Stability of MHD Viscous Flow in a Curved Channel*

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The purpose of this paper is to consider by the method of small perturbations how an external magnetic field suppresses the onset of a secondary vortex flow in a two-dimensional, incompressible, electrically conducting viscous flow in a curved channel. The magnetic field is applied in the direction perpendicular to the channel walls. The results show: (1) the critical Dean number K_C begins to increase as the Hartmann number H_a becomes larger than about 2, and in the limit $H_a \rightarrow \infty$ the Dean number K_C tends to a relation proportional to $H_a^{3/2}$; (2) the associated dimensionless wavenumber σ_C of the vortices also increases with H_a , and the location of the vortex center comes closer to the outer concave wall. In addition the effect of electrical conductivity of the wall material on the present stability problem is discussed.

1. Introduction

For flows of electrically conducting fluid in a straight channel, the critical Reynolds number for the onset of Tollmien-Schlichting wavelike disturbances is remarkably increased by application of magnetic field. This fact has been theoretically and also experimentally verified for various sectional forms of the channel and for application of various magnetic field directions. It might, therefore, be said that the state of laminar flow is to appear more often in such MHD channel flows than in ordinary fluid flows.

For curved channel flows, another centrifugal instability should be considered, as the result of which a secondary vortex flow appears. This problem was first studied by Dean⁽¹⁾ for an ordinary fluid flow in a channel formed by two concentric cylinders. He found that the instability will first arise when the parameter $(2y_0 U_m / \nu) \sqrt{2y_0 / R_1}$ (U_m : the mean velocity of the unperturbed flow, $2y_0$: the spacing between the cylinders, R_1 : the radius of the inner cylinder, ν : viscosity) exceeds a value of about 36, and that a cellular pattern of motion ensues. After Yih and Sangster⁽²⁾ obtaining another result different from Dean's critical value, Reid⁽³⁾ and Hämmerlin⁽⁴⁾ more recently re-examined the same problem, the results of which were in close agreement with Dean's one. In these previous studies it was assumed that the spacing $2y_0$ is small compared with the radius R_1 of the inner cylinder ($y_0 / R_1 \ll 1$). Experimental verification was carried out by Brewster et al.⁽⁵⁾

In previous papers,⁽⁶⁾⁻⁽⁸⁾ we showed that the onset of longitudinal vortices in laminar boundary layers of electrically conducting fluid along concave walls is remarkably

suppressed by external magnetic fields. From the results it might be suggested that the magnetic field applied to curved channel flows suppresses the onset of a secondary vortex flow. Chandrasekhar et al.⁽⁹⁾ considered this problem in a case where the external magnetic field is applied in the direction of the cylindrical axis. It is, however, expected from the results of the boundary layer case⁽⁸⁾ that a stronger stabilizing effect can be obtained by application of magnetic field in the direction perpendicular to the cylindrical surfaces than in the axial direction.

This paper purports to consider by the method of small perturbations how the instability of MHD viscous flow in a two-dimensional curved channel is affected in the presence of a magnetic field applied in the direction perpendicular to the curved surfaces. The effects of the curvature of the channel as well as the electrical conductivity of the channel wall on the present problem are also discussed.

2. Velocity profiles of MHD viscous flow

Prior to considering instability for the onset of a secondary vortex flow, it is necessary to know the velocity profiles of the unperturbed two-dimensional MHD viscous flow (primary flow). The curved channel is formed by two coaxial cylindrical surfaces. The radius of curvature is R_1 for the inner wall, R_0 for the central surface of the channel and R_2 for the outer wall. An external magnetic field is applied in the direction normal to the wall surfaces. Cylindrical coordinates (r, φ, z) are adopted, where the z -axis coincides with the axis of the cylindrical surfaces. Denote by (u_r, u_φ, u_z) the components of the velocity vector $\mathbf{u}$ and by (H_r, H_φ, H_z) the components of the magnetic field $\mathbf{H}$ in the r -, φ -, and z -directions respectively. The velocity and the magnetic field of the primary flow are independent of φ and z , while the pressure p varies across the curved channel and falls steadily as φ increases. The di-

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vergence of the magnetic field, $\text{div } \mathbf{H} = 0$, becomes then $dH_r/dr + H_r/r = 0$, which gives a solution $H_r = H_0 R_0/r$, where H_0 denotes the magnetic field at the central surface of the channel, $r = R_0$. From the relation of $H_r(r)$, it can be recognized that the velocity field independent of the φ -direction is obtained in a case where the intensity of radial magnetic field is inversely proportional to r .

The equation of motion reduces to the following expression for the present primary flow:

$$-\frac{1}{\rho} \frac{1}{r} \frac{\partial p}{\partial \varphi} + \frac{\kappa}{\rho} \mu H_r (E_z - \mu H_r u_\varphi) + \nu \left(\frac{d^2 u_\varphi}{dr^2} + \frac{1}{r} \frac{du_\varphi}{dr} - \frac{u_\varphi}{r^2} \right) = 0 \quad (1)$$

where ρ denotes the density of the fluid, μ its magnetic permeability, ν the coefficient of kinematic viscosity, κ the electrical conductivity of the fluid, and E_z the z -component of electric field. Introducing the relation of $H_r(r)$ into Eq.(1) and replacing $u_\varphi = U$, we have the differential equation:

$$\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{a}{r^2} U = \frac{F}{r} \quad (2)$$

in which

$$\left. \begin{aligned} a &= 1 + H_a^2 \left(\frac{R_0}{y_0} \right)^2, \quad H_a = \mu H_0 y_0 \sqrt{\frac{\kappa}{\rho \nu}} \\ F &= \frac{1}{\rho \nu} \left(\frac{\partial p}{\partial \varphi} \right) - H_a^2 \frac{E_z}{\mu H_0 y_0} \left(\frac{R_0}{y_0} \right) \end{aligned} \right\} \quad (3)$$

H_a is the Hartmann number. Under the boundary conditions $U = 0$ at $r = R_1$ and $r = R_2$, Eq.(2) gives an exact solution for the velocity distribution $U(r)$ in the curved channel as follows:

$$U = \frac{-F}{(a-1)(\gamma\sqrt{a}-\gamma^{-\sqrt{a}})} \left[R_1 \left\{ \left(\frac{r}{R_2} \right)^{-\sqrt{a}} - \left(\frac{r}{R_2} \right)^{\sqrt{a}} \right\} + R_2 \left\{ \left(\frac{r}{R_1} \right)^{\sqrt{a}} - \left(\frac{r}{R_1} \right)^{-\sqrt{a}} \right\} - r(\gamma\sqrt{a}-\gamma^{-\sqrt{a}}) \right] \quad (4)$$

Fig.1 Velocity profiles of two-dimensional MHD viscous flow in a curved channel

where $\gamma = R_2/R_1$. The mean flow velocity U_m is calculated from Eq.(4) as

$$U_m = \frac{1}{(R_2 - R_1)} \int_{R_1}^{R_2} U dr = \frac{-FR_1/(a-1)^2}{(\gamma-1)(\gamma\sqrt{a}-\gamma^{-\sqrt{a}})} \times \left[4\sqrt{a}\gamma + \frac{1}{2}\gamma\sqrt{a}(\gamma^2(\sqrt{a}-1)^2 - (\sqrt{a}+1)^2) - \frac{1}{2}\gamma^{-\sqrt{a}}(\gamma^2(\sqrt{a}+1)^2 - (\sqrt{a}-1)^2) \right] \quad (5)$$

Figure 1 shows the velocity profiles U/U_m of the curved channel with the curvature y_0/R_0 of 0.1, 0.01 and 0 for several values of the Hartmann number H_a , in which η is a dimensionless expression of r ; $\eta = (R_0 - r)/y_0$. The velocity profiles for $y_0/R_0 \rightarrow 0$ agrees with that of the Hartmann flow for a straight channel.

The coefficients of shear stress, C_{f1} and C_{f2} , at the inner and outer walls and also the friction coefficient λ of the curved channel are given in Fig.2. They were obtained from the relations:

$$\left. \begin{aligned} \tau_{w1} &= \rho \nu \left(\frac{dU}{dr} \right)_{r=R_1} \equiv C_{f1} \frac{1}{2} \rho U_m^2 \\ \tau_{w2} &= -\rho \nu \left(\frac{dU}{dr} \right)_{r=R_2} \equiv C_{f2} \frac{1}{2} \rho U_m^2 \\ \lambda &= C_{f1} + C_{f2} \end{aligned} \right\} \quad (6)$$

where τ_{w1} and τ_{w2} are the corresponding wall shear stresses and $Re = U_m y_0 / \nu$. Figure 2 shows that the friction coefficient λ varies little as the curvature of the channel (y_0/R_0) changes from 0 to 0.1, though there exist differences between the coefficients C_{f1} and C_{f2} of the two wall shear stresses.

3. Perturbation equations

Rayleigh's criterion for the centrifugal instability of inviscid curved flows is expressed with $d(rU)^2/dr < 0$. In the present curved channel, the flow region near the outer wall ($\eta = -1.0$) is found to be under this condition, as seen in Fig.1. The cri-

Fig.2 Coefficients of wall shear stresses ($Re C_{f1}$, $Re C_{f2}$) and friction coefficient ($Re \lambda$) of the curved channel

Fig. 3 Co-ordinate system

terion in viscous flows will, however, be more complicated. We write the equations of motion, continuity and induction for incompressible, electrically conducting, viscous fluid with the curvilinear coordinate system (x, y, z) shown in Fig. 3 by using the following transformations:

$$\left. \begin{aligned} r &= R_0 - y, & dr &= -dy, & rd\varphi &= -dx \\ dz &= -dx, & u_r &= -v, & u_\varphi &= -u \\ u_z &= -w, & U &= -U \end{aligned} \right\} \dots\dots\dots (7)$$

The radius R_0 of curvature is assumed to be large compared with the spacing $2y_0$ of the channel

We shall now suppose that the primary two-dimensional MHD laminar flow is slightly perturbed with small disturbances of secondary vortex flow type. The components of the velocity vector and of the magnetic field as well as the pressure in the perturbed motion are expressed as

$$\left. \begin{aligned} u &= U(y) + \hat{u}(y)e^{\omega t} \cos \alpha x \\ v &= \hat{v}(y)e^{\omega t} \cos \alpha x \\ w &= \hat{w}(y)e^{\omega t} \sin \alpha x \\ p &= P(x, y) + \hat{p}(y)e^{\omega t} \cos \alpha x \\ H_z &= \hat{H}_z(y) + \hat{h}_z(y)e^{\omega t} \cos \alpha x \\ H_y &= H_0 + \hat{h}_y(y)e^{\omega t} \cos \alpha x \\ H_x &= \hat{h}_x(y)e^{\omega t} \sin \alpha x \end{aligned} \right\} \dots\dots\dots (8)$$

where $U(y)$ is the unperturbed velocity distribution, $P(x, y)$ the corresponding static pressure, t the time, ω a growth rate of the disturbances with the time, and α the wave-number of the vortices, which is related to the wavelength T with $\alpha = 2\pi/T$. Substituting Eq. (8) into the system of the aforementioned basic equations, and linearizing the obtained equations assuming that squares and products of the perturbation quantities are negligible, we get then a set of perturbation equations as follows:

$$\left. \begin{aligned} \omega \hat{u} + \left(\frac{dU}{dy} - \frac{U}{R_0} \right) \hat{v} - \frac{\mu}{\rho} \left(H_0 \frac{d\hat{h}_z}{dy} + \frac{d\hat{H}_z}{dy} \hat{h}_y \right) &= \nu \left(\frac{d^2 \hat{u}}{dy^2} - \alpha^2 \hat{u} \right) \dots\dots\dots (9) \\ \omega \hat{v} + \frac{2U}{R_0} \hat{u} + \frac{\mu}{\rho} \frac{d(\hat{H}_z \hat{h}_x)}{dy} &= \nu \left(\frac{d^2 \hat{v}}{dy^2} - \alpha^2 \hat{v} \right) \dots\dots\dots (10) \end{aligned} \right\}$$

$$= -\frac{1}{\rho} \frac{d\hat{p}}{dy} + \nu \left(\frac{d^2 \hat{v}}{dy^2} - \alpha^2 \hat{v} \right) \dots\dots\dots (10)$$

$$\left. \begin{aligned} \omega \hat{w} - \frac{\mu}{\rho} \left\{ H_0 \left(\frac{d\hat{h}_x}{dy} + \alpha \hat{h}_y \right) + \alpha \hat{H}_z \hat{h}_z \right\} &= \nu \left(\frac{d^2 \hat{w}}{dy^2} - \alpha^2 \hat{w} \right) \dots\dots\dots (11) \\ \omega \hat{h}_z + \frac{d\hat{H}_z}{dy} \hat{v} - \frac{dU}{dy} \hat{h}_y - H_0 \frac{d\hat{u}}{dy} &= \nu_H \left(\frac{d^2 \hat{h}_z}{dy^2} - \alpha^2 \hat{h}_z \right) \dots\dots\dots (12) \end{aligned} \right\}$$

$$\left. \begin{aligned} \omega \hat{h}_y - H_0 \frac{d\hat{v}}{dy} &= \nu_H \left(\frac{d^2 \hat{h}_y}{dy^2} - \alpha^2 \hat{h}_y \right) \dots\dots\dots (13) \\ \frac{d\hat{v}}{dy} + \alpha \hat{w} &= 0 \dots\dots\dots (14) \\ \frac{d\hat{h}_y}{dy} + \alpha \hat{h}_z &= 0 \dots\dots\dots (15) \end{aligned} \right\}$$

where $\nu_H (= 1/(\kappa\mu))$ denotes the coefficient of magnetic viscosity. Order estimation of magnitudes in the perturbation equations indicates that the following quantities are of the same order at first sight:

$$\left. \begin{aligned} \frac{\hat{u}}{U_m} \approx R_m \frac{\hat{v}}{U_m} \approx R_m \frac{\hat{w}}{U_m} \approx \frac{1}{R_m} \frac{\hat{h}_z}{H_0} \\ \approx \frac{R_m}{R_m} \frac{\hat{h}_y}{H_0} \approx \frac{R_m}{R_m} \frac{\hat{h}_x}{H_0} \dots\dots\dots (16) \end{aligned} \right\}$$

where $R_m (= U_m y_0 / \nu_H)$ is the magnetic Reynolds number. After eliminating the perturbed quantities $\hat{w}$, $\hat{p}$, $\hat{h}_x$, $\hat{h}_y$ and $\hat{h}_z$ from Eqs. (9) to (15), we introduce the dimensionless expressions defined as follows:

$$\left. \begin{aligned} \bar{U} &= \frac{U}{U_m}, & \bar{u} &= \frac{\hat{u}}{U_m}, & \bar{v} &= R_m \frac{\hat{v}}{U_m} \\ \bar{w} &= R_m \frac{\hat{w}}{U_m}, & \bar{h}_z &= \frac{1}{R_m} \frac{\hat{h}_z}{H_0} \\ \bar{h}_y &= \frac{R_m}{R_m} \frac{\hat{h}_y}{H_0}, & \bar{h}_x &= \frac{R_m}{R_m} \frac{\hat{h}_x}{H_0} \\ \eta &= \frac{y}{y_0}, & \sigma &= \alpha y_0, & K &= \frac{U_m y_0}{\nu} \sqrt{\frac{y_0}{R_0}} \end{aligned} \right\} \dots\dots\dots (17)$$

The magnetic Prandtl number $P_m (= R_m/R_e)$ is small compared with 1 for liquid metals. We now consider the state of neutral stability in which initial perturbations neither grow nor damp with the time. The state is characterized by the condition $\omega = 0$. We reach finally a system of perturbation differential equations:

$$\left. \begin{aligned} \{ (D^2 - \sigma^2)^2 - H_z^2 D^2 \} \bar{u} &= (D^2 - \sigma^2) \{ \bar{v} D \bar{U} \} \\ \{ (D^2 - \sigma^2)^2 - H_z^2 D^2 \} \bar{v} &= -2\sigma^2 K^2 \bar{U} \bar{u} \end{aligned} \right\} \dots\dots\dots (18)$$

where D denotes differential with respect to η and K is the Dean number.

The boundary conditions which follow from the requirement of no slip at the wall surfaces ($\eta = \pm 1.0$) are $\bar{u} = \bar{v} = D\bar{v} = 0$ in view of the equation of continuity. For two electrically insulating walls, the y -component of electric current (J_y) at the wall surfaces is $J_y = (\text{rot } H)_y = 0$, which becomes $\hat{h}_x = 0$ from Eq. (8) and then leads to the condition

$$D^2\bar{u} - (H_a^2 + \sigma^2)D\bar{u} = 0 \quad (19)$$

by using Eqs. (9) and (12) with $P_m \ll 1$. For perfectly conducting walls, the z-component of electric current (J_z) at the wall surfaces is $J_z = (\text{rot } \mathbf{H})_z = 0$, which reduces to $dh_x/dy = 0$ and then the condition

$$D^2\bar{u} = 0 \quad (20)$$

4. Transformation of perturbation equations to integral equations

The centrifugal instability of MHD viscous flow in the curved channel for the onset of a secondary vortex flow is now reduced to an eigenvalue problem to find the Dean number K as eigenvalue together with two components $\bar{u}(\eta)$ and $\bar{v}(\eta)$ of the perturbation velocities as eigenfunctions. In order to solve the eigenvalue problem, we transform the perturbation equations (18) into integral equations as follows:

$$\bar{u}(\eta) = - \int_{-1}^1 \frac{d\bar{U}(\xi)}{d\xi} \bar{v}(\xi) GU_{\pm}(\eta, \xi) d\xi \quad (21)$$

$$\bar{v}(\eta) = 2\sigma^2 K^2 \int_{-1}^1 \bar{U}(\xi) \bar{u}(\xi) GV_{\pm}(\eta, \xi) d\xi \quad (22)$$

where $GU_{\pm}(\eta, \xi)$ and $GV_{\pm}(\eta, \xi)$ are Green function, which are introduced from the reduced system of the differential equations (18) with the corresponding boundary conditions for $\bar{u}$ and $\bar{v}$. The Green functions $GU_{-}(\eta, \xi)$ and $GV_{-}(\eta, \xi)$ for $-1 \leq \eta \leq \xi$ are expressed as follows: for the case of electrically insulating wall,

$$GU_{-}(\eta, \xi) = \frac{1}{2A \sinh A} [\cosh \{A - \lambda_0(\xi - \eta)\} + \cosh \{A - \lambda_*(\xi - \eta)\} - \cosh \{\lambda_0\xi + \lambda_*\eta\} - \cosh \{\lambda_*\xi + \lambda_0\eta\}] \quad (23)$$

and for the case of perfectly conducting wall,

$$GU_{-}(\eta, \xi) = \frac{1}{2A \sinh 2\lambda_0 \sinh 2\lambda_*} \times [\sinh(2\lambda_*) [\cosh \{2\lambda_0 - \lambda_0(\xi - \eta)\} - \cosh \{\lambda_0(\xi + \eta)\}] + \sinh(2\lambda_0) [\cosh \{2\lambda_* - \lambda_*(\xi - \eta)\} - \cosh \{\lambda_*(\xi + \eta)\}]] \quad (24)$$

while for both cases

$$GV_{-}(\eta, \xi) = \frac{1}{\sigma^2 k H_a A} [2\sigma^2 \sinh A \{2\lambda_* \cosh \{A - \lambda_0(\xi - \eta)\} - 2\lambda_0 \cosh \{A - \lambda_*(\xi - \eta)\} + H_a \cosh \{\lambda_0\xi + \lambda_*\eta\} + H_a \cosh \{\lambda_*\xi + \lambda_0\eta\}\} - \lambda_* A \sinh(2\lambda_*) [A \cosh \{2\lambda_0 - \lambda_0(\xi - \eta)\} + H_a \cosh \lambda_0(\xi + \eta)] + \lambda_0 A \sinh(2\lambda_0) \times [A \cosh \{2\lambda_* - \lambda_*(\xi - \eta)\} - H_a \cosh \lambda_*(\xi + \eta)] - 2\sigma^2 A \sinh H_a \{ \cosh \{\lambda_*\xi - \lambda_0\eta\} + \cosh \{\lambda_0\xi - \lambda_*\eta\} \}] \quad (25)$$

in which

$$\left. \begin{aligned} A &= \sqrt{H_a^2 + 4\sigma^2} \\ \lambda_0 &= \frac{1}{2}(A + H_a), \quad \lambda_* = \frac{1}{2}(A - H_a) \end{aligned} \right\} \quad (26)$$

$$k = 8\sigma^2 \sinh^2 A - 2A^2 \sinh 2\lambda_0 \sinh 2\lambda_*$$

In view of the fact that the reduced system of the differential equations (18) is self-adjoint and that the obtained Green functions possess symmetry, we can introduce the other Green functions $GU_{+}(\eta, \xi)$ and $GV_{+}(\eta, \xi)$ for $\xi \leq \eta \leq 1$ directly from the Green functions obtained above for $-1 \leq \eta \leq \xi$ by using the relations:

$$\left. \begin{aligned} GU_{+}(\eta, \xi) &= GU_{-}(\xi, \eta) \\ GV_{+}(\eta, \xi) &= GV_{-}(\xi, \eta) \end{aligned} \right\} \quad (27)$$

5. Results and discussions

The simultaneous integral equations (21) and (22) were numerically solved for various values of the dimensionless wavenumber σ and the Hartmann number H_a by iterative procedure. The larger part of the numerical calculations was carried out using the digital computer NEAC-2200/700 in the Computer Center of Tohoku University. The component $\bar{w}(\eta)$ of the perturbation velocity can be obtained from the relation $\bar{w} = -(\bar{d}\bar{v}/d\eta)/\sigma$, which is a dimensionless expression of Eq. (14).

Figure 4 shows neutral stability curves in relation of the Dean number K to the dimensionless wavenumber σ of the vortices for y_0/R_0 small in a case of the electrically insulating wall. The stable range is below the neutral curve for each value of the Hartmann number H_a . The figure includes the previous results for ordinary fluid ($k=0$,

Fig. 4 Neutral stability curves in relation between Dean number K and dimensionless wavenumber σ of vortices (insulating wall, $y_0/R_0 \rightarrow 0$)
Curve a: critical condition for various values of Hartmann number H_a
 $H_a = 0$: $\circ$ Dean⁽¹⁾, $\times$ Hämmerlin⁽⁴⁾, $\triangle$ Reid⁽³⁾ (first approximation), $\bullet$ Brewster et al.⁽⁵⁾ (experiment)

$H_a = 0$). It can be noticed from Fig. 4 that the stable range becomes larger as the Hartmann number H_a is increased. The curve gives the critical Dean number (K_C), which is the minimum value of the Dean number K as a function of the wavenumber σ for each value of H_a .

Figures 5 and 6 show that the critical Dean number K_C and the associated wavenumber σ_C become larger with an increasing Hartmann number H_a in the two cases of the electrically insulating wall and the completely conducting wall. It should be noted that K_C and σ_C for $y_0/R_0 = 0.1$ in these figures may have less accuracy, since the approxi-

Fig. 5 Critical Dean number K_C and the associated wavenumber σ_C as a function of Hartmann number H_a (insulating wall)
 —: $y_0/R_0 \rightarrow 0$, ---: $y_0/R_0 = 0.01$
 - · - ·: $y_0/R_0 = 0.1$, · · · ·: magnetic field applied in the z -direction⁽⁹⁾ ($y_0/R_0 \rightarrow 0$)

Fig. 6 Critical Dean number K_C and the associated wavenumber σ_C as a function of Hartmann number H_a (perfectly conducting wall)
 —: $y_0/R_0 \rightarrow 0$, ---: $y_0/R_0 = 0.01$
 - · - ·: $y_0/R_0 = 0.1$, · · · ·: magnetic field applied in the z -direction⁽⁹⁾ ($y_0/R_0 \rightarrow 0$)

mation $y_0/R_0 \ll 1$ was adopted in Eqs. (21) and (22), although the exact formula (4) was used as primary velocity profile. The previous results of K_C and σ_C for y_0/R_0 small in a case of ordinary fluid ($H_a = 0$) were, respectively, 12.7 and 1.98 by Dean,⁽¹⁾ 12.705 and 1.98 by Reid,⁽³⁾ 12.6 and 2 by Hämmerlin,⁽⁴⁾ and also the experiment by Brewster et al.⁽⁵⁾ gave 12.9 ± 0.4 and 2.5 ± 0.5 , while the present results are 12.70 and 1.98. The broken lines in Figs. 5 and 6 indicate the results obtained by Chandrasekhar et al.⁽⁹⁾ for a case where a uniform external magnetic field is applied in the direction of the z -axis and for small y_0/R_0 . It is clear from the comparison of the critical Dean number K_C that the application of the magnetic field in the y -direction gives a more remarkable stabilizing effect than that in the z -direction at the same value of the Hartmann number H_a . The similar results had been obtained for MHD boundary layer along a concave wall.⁽⁶⁾ ~ (8) The critical Dean number K_C varies little for the Hartmann number H_a ranging from 0 to about 1, while it begins to increase considerably as H_a becomes larger than 2. The associated wavenumber σ_C of the vortices also increases with H_a . This behavior of σ_C is in contrast to that⁽⁹⁾ for the magnetic field in the z -direction, in the case of which σ_C decreases as H_a is increased. It can also be said that deviations of the critical Dean number K_C for the curvature $y_0/R_0 < 10^{-2}$ from the value of K_C for $y_0/R_0 \rightarrow 0$ remain less than 2.8% when $H_a \leq 50$, and that the critical values K_C and σ_C for the conducting wall are greater than those for the insulating wall.

In Fig. 7, the momentum thickness (θ/y_0), calculated from the unperturbed velocity distribution, is shown as a function of the Hartmann number H_a , in which the Hartmann number $H_a\theta$, the Dean number $K_C\theta$ and the associated wavenumber $\sigma_C\theta$, based on the momentum thickness θ instead of y_0 , are also

Fig. 7 Momentum thickness θ/y_0 of the primary flow, the critical Dean number $K_C\theta$ and the associated wavenumber $\sigma_C\theta$ as well as Hartmann number $H_a\theta$ (based on the momentum thickness θ) as functions of Hartmann number H_a ($y_0/R_0 \rightarrow 0$)
 Curve a: insulating wall
 Curve b: perfectly conducting wall

presented. As $H_a \rightarrow \infty$, the momentum thickness becomes thinner and thinner with a relation $\theta/y_0 = 1/(2H_a)$, and then the modified Hartmann number $H_a\theta$ tends to 0.5. In this limiting case, the primary flow field consists of a mid core with a constant flow velocity U_{max} and two boundary layers of the momentum thickness θ with the velocity distribution

$$\frac{U}{U_{max}} = 1 - e^{-y/2\theta} \dots (28)$$

The modified critical Dean numbers $K_C\theta$, for $H_a\theta = 0.5$ are 0.77 and 1.10 for the insulating wall and for the completely conducting wall, respectively, and the associated wave-number $\sigma_C\theta$ becomes 0 and 0.23. It should be noticed that these values are similar to the previous results of MHD boundary layer problem.^{(8)*} As $H_a \rightarrow \infty$, the critical Dean number $K_C (= Re\sqrt{y_0/R_0})$ tends to a relation proportional to $H_a^{3/2}$; that is,

$$K_C = 2.2H_a^{3/2} \dots (29)$$

for the insulating wall, and

$$K_C = 3.1H_a^{3/2} \dots (30)$$

for the completely conducting wall. Lock⁽¹⁰⁾ showed for a Hartmann flow through a straight channel (a uniform magnetic field is applied in the direction perpendicular to the channel walls) that the critical Reynolds number $Re_C (= U_{max}y_0/\nu)$, at which the flow becomes unstable for disturbances of Tollmien-Schlichting waves, increases with the Hartmann number H_a in a relation: $Re_C = 5 \times 10^4 H_a$ for $H_a > 20$. Experiments gave less influence of H_a ; for example, transition Reynolds number $Re_C \approx 225 H_a$.⁽¹¹⁾ From a comparison of both the instabilities, we can, therefore, say that the critical Dean number K_C for the onset of a secondary vortex flow increases much less with the Hartmann number H_a than the critical Reynolds number Re_C for the Tollmien-Schlichting waves.

Figures 8(a), (b) and (c) show the distributions of the perturbation amplitudes $\bar{u}(\eta)$, $\bar{v}(\eta)$ and $\bar{w}(\eta)$ for the insulating wall at the critical state (corresponding to the curve a in Fig.4)

* The MHD boundary layer study⁽⁸⁾ gives $K_C\theta = 0.70$ and 1.00 and also $\sigma_C\theta = 0$ and 0.25 for the insulating wall and the completely conducting wall, respectively, in which the Blasius profile was adopted as the unperturbed velocity distribution. It is clear from Figs.13 and 14 of the paper that a variation of the unperturbed velocity distribution from the Blasius profile to the asymptotic profile (Eq. (28)) leads to a slight increase of the critical value $K_C\theta$.

where a vortical motion ensues. The similar distributions were also obtained for the completely conducting wall. In these figures, the ordinates are normalized by reference to the maximum value $|\bar{u}|_{max}$ and $\bar{v}_{max}$, respectively, and $\bar{w}$ to $\bar{v}_{max}$. The ratio of the two maximum values $\bar{v}_{max}/|\bar{u}|_{max}$ ($= Re\bar{v}_{max}/|\bar{u}|_{max}$), which is shown in Fig.9, becomes larger as the Hartmann number H_a is increased. Fig.8 shows that the location of $|\bar{u}|_{max}$ and $\bar{v}_{max}$ approaches the outer concave wall ($\eta = -1.0$) of the curved channel with an increasing value of the Hartmann number H_a . It comes from the fact that the region having positive velocity gradient $dU/d\eta > 0$, where the centrifugal instability is induced, is limited closer to the outer wall, as seen in Fig.1. The center of the vortices is located at the point (η_C) where $\bar{w}(\eta) = 0$ in Fig.8(c). Figure 10 shows the location η_C of the vortex center with relation to the Hartmann number H_a . Figure 11 illustrates streamlines of the secondary vortex flow in the y - z plane for $H_a = 0.1$ and 10, which are obtained using the stream-

Fig.8 Perturbation velocity distributions $\bar{u}(\eta)$, $\bar{v}(\eta)$ and $\bar{w}(\eta)$ at the onset of secondary vortex flow under critical condition (curve a in Fig.4) (insulating wall)

— : $y_0/R_0 \rightarrow 0$, — — : $y_0/R_0 = 0.01$
 - · - : $y_0/R_0 = 0.1$

Fig. 9 Ratio $\bar{v}_{max}/|\bar{u}|_{max}$ of the perturbation velocities under critical condition
 Curve a: insulating wall
 Curve b: perfectly conducting wall
 — : $y_0/R_0 \rightarrow 0$, — — : $y_0/R_0 = 0.01$
 - · - : $y_0/R_0 = 0.1$

Fig. 10 Variations of the location η_c of vortices with the Hartmann number H_a
 Curve a: insulating wall
 Curve b: perfectly conducting wall
 — : $y_0/R_0 \rightarrow 0$, — — : $y_0/R_0 = 0.01$
 - · - : $y_0/R_0 = 0.1$

Fig. 11 Streamlines of the secondary vortex flow appearing under the critical condition (insulating wall, $y_0/R_0 \rightarrow 0$)
 a: inner wall, b: outer wall
 - - - : $H_a = 0.1$, — : $H_a = 10$

function calculated from Fig. 8(b). It is again evident from these figures 10 and 11 that the location η_c of the vortex center approaches the outer concave wall and the wavelength of the vortices shortens as the Hartmann number H_a is increased.

6. Conclusions

For electrically conducting viscous flows through a two-dimensional, slightly curved channel, it was considered by using the linear stability theory how a magnetic field, applied in a direction perpendicular to the channel walls, affects the centrifugal instability for the onset of a secondary vortex flow.

The results are summarized as follows:

(1) In the stability diagram, which is related with the Dean number K and the dimensionless wavenumber σ of the secondary vortex flow, the stable range becomes larger as the Hartmann number H_a is increased, as shown in Fig. 4, and the primary flow is stabilized by the magnetic field.

(2) The critical Dean number K_c increases remarkably as the Hartmann number H_a becomes larger than 2, and tends to a relation $K_c \propto H_a^{3/2}$ (Figs. 5 and 6). The associated wavenumber σ_c becomes also larger. As $H_a \rightarrow \infty$, the modified critical Dean number $K_{c\theta}$ and the wavenumber $\sigma_{c\theta}$, based on the momentum thickness θ , tend to a constant value, respectively (Fig. 7).

(3) The critical Dean number K_c and the wavenumber σ_c are larger for the electrically conducting wall than for the insulating wall. The increase in K_c with H_a is more remarkable in the present case of the magnetic field applied perpendicular to the channel walls, compared with the case of axial magnetic field (Figs. 5 and 6).

(4) The center of vortices approaches the outer concave wall as the Hartmann number H_a becomes large (Fig. 10). It comes from the fact that the velocity profile of the primary flow is flattened, so that the region of the positive velocity gradient which induces the centrifugal instability is limited closer to the outer wall.

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