539.4.013:539.38

Some Mixed Boundary Value Problems for an Infinite Solid Containing a Flat Elliptical Crack*

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The present paper deals with mixed boundary value problems for an infinite solid containing a flat elliptical crack under internal pressure and bending. In each case, the author maps conformally the crack region onto the unit circle. In the image plane, he shows that the deformation of the crack surface is of the same form as that for a circular crack in the use of the variables of the plane, and obtains a rigorous solution of the problem in the closed form.

1. Introduction

Concerning fracture mechanics, there are many investigations in the crack problem. Generally, the problem is reduced to finding the solution of dual integral equations. In the case of an infinite solid containing a circular crack, a general solution of the dual integral equations is given by Busbridge⁽¹⁾. Therefore, analytic solutions are obtained in the cases of internal pressure (or tension)⁽²⁾, bending⁽³⁾, torsion⁽⁴⁾ and shear⁽⁵⁾⁽⁶⁾. On the other hand, in the case of an infinite solid containing an elliptical crack, Green-Sneddon⁽⁷⁾ analyzed, at first, the problem under internal pressure by using the elliptical co-ordinates. Since then Eshelby⁽⁸⁾, Kassir-Sih⁽⁹⁾ and Smith-Sorensen⁽¹⁰⁾ considered the problem under shear by the same method. Although the crack surface is represented easily with use of the elliptical co-ordinates, it is difficult to extend such problems to the case of a thick plate. Moreover, considering that an arbitrary shaped crack can be mapped onto a unit circle by conformal mapping, it is important to analyze such problems with use of the Cartesian co-ordinates.

Recently, the author⁽¹¹⁾ analyzed the elliptical punch problems under indentation, bending and torsion with use of conformal mapping and Cartesian co-ordinates. In the present paper, the author has applied the similar method to the elliptical crack problems, in which the conformal mapping function is represented in the simplest form. The solution under internal pressure or bending is obtained in closed form.

2. Stress functions

If (u, v, w) denote the displacement components in the Cartesian co-ordinates (x, y, z) , a general solution of the elastic equations without body forces can be re-

presented by Boussinesq's stress functions φ_0 , φ_1 and λ_1 as follows:

$$\left. \begin{aligned} u &= \frac{\partial \varphi_0}{\partial x} + z \frac{\partial \varphi_1}{\partial x} + \frac{\partial \lambda_1}{\partial y} \\ v &= \frac{\partial \varphi_0}{\partial y} + z \frac{\partial \varphi_1}{\partial y} - \frac{\partial \lambda_1}{\partial x} \\ w &= \frac{\partial \varphi_0}{\partial z} + z \frac{\partial \varphi_1}{\partial z} - (3-4\nu)\varphi_1 \end{aligned} \right\} \dots\dots\dots (1)$$

where $\nabla^2 \varphi_0 = \nabla^2 \varphi_1 = \nabla^2 \lambda_1 = 0$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and ν is Poisson's ratio.

In the case without torsion, we may put $\lambda_1 = 0$. The stresses corresponding to φ_0 and φ_1 are

$$\left. \begin{aligned} \frac{\sigma_x}{2G} &= \frac{\partial^2 \varphi_0}{\partial x^2} + z \frac{\partial^2 \varphi_1}{\partial x^2} - 2\nu \frac{\partial \varphi_1}{\partial z} \\ \frac{\sigma_y}{2G} &= \frac{\partial^2 \varphi_0}{\partial y^2} + z \frac{\partial^2 \varphi_1}{\partial y^2} - 2\nu \frac{\partial \varphi_1}{\partial z} \\ \frac{\sigma_z}{2G} &= \frac{\partial^2 \varphi_0}{\partial z^2} + z \frac{\partial^2 \varphi_1}{\partial z^2} - 2(1-\nu) \frac{\partial \varphi_1}{\partial z} \\ \frac{\tau_{xy}}{2G} &= \frac{\partial^2 \varphi_0}{\partial x \partial y} + z \frac{\partial^2 \varphi_1}{\partial x \partial y} \\ \frac{\tau_{xz}}{2G} &= \frac{\partial^2 \varphi_0}{\partial x \partial z} + z \frac{\partial^2 \varphi_1}{\partial x \partial z} - (1-2\nu) \frac{\partial \varphi_1}{\partial x} \\ \frac{\tau_{yz}}{2G} &= \frac{\partial^2 \varphi_0}{\partial y \partial z} + z \frac{\partial^2 \varphi_1}{\partial y \partial z} - (1-2\nu) \frac{\partial \varphi_1}{\partial y} \end{aligned} \right\} \dots\dots\dots (2)$$

where G is the shear modulus.

3. The elliptical crack under internal pressure

Suppose that an elliptical crack under internal pressure p_0 is located at $z=0$ in an elastic solid, as shown in Fig.1. Since the problem is symmetrical with respect to $z=0$, it is sufficient to consider only the solution for the upper half-space $z \geq 0$. Then, the boundary conditions of the problem may be written as

- $$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= -p_0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \\ (b) \quad (w)_{z=0} &= 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \\ (c) \quad (\tau_{xz})_{z=0} &= (\tau_{yz})_{z=0} = 0 \\ (d) \quad \text{All components of displacement} & \text{ and stress vanish at infinity.} \end{aligned} \right\} \dots\dots\dots (3)$$

In order to satisfy the boundary conditions (c) and (d) of Eq.(3), and considering that $(w)_{z=0}$ is even in x and y , we express the stress functions as follows:

* Received 15th October, 1975.

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$$\left. \begin{aligned} 2G\varphi_0 &= -(1-2\nu) \int_0^\infty \int_0^\infty \frac{f(\lambda, \xi)}{\sqrt{\lambda^2 + \xi^2}} \cos \lambda x \cos \xi y \\ &\quad \times \exp(-x\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \\ 2G\varphi_z &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \cos \xi y \\ &\quad \times \exp(-x\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \end{aligned} \right\} \dots\dots\dots (4)$$

where $f(\lambda, \xi)$ is an arbitrary function of λ and ξ . Using the boundary conditions (a) and (b) of Eq.(3), we can find that the function $f(\lambda, \xi)$ satisfies the dual integral equations

$$\left. \begin{aligned} (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \cos \lambda x \cos \xi y \\ &\quad \times d\lambda d\xi = -p_0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \\ -\frac{G(w)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \cos \xi y \\ &\quad \times d\lambda d\xi = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \end{aligned} \right\} \dots\dots\dots (5)$$

Now, we represent the mapping function by which the crack surface is mapped onto the unit circle of the plane $\zeta = \rho e^{i\theta}$ by

$$x + iy = R(\zeta + m\bar{\zeta}) \quad (0 \leq m < 1) \quad \dots\dots\dots (6)$$

where $\bar{\zeta}$ is a conjugate value of ζ , and

$$\left. \begin{aligned} x &= R\rho(1+m)\cos\theta, \quad y = R\rho(1-m)\sin\theta \\ a &= R(1+m), \quad b = R(1-m) \end{aligned} \right\} \dots\dots\dots (7)$$

We can extend the mapping function to the exterior of the unit circle. Points of $\rho = \text{const.}$ correspond to points on

$$\frac{x^2}{R^2\rho^2(1+m)^2} + \frac{y^2}{R^2\rho^2(1-m)^2} = 1 \quad \dots\dots\dots (8)$$

The cases of $\rho < 1$, $\rho = 1$ and $\rho > 1$ denote respectively the interior, the boundary and the exterior regions of the elliptical crack.

About the mapping plane, we can say that the present problem is analogous to the case of a circular crack of unit radius under internal pressure. Then, we consider that $(w)_{z=0}$ is given, in the mapping plane, by

$$(w)_{z=0} = A\sqrt{1-\rho^2} H(1-\rho) \quad \dots\dots\dots (9)$$

where A is a constant and $H(\rho)$ is a step function.

Performing inverse-Fourier transform of the second equation of Eq.(5) and Eq.(9), we obtain

$$f(\lambda, \xi) = -\frac{GA}{1-\nu} \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \sqrt{1-\rho^2} \cos \lambda x \cos \xi y dx dy \quad \dots\dots\dots (10)$$

Changing the variables x and y to ρ and θ , and putting

$$R(1+m)\lambda = \rho \sin \phi, \quad R(1-m)\xi = \rho \cos \phi \quad \dots\dots\dots (11)$$

we get

condition of $(w)_{z=0}$ is satisfied. Substituting Eq.(15) into $(\sigma_z)_{z=0}$, we obtain

$$(\sigma_z)_{z=0} = -\frac{2}{\pi} \frac{GA}{1-\nu} R^2(1-m^2) \int_0^\infty \int_0^\infty \frac{\sin p - p \cos p}{p^3} \sqrt{\lambda^2 + \xi^2} \cos \lambda x \cos \xi y d\lambda d\xi \quad \dots\dots\dots (16)$$

Changing the variables λ and ξ to p and ϕ , we obtain $(\sigma_z)_{z=0}$ as follows. Additionally, we should notice that the Jacobian of Eq.(11) is negative.

$$\begin{aligned} (\sigma_z)_{z=0} &= \frac{2}{\pi} \frac{GA}{1-\nu} R^2(1-m^2) \int_0^{\pi/2} \int_0^\infty \frac{\sin p - p \cos p}{p^3} \frac{p}{R(1-m)} \sqrt{1-k^2 \sin^2 \phi} \\ &\quad \times \cos \{p\rho \sin \phi \cos \theta\} \cos \{p\rho \cos \phi \sin \theta\} \left\{ \frac{\partial \lambda}{\partial p} \frac{\partial \xi}{\partial \phi} - \frac{\partial \lambda}{\partial \phi} \frac{\partial \xi}{\partial p} \right\} dp d\phi \\ &= -\frac{GA}{2(1-\nu)R(1-m)} \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi} [H(1-|\rho \sin(\phi+\theta)|) + H(1-|\rho \sin(\phi-\theta)|) \\ &\quad - \delta(1-\rho \sin(\phi+\theta)) - \delta(1+\rho \sin(\phi+\theta)) - \delta(1-\rho \sin(\phi-\theta)) - \delta(1+\rho \sin(\phi-\theta))] d\phi \quad \dots\dots\dots (17) \end{aligned}$$

(a) Elliptical crack, $(y=0)$.

(b) The plane of $z=0$.

(c) The mapping plane.

Fig.1. Elliptical crack under internal pressure.

$$\begin{aligned} f(\lambda, \xi) &= -\frac{GA}{(1-\nu)\pi^2} \int_0^1 \int_0^{2\pi} \sqrt{1-\rho^2} \\ &\quad \times \cos \{p\rho \cos \theta \sin \phi\} \cos \{p\rho \sin \theta \cos \phi\} \\ &\quad \times \left\{ \frac{\partial x}{\partial \rho} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \rho} \right\} d\rho d\theta \\ &= -\frac{GAR^2(1-m^2)}{2\pi^2(1-\nu)} \int_0^1 \rho \sqrt{1-\rho^2} d\rho \\ &\quad \times \int_0^{2\pi} [\cos \{p\rho \sin(\theta+\phi)\} + \cos \{p\rho \sin(\theta-\phi)\}] d\theta \quad \dots\dots\dots (12) \end{aligned}$$

Using formulae

$$\int_0^{2\pi} \cos \{p\rho \sin(\theta+\phi)\} d\theta = 2\pi J_0(p\rho) \quad \dots\dots\dots (13)$$

$$\int_0^1 \rho \sqrt{1-\rho^2} J_0(p\rho) d\rho = \frac{\sin p - p \cos p}{p^3} \quad \dots\dots\dots (14)$$

we get $f(\lambda, \xi)$ as

$$f(\lambda, \xi) = -\frac{2}{\pi} \frac{GA}{1-\nu} R^2(1-m^2) \frac{\sin p - p \cos p}{p^3} \quad \dots\dots\dots (15)$$

When $f(\lambda, \xi)$ is given by Eq.(15), the boundary

where $\delta(\rho)$ is a Dirac delta function, and $k^2 = 1 - ((1-m)/(1+m))^2$.

On the crack surface ($\rho < 1$), each step function in Eq.(17) is always unity for arbitrary values of ρ and θ , and each delta function is always zero, and $(\sigma_z)_{z=0}$ is

$$(\sigma_z)_{z=0} = -\frac{GA}{(1-\nu)R(1-m)} E(k), \quad (\rho < 1) \quad (18)$$

Then, $f(\lambda, \xi)$ given by Eq.(15) satisfies completely all boundary conditions, where $E(k)$ is a complete elliptic integral of second kind, and

$$A = \frac{(1-\nu)R(1-m)}{G} \frac{1}{E(k)} p_0$$

As the result, we know that the solution of the dual integral equations(5) is

$$f(\lambda, \xi) = -\frac{2R^2(1-m)^2(1+m)}{\pi} \frac{\sin p - p \cos p}{E(k)} \frac{1}{p^3} p_0 \quad (19)$$

The stress $(\sigma_z)_{z=0}$, ($\rho > 1$) is

$$\left. \begin{aligned} (\sigma_z)_{z=0, y=0} &= -\frac{p_0}{E(k)} \left[E\left(\sin^{-1} \frac{1}{\rho}, k\right) - \frac{\sqrt{\rho^2 - k^2}}{\rho \sqrt{\rho^2 - 1}} \right], \quad (\rho > 1) \\ (\sigma_z)_{z=0, z=0} &= -\frac{p_0}{E(k)} \left[E(k) - E\left(\cos^{-1} \frac{1}{\rho}, k\right) - \frac{\sqrt{k^2 + k'^2 \rho^2}}{\rho \sqrt{\rho^2 - 1}} \right], \quad (\rho > 1) \end{aligned} \right\} (20)$$

on the x-axis, and

on the y-axis, where $k'^2 = 1 - k^2$ and $E(\phi, k)$ is an elliptic integral of second kind.

As shown in Fig.1(b), the distance from $P(a \cos \theta, b \sin \theta)$ on the edge of the crack to

the point P' normal to the crack is put by r , and putting $P' \rightarrow P$, we get

$$\sqrt{\rho^2 - 1} = \frac{\sqrt{2r}}{\sqrt{ab}} (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/4} \quad (21)$$

Then, the stress intensity factor K_I is given by

$$K_I = \lim_{r \rightarrow 0} (\sigma_z)_{z=0} \sqrt{2r} = -\frac{p_0}{E(k)} \left(\frac{b}{a} \right)^{1/2} (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/4} \quad (22)$$

In particular, the stress intensity factors K_{Ia} at $(a, 0)$ and K_{Ib} at $(0, b)$ are

$$K_{Ia} = \frac{p_0}{E(k)} \frac{b}{\sqrt{a}}, \quad K_{Ib} = \frac{p_0}{E(k)} \sqrt{b} \quad (23)$$

In the special case of $m \rightarrow 0$, the results are

$$\left. \begin{aligned} a=b=R, \quad k=0, \quad E(k) &= \frac{\pi}{2}, \quad E(\phi, k) = \phi \\ A &= \frac{2(1-\nu)R}{\pi G} \\ f(\lambda, \xi) &= -\frac{4}{\pi^2} \frac{\sin p - p \cos p}{p^3} p_0 R^3 \\ (w)_{z=0} &= \frac{2(1-\nu)R}{\pi G} \sqrt{1-\rho^2} H(1-\rho) \\ (\sigma_z)_{z=0} &= -p_0 \quad (\rho < 1) \\ &= -\frac{2}{\pi} p_0 \left[\sin^{-1} \frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 - 1}} \right] \quad (\rho > 1) \\ K_I &= \frac{2}{\pi} p_0 \sqrt{R} \end{aligned} \right\} \quad (24)$$

and agree with those of circular crack⁽²⁾.

4. The elliptical crack under bending

We now assume that an infinite solid containing an elliptical crack is subjected to bending around x-axis, as shown in Fig.2. An elastic solution of the solid without a crack is

$$\left. \begin{aligned} u &= -\nu \frac{k_y x y}{2G(1+\nu)}, \quad v = -\frac{k_y [x^2 + \nu(y^2 - x^2)]}{4G(1+\nu)} \\ w &= \frac{k_y y z}{2G(1+\nu)}, \quad \sigma_z = k_y y \\ \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} &= 0 \end{aligned} \right\} (25)$$

where k_y is the slope of σ_z in the y-direction.

A solution of the problem is obtained by superposing Eq.(25)

to the stress state satisfying the following boundary conditions

$$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= -k_y y, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \quad (b) \quad (w)_{z=0} = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1, \quad (c) \quad (\tau_{xz})_{z=0} = (\tau_{yz})_{z=0} = 0, \\ (d) \quad \text{All components of displacement and stress} &\text{vanish at infinity.} \end{aligned} \right\} \quad (26)$$

Considering that $(\sigma_z)_{z=0}$ is even in x and odd in y , we express the stress functions as follows:

$$\left. \begin{aligned} 2G\varphi_0 &= -(1-2\nu) \int_0^\infty \int_0^\infty \frac{f(\lambda, \xi)}{\sqrt{\lambda^2 + \xi^2}} \cos \lambda x \sin \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \\ 2G\varphi_3 &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \sin \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \end{aligned} \right\} \quad (27)$$

Eq.(27) satisfies the boundary conditions (c) and (d) of Eq.(26). Using the conditions (a) and (b), we can reduce the problem to finding a solution of dual integral equations

$$\left. \begin{aligned} (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \cos \lambda x \sin \xi y d\lambda d\xi = -k_y y, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right) \\ -\frac{G(w)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \cos \lambda x \sin \xi y d\lambda d\xi = 0, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \right) \end{aligned} \right\} \quad (28)$$

Fig.2. Elliptical crack under bending around x-axis.

Similarly to the previous section, on the mapping plane, we see that the present problem is analogous to one of a circular crack. Then, we consider the case that $(w)_{z=0}$ is given by

$$(w)_{z=0} = A\rho\sqrt{1-\rho^2}\sin\theta H(1-\rho) \quad (29)$$

Performing inverse-Fourier transform of the second equation of Eq.(28) and Eq.(29), we get

$$f(\lambda, \xi) = -\frac{GA}{1-\nu} \frac{1}{\pi^2} \int_0^1 \int_{\rho \leq 1} \rho \sqrt{1-\rho^2} \sin\theta \cos\lambda x \sin\xi y dx dy \quad (30)$$

Using formulae

$$\int_0^{2\pi} \sin\theta \sin(p\rho \sin(\theta+\phi)) d\theta = 2\pi \cos\phi J_1(p\rho) \quad (31)$$

$$\int_0^1 \rho^2 \sqrt{1-\rho^2} J_1(p\rho) d\rho = \frac{1}{p^2} \left\{ \left(\frac{3}{p^2} - 1 \right) \sin p - \frac{3}{p} \cos p \right\} \quad (32)$$

we obtain

$$f(\lambda, \xi) = -\frac{2}{\pi} \frac{GA}{1-\nu} R^2(1-m^2) \cos\phi \frac{1}{p^2} \left\{ \left(\frac{3}{p^2} - 1 \right) \sin p - \frac{3}{p} \cos p \right\} \quad (33)$$

Substituting Eq.(33) into $(\sigma_z)_{z=0}$, we get

$$(\sigma_z)_{z=0} = -\frac{AG}{\pi(1-\nu)} \frac{1}{R(1-m)} \int_0^{\pi/2} \cos\phi \sqrt{1-k^2 \sin^2\phi} d\phi \int_0^\infty \left\{ \left(\frac{3}{p^2} - 1 \right) \sin p - \frac{3}{p} \cos p \right\} \\ \times [\sin(p\rho \sin(\phi+\theta)) - \sin(p\rho \sin(\phi-\theta))] dp \quad (34)$$

Using the formula

$$\frac{2}{\pi} \int_0^\infty \left\{ \left(\frac{3}{p^2} - 1 \right) \sin p - \frac{3}{p} \cos p \right\} \sin ap dp = 2 \operatorname{sgn}(a) [\min(1, |a|) - H(|a|-1)] \\ + a[H(1-|a|) - \delta(1-|a|)] \quad (35)$$

we obtain

$$(\sigma_z)_{z=0} = -\frac{GA}{1-\nu} \frac{1}{R(1-m)} \int_0^{\pi/2} \cos\phi \sqrt{1-k^2 \sin^2\phi} \left\{ \operatorname{sgn}(\rho \sin(\phi+\theta)) [\min(1, |\rho \sin(\phi+\theta)|)] \right. \\ - H(|\rho \sin(\phi+\theta)|-1) - \operatorname{sgn}(\rho \sin(\phi-\theta)) [\min(1, |\rho \sin(\phi-\theta)|)] - H(|\rho \sin(\phi-\theta)|-1) \\ \left. + \frac{1}{2} \rho \sin(\phi+\theta) [H(1-|\rho \sin(\phi+\theta)|) - \delta(1-|\rho \sin(\phi+\theta)|)] \right. \\ \left. - \frac{1}{2} \rho \sin(\phi-\theta) [H(1-|\rho \sin(\phi-\theta)|) - \delta(1-|\rho \sin(\phi-\theta)|)] \right\} d\phi \quad (36)$$

On the crack surface ($\rho < 1$), $(\sigma_z)_{z=0}$ is

$$(\sigma_z)_{z=0} = -\frac{GA}{1-\nu} \frac{1}{R(1-m)} \int_0^{\pi/2} \cos\phi \sqrt{1-k^2 \sin^2\phi} \frac{3}{2} [\rho \sin(\phi+\theta) - \rho \sin(\phi-\theta)] d\phi \\ = -\frac{GA}{1-\nu} \frac{\nu}{R^2(1-m)^2} \frac{(k^2+1)E(k) - k'^2 K(k)}{k^2} \propto \nu, \quad (\rho < 1) \quad (37)$$

Then, $f(\lambda, \xi)$ given by Eq.(33) satisfies completely all boundary conditions, where $K(k)$ is a complete elliptic integral of first kind, and

$$A = \frac{1-\nu}{G} R^2(1-m)^2 \frac{k^2}{(k^2+1)E(k) - k'^2 K(k)} k_\nu$$

As the result, we know that the solution of the dual integral equations (28) is

$$f(\lambda, \xi) = -\frac{2k_\nu}{\pi} R^4(1+m)(1-m)^3 \frac{k^2}{(k^2+1)E(k) - k'^2 K(k)} \cos\phi \frac{(3-p^2) \sin p - 3p \cos p}{p^4} \quad (38)$$

Superposing Eq.(25) and Eq.(26), we obtain

$$(\sigma_z)_{z=0, z=0} = k_\nu y - \frac{GA}{1-\nu} \frac{1}{R(1-m)} \left[3\rho \int_{\cos^{-1}(1/\rho)}^{\pi/2} \cos^2\phi \sqrt{1-k^2 \sin^2\phi} d\phi - \frac{\sqrt{k^2+k'^2\rho^2}}{\rho\sqrt{\rho^2-1}} \right], \quad (\rho > 1) \quad (39)$$

The stress intensity factor K_I is

$$K_I = \lim_{r \rightarrow 0} \sqrt{2r} (\sigma_z)_{z=0} = \frac{k^2}{(k^2+1)E(k) - k'^2 K(k)} k_\nu b \sqrt{\frac{b}{a}} (a^2 \sin^2\theta + b^2 \cos^2\theta)^{1/4} \sin\theta \quad (40)$$

At the point $(0, b)$ on the y-axis, it is

$$K_{Ib} = \frac{k^2}{(k^2+1)E(k) - k'^2 K(k)} k_\nu b^{3/2} \quad (41)$$

In the special case of the circular crack ($m \rightarrow 0$), we get

$$\left. \begin{aligned} a=b=R, \quad A &= \frac{4}{3\pi} \frac{1-\nu}{G} k_\nu R^2, \quad f(\lambda, \xi) = -\frac{8k_\nu R^4 \cos\phi}{3\pi^2} \frac{(3-p^2) \sin p - 3p \cos p}{p^4} \\ (w)_{z=0} &= \frac{4}{3\pi} \frac{1-\nu}{G} k_\nu R^2 \rho \sqrt{1-\rho^2} \sin\theta H(1-\rho) \\ (\sigma_z)_{z=0, z=0} &= k_\nu y - \frac{GA}{1-\nu} \frac{1}{R} \left[\frac{5}{2} \rho \sin^{-1} \frac{1}{\rho} - \frac{3}{2} \frac{\sqrt{\rho^2-1}}{\rho} - \frac{1}{\sqrt{\rho^2-1}} \right], \quad (\rho > 1) \\ K_I &= \frac{4}{3\pi} k_\nu R^{3/2} \sin\theta \end{aligned} \right\} \quad (42)$$

Next, we consider the case that the solid is subjected to bending around y-axis, as shown in Fig.3. The elastic solution in the case without a crack is

$$\left. \begin{aligned} u &= -\frac{k_z[z^2 + \nu(x^2 - y^2)]}{4G(1+\nu)} \\ v &= -\nu \frac{k_z xy}{2G(1+\nu)}, \quad w = \frac{k_z xz}{2G(1+\nu)} \\ \sigma_x &= k_z x, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \end{aligned} \right\} \dots (43)$$

where k_z is the slope of σ_z in the x-direction. A solution of the problem is obtained by superposing Eq.(43) to the stress state satisfying the following boundary conditions

$$\left. \begin{aligned} (a) \quad (\sigma_z)_{z=0} &= -k_z x, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \\ (b) \quad (w)_{z=0} &= 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \\ (c) \quad (\tau_{xz})_{z=0} &= (\tau_{yz})_{z=0} = 0 \\ (d) \quad &\text{All components of displacement and stress vanish at infinity.} \end{aligned} \right\} \dots (44)$$

Considering that $(\sigma_z)_{z=0}$ is odd in x and even in y, we express the stress functions as follows:

$$\left. \begin{aligned} 2G\varphi_0 &= -(1-2\nu) \int_0^\infty \int_0^\infty \frac{f(\lambda, \xi)}{\sqrt{\lambda^2 + \xi^2}} \sin \lambda x \cos \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \\ 2G\varphi_1 &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \sin \lambda x \cos \xi y \exp(-z\sqrt{\lambda^2 + \xi^2}) d\lambda d\xi \end{aligned} \right\} \dots (45)$$

Eq.(45) always satisfies the boundary conditions (c) and (d) of Eq.(44). Using the conditions (a) and (b), we can reduce the problem to finding a solution of the following dual integral equations

$$\left. \begin{aligned} (\sigma_z)_{z=0} &= \int_0^\infty \int_0^\infty \sqrt{\lambda^2 + \xi^2} f(\lambda, \xi) \sin \lambda x \cos \xi y d\lambda d\xi = -k_z x, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right) \\ -\frac{G(w)_{z=0}}{1-\nu} &= \int_0^\infty \int_0^\infty f(\lambda, \xi) \sin \lambda x \cos \xi y d\lambda d\xi = 0, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \right) \end{aligned} \right\} \dots (46)$$

Similarly to the case of Fig.2, on the mapping plane, the problem is analogous to one of a circular crack under bending around y-axis. Then, we consider the case that $(w)_{z=0}$ is given by

$$(w)_{z=0} = A\rho\sqrt{1-\rho^2} \cos \theta H(1-\rho) \dots (47)$$

Performing inverse-Fourier transform of the second equation of Eq.(46) and Eq.(47), and using Eq.(32) and the formula

$$\int_0^{2\pi} \cos \theta \sin \{p\rho \sin(\theta + \phi)\} d\theta = 2\pi \sin \phi J_1(p\rho) \dots (48)$$

we obtain $f(\lambda, \xi)$ as

$$f(\lambda, \xi) = -\frac{2}{\pi} \frac{GA}{1-\nu} \frac{R^2(1-m^2)}{R^2(1-m^2)} \sin \phi \frac{(3-p^2) \sin p - 3p \cos p}{p^4} \dots (49)$$

Substituting Eq.(49) into $(\sigma_z)_{z=0}$, we get $(\sigma_z)_{z=0}$ as

$$\begin{aligned} (\sigma_z)_{z=0} &= -\frac{GA}{1-\nu} \frac{1}{R(1-m)} \int_0^{\pi/2} \sin \phi \sqrt{1-k^2 \sin^2 \phi} \left[\operatorname{sgn}(\rho \sin(\phi + \theta)) [\min\{1, |\rho \sin(\phi + \theta)|\}] \right. \\ &\quad + H(|\rho \sin(\phi + \theta)| - 1) + \operatorname{sgn}(\rho \sin(\phi - \theta)) [\min\{1, |\rho \sin(\phi - \theta)|\}] - H(|\rho \sin(\phi - \theta)| - 1) \\ &\quad + \frac{1}{2} \rho \sin(\phi + \theta) [H(1 - |\rho \sin(\phi + \theta)|) - \delta(1 - |\rho \sin(\phi + \theta)|)] \\ &\quad \left. + \frac{1}{2} \rho \sin(\phi - \theta) [H(1 - |\rho \sin(\phi - \theta)|) - \delta(1 - |\rho \sin(\phi - \theta)|)] \right] d\phi \dots (50) \end{aligned}$$

On the crack surface ($\rho < 1$), we obtain

$$(\sigma_z)_{z=0} = -\frac{GA}{1-\nu} \frac{x}{R^2(1-m^2)} \frac{(2k^2-1)E(k) + k'^2K(k)}{k^2} \propto x, \quad (\rho < 1) \dots (51)$$

Then, $f(\lambda, \xi)$ given by Eq.(49) satisfies completely all boundary conditions, when

$$A = \frac{1-\nu}{G} \frac{R^2(1-m^2)}{(2k^2-1)E(k) + k'^2K(k)} k_z$$

As the result, the solution of the dual integral equations (46) is

$$f(\lambda, \xi) = -\frac{2k_z}{\pi} \frac{R^4(1-m)^2(1+m)^2}{(2k^2-1)E(k) + k'^2K(k)} \sin \phi \frac{(3-p^2) \sin p - 3p \cos p}{p^4} \dots (52)$$

Superposing Eqs.(43) and (44), we get $(\sigma_z)_{z=0}$ on the x-axis

Fig.3. Elliptical crack under bending around y-axis.

$$(\sigma_z)_{z=0, y=0} = k_z x - \frac{GA}{1-\nu} \frac{1}{R(1-m)} \left[\int_0^{\sin^{-1}(1/\rho)} 3\rho \sin^2 \phi \sqrt{1-k^2 \sin^2 \phi} d\phi - \frac{\sqrt{\rho^2-k^2}}{\rho \sqrt{\rho^2-1}} \right], \quad (\rho > 1) \quad (53)$$

The stress intensity factor $K_I = \lim_{r \rightarrow 0} \sqrt{2r} (\sigma_z)_{z=0}$ is

$$K_I = \frac{k^2}{(2k^2-1)E(k) + k'^2 K(k)} k_z a \left(\frac{b}{a}\right)^{1/2} (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/4} \cos \theta \quad (54)$$

and, at the point $(a, 0)$ on the x-axis, it is

$$K_{Ia} = \frac{k^2}{(2k^2-1)E(k) + k'^2 K(k)} k_z a^{1/2} b \quad (55)$$

In the special case of the circular crack ($m \rightarrow 0$), the results are obtained by replacing $k_y \rightarrow k_z$, $y \rightarrow x$, $\theta \rightarrow \pi/2 - \theta$, $\phi \rightarrow \pi/2 - \phi$ in Eq. (42).

Fig. 4 shows the relations between the stress intensity factors and the ratio b/a . In the case of internal pressure, K_{Ib} at $(0, b)$ is larger than K_{Ia} at $(a, 0)$. In the case of bending when $k_z = k_y$, K_{Ia} is larger than K_{Ib} .

Acknowledgements

The author is deeply indebted to Prof. Ichiro Nakahara and Prof. Takashi Koizumi of the Tokyo Institute of Technology for many valuable suggestions in the preparation of this paper.

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Fig. 4. Stress intensity factors,

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