Analysis of Lateral Vibration Characteristics of Rotating Shafts with Flexible and Axi-asymmetric Bearings

By Keishi KAWAMO** Yoshitaka MATSUKURA** Toshio INOUE***

A method is presented to calculate the unbalance vibration behaviours using transfer matrix method in which such factors related to the vibration are covered as the interaction characteristics of load displacement between horizontal and vertical directions at oil film bearings, the dynamic stiffness of bearing support structures, etc. In case of the rotor systems with multi-spans, the numerical accuracy is occasionally deteriorated when number of sections, number of bearings and rotating speed exceed a certain boundary as far as the conventional method is applied. The authors developed a method to suppress the propagation of the error by modifying the procedure of matrix operations and solving the simultaneous linear equations at the last step of the computation works.

1. Introduction

The unbalance vibration problems of rotor systems of varying flexible rigidity supported by the oil film bearings were long studied by Koening et al.(1)(2), Lund et al.(3), Authors(4), Kikuchi(5)(6), and others, through the analysis on the rotor systems with multi-spans, multi-disks and multi-beams.

The analysis to simulate the bearings as symmetric systems, which consist of the linear spring and damper systems, was accomplished by Koening(1)(2) and the Authors (4). However, the actual bearings have different vibration characteristics between the vertical and horizontal directions, namely, asymmetry. In the case of the oil film bearings, moreover, the interactive characteristics are observed, where the vertical load causes a horizontal displacement. An analysis to include the effects of such asymmetric and interactive characteristics was established by developing the basic equations described by means of the complex numbers under the three dimensional space (3).

Width of the bearing may generate a reaction moment, a so-called rotational reaction due to the inclination of the rotor, in addition to the reaction force to the rotor displacement, a so-called translational reaction. Consideration of such a rotational reaction was made by Di Taranto(7) and the Authors. Further improvement was carried out in the analysis by establishing models of bearings for the translational and rotational movement and by simulating bearing supports to vibration systems with one degree of freedom with respect to the translational movement (8)(9).

Following the analysis of Lund and Kikuchi, this paper describes a further development of the analysis integrated by representing the bearing-bearing support systems with linear spring-damper models which have the asymmetric and interactive characteristics of the translational and rotational movements of the rotor.

The transfer matrix method has been often employed as a useful method for calculations of the unbalance vibration and critical speeds of the rotating shaft systems. However this method is known to be apt to cause a numerical error. Therefore, Koening et al. developed the following method. The simultaneous solution of the state vectors at the sections with bearings for the whole rotor system yielded the boundary values of each span. With the boundary values known, repeated matrix operations were performed through each span to obtain the state vectors. In this way the propagation of the error was suppressed. In calculating the natural frequencies of axi-symmetrical shells, a structures were divided into several portions and the matrix operations were made in portion-by-portion manner (10). It was tried to lessen the round-off error by this method.

A numerical calculation method is developed to suppress the propagation of the error by means of improvement of the procedures of matrix operations and by solving the simultaneous linear equations at the last step of a series of numerical works. In this way, the vibration analysis of the rotor systems with multi-spans is made possible with a high accuracy.

This report covers the analytical theory and numerical calculation method by the transfer matrix method on the un-
balance vibration problems of the rotating shaft systems. At the same time, the results of numerical experiments of the conventional numerical method and the numerical method developed by the Authors, are described in regard to some examples of the rotating shaft systems.

Nomenclature

- \( E_I \) = bending rigidity of rotor
- \( F_x, F_y \) = external forces acting on rotor in \( x \) and \( y \) directions
- \( G_A \) = shear rigidity of rotor
- \( I_p, I_d \) = polar mass moment of inertia, and transverse mass moment of rotor element
- \( M \) = mass attached to rotor
- \( c_B, c_p \) = damping coefficients of bearing oil film and bearing support
- \( k_s, k_p \) = spring constants of bearing and bearing support
- \( k_{Bxy} \) = bearing spring constant in \( x \) direction by displacement of \( y \) direction
- \( l \) = length of rotor element
- \( m \) = mass of rotor element
- \( m_{px}, m_{py} \) = effective masses of bearing support in \( x \) and \( y \) directions
- \( k \) = shear coefficient
- \( \Omega, \omega \) = angular frequencies of whirling and revolution

2. Lateral vibration analysis

As is customary, the rotors are assumed to be symmetrical in flexural rigidity and to be negligibly small in internal friction. Furthermore, the effect of torsional vibration is ignored on the assumption that the lateral vibration is not coupled with the torsional vibration. Gyroscopic effect and shear deflection effect of the rotor are considered.

Under the above assumptions, the analysis is proceeded as follows: A series of rotating shaft systems is modeled by a Myklestäd beam, that is, a lumped parameter system. The system is made up of a large number of rotor elements, each of which consists of three kinds of elements. They are massless beam elements of uniform flexural rigidity, inertia elements of concentrated balanced mass and moment of inertia with unbalanced mass attached, and bearing-bearing support elements.

A series of rotating shaft systems is divided into \( N \) pieces of rotor elements. The vibration condition of section \( i \) is expressed by the following equation of the state vector, whose components are deflection, slope, bending moment and shearing force.

\[
(Z)_i = \begin{bmatrix} u_x, v_y, M_y, -v_x, \theta_x, M_x, v_y \end{bmatrix}^T
\]

(1)

As shown in Fig. 1, \( z \) axis is taken along the length of shaft, while \( x \) and \( y \) axes are arranged in two directions crossing at right angles with \( z \) axis in Cartesian righthanded coordinate system. Since the bearing-bearing support systems have asymmetry and damping effect, all the vector components of Eq.(1) are independent complex variables. Following relation is established between the state vectors \((Z)_{i-1}\) and \((Z)_i\).

\[
(Z)_i = [E]_i [Z]_{i-1}
\]

or

\[
(Z)_i = [E]_i (Z)_{i-1} + [W]_i
\]

(2)

Here, \([E]_i\) are the 9th order square matrices of rotor element which relate the state vectors of sections \( i-1 \) and \( i \), and are determined by the physical characteristics of rotor element between both sections. The elements of matrices are complex number elements. Equation(2)' is another expression of Eq.(2). \((Z)_i\) represent the state vectors made of the first 8 components of matrices \((Z)_i\). \([E]_i\) are the rotor element matrices, which are composed of the elements of the first 8 rows by 8 columns of \([E]_i\). \([W]_i\) are the vectors, consisting of the first 8 components of the 9th column. These express the vibration exciting forces to the rotor systems.

The rotor element matrices are explained already in detail(5) and further explanation shall be omitted. For the convenience of the calculation, the ele-
ments of the above three kinds are made in one set, which can be taken as a unit of the rotor element, as shown in Fig. 1. Then, the rotor element matrices defined in Eq. (2) are expressed as the products of the beam, inertia and bearing-bearing support elements. This is represented in Eq. (3) which is shown in the end of this paper, where

\[
\begin{align*}
T_{xx} &= -T'_{xx} + (\Omega^2)\Omega^2 \\
T_{yy} &= -T'_{yy} + (\Omega^2)\Omega^2 \\
T_{xy} &= T'_{xy} + T'_{yx} \\
T_{xy} &= T'_{xy} + j\Omega\Omega \\
T_{yx} &= T'_{yx} + j\Omega\Omega \\
T_{x0} &= T'_{x0} - I_d\Omega^2 \\
T_{y0} &= T'_{y0} - I_d\Omega^2
\end{align*}
\]  

(3)

\[T'_{xy}\] and \[T'_{x0}\] are the dynamic stiffness in x axis direction by y axis displacement, and the dynamic stiffness around x axis by the rotation around y axis, respectively. Since the oil film bearings have the damping effect, the dynamic stiffness is of all complex numbers. 

The contents of \[T_{xy}\], etc. are abbreviated herein.

The state vectors at both ends of the whole rotor system can be combined from Eq. (2) as follows;

\[
[C]_N = [E][N][E][N-1][E][N-2]...[E][E][1][Z]_0
\]

(4)

The boundary conditions at both ends are set into Eq. (4) and the unbalanced vibration is calculated as explained later.

3. Numerical analysis method

Usually a boundary condition at the rotor end prescribes four state quantities; for example, \(x_N, y_N, x_N, y_N\) are equal to zero for "free" boundary condition. In general case, \(s\) pieces of prescribing relations of state quantities are given at the left end, section \(O\), of the rotor and \(r\) pieces of the relations are given at the right end, section \(N\), as the boundary conditions. \(s + r = 8\) must be satisfied.

The Authors' numerical analysis method is composed of the following three steps.

Step 1: Calculation of the coefficient matrices \([8]_4\) and vectors \([q]_4\) from the left end to the right end.

Step 2: Calculation of \([8']_4\), \([q']_4\) from the right end to the left end.

Step 3: Calculation to solve the simultaneous equations to obtain the state vectors \([Z]_4\).

The details of above procedures are described below.

![Fig. 2 Experimental results for Model 1](image)

![Fig. 3 Experimental Results for Model 2](image)
Step 1: In the numerical analysis from the left end if the boundary condition is "free", the four components of the state vector at section 0, \( \{z\}_0 \) should be prescribed as

\[
\begin{align*}
(Mx)_0 &= (My)_0 = 0 \\
(Vx)_0 &= (Vy)_0 = 0
\end{align*}
\tag{5}
\]

This can be expressed in matrix form and the following equation is obtained.

\[
\begin{align*}
&4 \times 8 \quad 8 \times 1 \\
&4 \times 1 \\
[R]_0 \quad \{z\}_0 &= \{q\}_0
\end{align*}
\tag{5'}
\]

Where,

\[
[R]_0, \quad \{q\}_0 \text{ may be given as follows;}
\]

\[
[R]_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\tag{6}
\]

The symbols over the matrices and the vectors represent numbers of rows and columns. Generally speaking, however, the boundary conditions are not necessarily "free". The state vector \( \{z\}_0 \) at section 0 can be expressed by the following 8 pieces of linear combination equations;

\[
\begin{align*}
&s \times 8 \quad s \times 1 \\
&[R]_0 \quad \{z\}_0 &= \{q\}_0
\end{align*}
\tag{7}
\]

Now, it is supposed that the following equation is established in general as to the unknowns \( [R]_4, \{q\}_4 \) in the same form as Eq. (7) at section 1.

\[
\begin{align*}
&s \times 8 \quad s \times 1 \\
&[R]_4 \quad \{z\}_4 &= \{q\}_4
\end{align*}
\tag{8}
\]

From equations (2)', (8) and (8) written for section 1-1, i.e., [R]_{i-1} \{z\}_{i-1} = \{q\}_{i-1}, the following equations are established.

\[
\begin{align*}
[R]_4 &= [R]_{i-1} \quad [E]^{-1}_i \\
\{q\}_4 &= \{q\}_{i-1} + [R]_4 \quad \{q\}_4
\end{align*}
\tag{9}
\]

Therefore, if \( \{q\}_0 \) and \( [R]_0 \) are given as the initial conditions, then the values of \( [R]_4 \) and \( \{q\}_4 \) at section 1 can be obtained in a step-by-step procedure from the left end to the right end.

Step 2: In numerical analysis from the right end, \( r \) pieces of the linear equations are obtainable at the section \( N \) as the boundary condition at the right end.

\[
\begin{align*}
&r \times 8 \quad r \times 1 \\
&[R]_N \quad \{z\}_N = \{q\}_N
\end{align*}
\tag{7}
\]

In the same form as Eq. (7)', the next equation can be introduced at section 1.

\[
\begin{align*}
&[R']_1 \quad \{z\}_1 = \{q\}'_1
\end{align*}
\tag{8}
\]

Then the following equations are obtained from equations (2)', (8)' and (8) at section 1-1.

\[
\begin{align*}
[R']_{i-1} &= [R']_1 \\
\{q\}'_{i-1} &= \{q\}'_1 + [R']_1 \quad \{q\}'_i
\end{align*}
\tag{9'}
\]

By use of the above equations, the values of \( [R']_i \) and \( \{q\}'_i \) at section i can be obtained from the right end to the left end under the initial value \( \{q\}'_N \) and \( [R']_N \).

Step 3: By combining Eq. (8) and (8)',

\[
\begin{align*}
&(s+r) \times 8 \\
&[R]_4 \quad \{z\}_4 = \{q\}_4
\end{align*}
\tag{10}
\]

is obtained. From values of \( [R]_4, [R']_i, \{q\}_i \) and \( \{q\}'_i \), which are obtained in the above Steps 1 and 2, the state vectors \( \{z\}_i \) at section i are acquired. As to the boundary condition, section 0 can have always the "free" boundary condition by providing a fictitious rotor element having the length 0 at the left end of the rotor system model. On the other hand, the right end has always "free" boundary condition in view of the characteristics of the modeling.

Therefore, the following conditions are satisfied.

\[
\begin{align*}
&(Mx)_N = (My)_N = 0 \\
&(Vx)_N = (Vy)_N = 0
\end{align*}
\tag{11}
\]

Each component of \( \{z\}_0 \) at the left end is linearly dependent. Therefore if zero vector is selected for \( \{q\}_0 \), the coefficient matrix \( [R]_0 \) will take such matrix, where no elements are necessarily zero. There are many matrices which satisfy condition (5) as \( [R]_0 \). However, the most simple one is Eq. (6). The same is applied to the right end, and \( \{q\}'_N \) and \( [R']_N \) can be selected in the same form as Eq. (6).

4. Numerical experiments

Following numerical experiments on three kinds of simple model rotors were performed in order to study numerical accuracy of the Authors' method in comparison with the conventional method.

1. Model 1
   
   Total length: 1000 mm
   Diameter: 10 mm
   Bearings: 2 axi-symmetrical bearings with springs and dampers at both ends of the rotor
   Rotor elements: 10 pieces of the elements having the length 100 mm
   Material: Steel (weight of the rotor to be neglected)
   Mass: 1 unbalance mass on a disk attached to the rotor at its center
(2) Model 2
Total length: 1000 mm
Diameter: 10 mm
Bearings: 7 axi-symmetrical bearings with springs and dampers at every 200 mm interval
Rotor elements: 10 pieces of the elements having the length 100 mm
Material: Steel
Mass: Same as Model 1

(3) Model 3
Total length: 2800 mm
Diameter: 10 mm
Bearings: 15 axi-symmetrical bearings with springs and dampers at every 200 mm interval
Rotor elements: 28 pieces of the elements having the length 100 mm
Material: Steel
Mass: Same as Model 1

In all the above models the bearing constants are same for all the bearings of each rotor.

It was indicated that the larger the mean order \( a \) of the elements of \([E]\) total of the system became, the larger numerical error should be accumulated. Therefore, the accuracy of the Authors' method in comparison with the conventional method was evaluated for various values of \( a \), which were varied by gradual increase of the spring constants and damping coefficients of the bearings.

The accuracy is evaluated on the following items with respect to the state vector components:

(i) Symmetrical property with respect to the central section.

Since each model forms a symmetric system, the state vectors of such sections as stand at the symmetrical positions with respect to the center should be identical.

(ii) Symmetrical property between the components of the horizontal and vertical directions.

Since the bearing constants are symmetrical, the vertical and horizontal quantities of the state vectors should be identical at the same section.

(iii) Cancelling of the slope at the central section.

Since the system is symmetrical, the slope at the central section should be zero.

(iv) Cancelling of the bending moments and shearing forces at both ends.

Since the boundary conditions are "free" at both ends, the bending moments and shearing forces should be zero.

The accuracy is evaluated for items (i) and (ii) by checking how many digits the state quantities are identical in, and for items (iii) and (iv) by checking how small the values of the state components are.

The experiments were carried out by using IBM 370/165 computer in single precision data input and double precision arithmetic.

The experimental results are shown in Fig. 2 for Model 1, in Table 1 and Fig. 3 for Model 2, and in Fig. 4 for Model 3.

Fig. 2 for Model 1 shows that the deflection at the central sections by the conventional method is identical with the analytical solution up to \( k = \omega = 0.1 \times 10^{37} \text{kg/mm} \). However, more rigid bearings causes the deflection to be diverged. On the other hand, the result obtained by the Authors' improved method shows the identity with the analytical solution up to \( k = \omega = 0.1 \times 10^{75} \text{kg/mm} \). In regard to the deflection at the right end, the conventional method results in divergence when \( k = \omega \geq 0.1 \times 10^{45} \text{kg/mm} \). However, the improved method gives the identical value with the analytical solution up to \( k = \omega = 0.1 \times 10^{75} \text{kg/mm} \).

In regard to the items (i)-(iv) of the above evaluation, Table 1 for Model 2 shows that the conventional method produces the physically meaningless solutions when the bearings are more rigid than \( k = \omega = 0.1 \times 10^{52} \text{kg/mm} \). On the other hand, the improved method allows the significant values up to \( k = \omega = 0.1 \times 10^{75} \text{kg/mm} \). Fig. 3 shows the deflection curves for the Table 1 in case of \( k = \omega = 0.1 \times 10^{52} \text{kg/mm} \). The conventional method brings about a divergence of the deflection at the sections near the right end. Meanwhile, the result obtained by the improved method is almost symmetric with respect to the center.

Fig. 4 gives the plotted deflection curves for the case of \( k = \omega = 0.1 \times 10^{46} \text{kg/mm} \). The conventional method causes an excessive divergence.

![Fig. 4 Experimental results for Model 3](k = \omega = 0.1 \times 10^{46} \text{kg/mm})
Table 1. Experiment results for Model 2

<table>
<thead>
<tr>
<th>S &amp; C</th>
<th>M. P. (W/m)</th>
<th>S. P. E.</th>
<th>O. S. C.</th>
<th>Order of</th>
<th>[ M_m, M_y, y_m, y_y ]</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 x 10^6</td>
<td>4 - 5 digits</td>
<td>more than 5 digits</td>
<td>- 10</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>-</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>4 - 5 digits</td>
<td>more than 5 digits</td>
<td>- 6</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>-</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>no digits</td>
<td>no digits</td>
<td>- 1</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>-</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>no digits</td>
<td>no digits</td>
<td>- 4</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>-</td>
</tr>
<tr>
<td>0.1 x 10^6</td>
<td>more than 5 digits</td>
<td>more than 5 digits</td>
<td>- 14</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>5.20</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>more than 5 digits</td>
<td>more than 5 digits</td>
<td>- 11</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>12.0</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>more than 5 digits</td>
<td>more than 5 digits</td>
<td>- 9</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>17.5</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>1 - 5 digits</td>
<td>more than 5 digits</td>
<td>- 6</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>21.5</td>
</tr>
<tr>
<td>\times 10^6</td>
<td>no digits</td>
<td>no digits</td>
<td>- 1</td>
<td>[ M_m, M_y, y_m, y_y ]</td>
<td>( 10^{-7}, 10^{-8} )</td>
<td>26.5</td>
</tr>
</tbody>
</table>

where
- *S. P. E.* = Symmetric property with respect to the central section
- **O. S. C.** = Order of the slope at the central section
- \( M_m, M_y \) = Bending moment at left and right and section respectively
- \( y_m, y_y \) = Shear force at left and right and section respectively
- \( \alpha \) = Mean order of components of \( [E]_m, [E]_y \)

5. Discussion

According to the conventional method, \( [E]_{total} \) as defined in Eq. (4)' is calculated as the first step. Then, the boundary conditions at both ends are applied to the equation and the simultaneous linear equations are solved to obtain the state vector \( [Z]_0 \) at the left end. In the third step, with the initial value \( [Z]_0 \) known, repeated applications of Eq. (2) are performed to compute the state vector at each section from the left to right end. In this case, the following numerical error may appear.

(i) Error in the numerical process of the matrix multiplications in the first step.

(ii) Round-off error of the simultaneous solution in the second step.

(iii) Error in the process of matrix operations in the third step.

The error in the (i)-(iii) matrix operation steps reduces the number of significant digits of the state vector components according to the next equation (8).

\[
\gamma = \beta - \alpha
\]

\( \alpha \) is the mean order of \( [E]_{total} \) and \( \beta \) is the number of significant digits of the computer used in this calculation, and \( \gamma \) is the number of significant digits which remain in the state vector components. \( \beta \) equals 16.5 for the computer used in this experiment.

As shown in Table 1, the larger bearing constants cause the larger value of \( \alpha \). From the experimental results of Models 1-3, though all of them are not described herein, it is recognized that an increase in the number of bearings and sections and the rotating speed also enlarges the value of \( \alpha \). Therefore, the decrease in the number of significant digits by this numerical operation cannot be ignored for the systems with multi-planes and multi-sections. The error in (iii) step is accumulated gradually since the initial value \( [Z]_0 \) which contains the errors generated in (i) and (ii) steps is used to obtain \( [Z]_1 \) at the successive sections. \( [Z]_1 \) occasionally diverge, when \( \alpha \) is large.

Fig. 5 shows again the experimental results of \( \gamma \) to \( \alpha \) represented in Table 1. Here, \( \alpha \) is defined as mean order of the components of \( [R]_0 \) or \( [R]_1 \) in Authors' method and of \( [E]_{total} \) in conventional method. \( \gamma \) is defined as the number of significant digits of bending moments or shearing forces at the right end section.

Fig. 5 Numerical accuracy of conventional method and authors' method

It is shown that the lower boundary of the solution accuracy roughly obeys the following criterion

\[
\gamma = \beta - \alpha/2
\]

in the Authors' method, while the accuracy obeys Eq. (12) in conventional method. From Eqs. (12) and (13), it is obvious that the numerical accuracy may be certainly improved by the adoption of the Authors' method. The improved method includes the above errors (i) and (ii), but does not the error (iii). This seems to contribute to reducing the error and not to bring about such a fatal phenomenon, where the
solutions are unstably diverged.

In carrying out the actual calculations, it may be favorable to adopt the procedure similar to what Koenig did. First the state vectors at the sections with bearings are obtained by the Authors' method. Next, by using the vectors as the boundary values of each span, repeated matrix operations are performed through each span to determine all the state vectors. This procedure is meaningful to save the memory of computer and reduce the number of solving simultaneous equations.

By application of the program coded on the Authors' method unbalance vibration can be calculated with a high degree of accuracy in regard to a large class of turbine-generator rotor systems in current use. Fig. 6 shows an example of plotted response modes.

![Fig. 6 Unbalance response model of a large class of turbine-generator rotor system](image)

6. Conclusions

In this paper, the theory is developed by considering almost all the factors which may influence the vibration, including the bearing asymmetry, interactive characteristics of the bearing, etc. Though the examples of numerical analysis of vibrational phenomena caused by these factors were not mentioned here, interesting results were obtained in each case.

Since the complex numbered matrices with 9 rows by 9 columns are handled in the above theory, the solution may occasionally show a divergence in the calculation routine as far as the conventional numerical method is applied. The Authors developed a new numerical analysis method, by which the propagation of the error was suppressed. In this method, the numerical procedures become somewhat complicated. However, the numerical experiments performed on 3 kinds of the models indicated an outstanding improvement of the numerical accuracy.

Based on the above numerical theory, a program was developed, and it was shown that the numerical analysis with enough accuracy could be made in regard to a large class of turbine-generator rotor systems.

Acknowledgement

The Authors wish to take this opportunity to express their heartfelt gratitude to Messrs. M. Shiki, H. Imai and N. Oishi who gave the opportunity to perform this study, and also to Messrs. R. Kato, Y. Anki, Y. Murai, Mrs. Y. Fujiiwara and Mr. K. Yagi of Mitsubishi Electric Corporation who made persevering efforts in developing the program.

References