z=0} = -\frac{1+m}{1-m} \frac{A\rho \sin \theta}{\sqrt{1-\rho^2}} H(1-\rho) \quad (52)$$

Then, if $f(\lambda, \xi)$ is given by Eq.(49), $(\tau_{xz})_{z=0}$ and $(\tau_{yz})_{z=0}$ satisfy their boundary conditions at the same time. Substituting Eq.(49) into $(\omega_z)_{z=0}$, we get

$$\begin{aligned} 2G(\omega_z)_{z=0} = & -\frac{A}{\pi} \frac{1+m}{1-m} \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi} d\phi \int_0^\infty \left(\frac{\sin p}{p} - \cos p \right) [\cos(p\rho \sin(\phi+\theta)) \\ & + \cos(p\rho \sin(\phi-\theta))] d\phi = -\frac{A}{2} \frac{1+m}{1-m} \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi} [H(1-|\rho \sin(\phi+\theta)|) \\ & + H(1-|\rho \sin(\phi-\theta)|) - \delta(1+|\rho \sin(\phi+\theta)|) - \delta(1-|\rho \sin(\phi+\theta)|) \\ & - \delta(1+|\rho \sin(\phi-\theta)|) - \delta(1-|\rho \sin(\phi-\theta)|)] d\phi \quad (53) \end{aligned}$$

where $\delta(\rho)$ is a Dirac delta function.

In the elliptical region ($\rho < 1$), the step and the delta functions in Eq.(53) are respectively 1 and zero for arbitrary values of ρ and θ . Then, $(\omega_z)_{z=0}$ in the region is

$$(\omega_z)_{z=0} = -\frac{A}{2G} \frac{1+m}{1-m} E(k) \quad (\rho < 1) \quad (54)$$

where

$$A = -\frac{1-m}{1+m} \frac{1}{E(k)} 2G\alpha_{xy}$$

So, $f(\lambda, \xi)$ given by Eq.(49) satisfies completely all boundary conditions. The resultant torque T is given by

$$T = -\int_{\rho < 1} \{y(\tau_{xz})_{z=0} - x(\tau_{yz})_{z=0}\} dx dy = \frac{8\pi}{3} R^3 (1-m^2)(1-m) \frac{1}{E(k)} G\alpha_{xy} \quad (55)$$

The displacements $(u)_{z=0}$ and $(v)_{z=0}$ are

$$\left. \begin{aligned} (u)_{z=0} = & \frac{AR(1+m)}{2G} \int_0^{\pi/2} \frac{\cos \phi}{\sqrt{1-k^2 \sin^2 \phi}} [\rho \sin(\phi+\theta) H(1-|\rho \sin(\phi+\theta)|) \\ & - \rho \sin(\phi-\theta) H(1-|\rho \sin(\phi-\theta)|)] d\phi \\ (v)_{z=0} = & -\frac{AR(1-m)}{2G} \int_0^{\pi/2} \frac{\sin \phi}{\sqrt{1-k^2 \sin^2 \phi}} [\rho \sin(\phi+\theta) H(1-|\rho \sin(\phi+\theta)|) \\ & + \rho \sin(\phi-\theta) H(1-|\rho \sin(\phi-\theta)|)] d\phi \end{aligned} \right\} \quad (56)$$

In the elliptical region ($\rho \leq 1$), they are

$$(u)_{z=0} = -2\alpha_{xy} y \frac{E(k) - (1-k^2)K(k)}{k^2 E(k)}, \quad (v)_{z=0} = 2\alpha_{xy} x \frac{(1-m)^2}{(1+m)^2} \frac{K(k) - E(k)}{k^2 E(k)} \quad (\rho \leq 1) \quad (57)$$

In the special case of the circular punch, we get

$$\left. \begin{aligned} a=b=R, \quad k=0, \quad A = -\frac{4}{\pi} G\alpha_{xy}, \quad T = -\frac{16}{3} R^3 G\alpha_{xy}, \quad f(\lambda, \xi) = \frac{16}{\pi^2} R^4 G\alpha_{xy} \frac{\sin p - p \cos p}{p^4} \\ \frac{(\tau_{xz})_{z=0}}{\sin \theta} = -\frac{(\tau_{yz})_{z=0}}{\cos \theta} = \frac{4}{\pi} G\alpha_{xy} \frac{\rho}{\sqrt{1-\rho^2}} H(1-\rho) \\ (\omega_z)_{z=0} = \begin{cases} \alpha_{xy}, & (\rho < 1) \\ \alpha_{xy} \left[\frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{1}{\pi} \frac{1}{\sqrt{\rho^2-1}} \right], & (\rho > 1) \end{cases} \\ (u)_{z=0} = \begin{cases} -\alpha_{xy} y, & (\rho \leq 1) \\ -\alpha_{xy} y \left[\frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{1}{\pi} \sin \left(2 \sin^{-1} \left(\frac{1}{\rho} \right) \right) \right], & (\rho \geq 1) \end{cases} \\ (v)_{z=0} = \begin{cases} \alpha_{xy} x, & (\rho \leq 1) \\ \alpha_{xy} x \left[\frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{1}{\pi} \sin \left(2 \sin^{-1} \left(\frac{1}{\rho} \right) \right) \right], & (\rho \geq 1) \end{cases} \end{aligned} \right\} \quad (58)$$

As shown in Eq.(57), although it is $(u)_{z=0} \propto y$ and $(v)_{z=0} \propto x$ in the contact region of the elliptical punch, the absolute values of their proportional coefficients are different from each other. So, the solution given by Eq.(49) can be applied to the case that the rotation around z -axis is constant in the contact region, but not to the case that the elliptical punch is fixed completely on the surface of the half-space. Both cases agree only in the case of the circular punch.

References

- [1] Harding, J.W. and Sneddon, I.N., Proc. Camb. Phil. Soc., 41(1945), 16.
- [2] Muki, R., Progress in Solid Mech., Vol.1, Chapt.8(1960), North-Holland.
- [3] Reissner, E. and Sagoci, H.F., J. Appl. Phys., 15(1944), 652.
- [4] Sneddon, I.N., J. Appl. Phys., 18(1947), 130.
- [5] Galin, G.A., P.M.M., 12(1948).
- [6] Green, A.E. and Sneddon, I.N., Proc. Camb. Phil. Soc., 46(1950), 159.
- [7] Lur'e, A.I., Three-Dimensional Problems of the Theory of Elasticity, Interscience Pub. (1964).
- [8] Hara, T. et al., Preprint of JSME., No.760-1(1976-4), 147.
- [9] Moriguchi, S. et al., Tables of Mathematical Formulae(in Japanese), (1963), Iwanami Shoten.