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Dynamic Responses of Circular Cylindrical Shells Having
Finite Lengths Subjected to Concentrated Impulsive Loads*

By Sadayuki UJIHASHI**, Masanori KODAIRA***

Hiroyuki MATSUMOTO**, Ichiro NAKAHARA**

In this paper, the dynamic displacements and stresses in circular cylindrical shells having finite lengths with both ends free and both ends clamped, which are suddenly subjected to equal and opposite radial concentrated loads, are analytically investigated on the basis of Donnell's shell equations including the effect of radial inertia force.

The values of the dynamic factors at the loaded point, defined by the ratio of the dynamic maximum radial displacement to the static one, decrease as the cylindrical shells become long, but do not coincide with the result in the case of an infinite cylindrical shell unless the lengths are sufficiently great.

Applying the method of solution proposed in this paper, the dynamic responses of finite circular cylindrical shells may be analyzed under arbitrary loading and boundary conditions.

1. Introduction

The basic equations for thin-walled circular cylindrical shells were originally derived by Love [1] and were presented in the form most frequently used today by Flügge [2]. Later, considerably simplified sets of equations were derived by Donnell [3] and Morley [4]. There are indications that, under most conditions of loading, the displacements and stresses calculated from these simplified equations differ little from those deducible from the generally accepted equations.

But there is even some doubt whether the governing equations of Donnell and Morley have always the same accuracy as the classical equations. Therefore, the accuracy of these simplified equations was examined by using various practical methods.

Hoff [5] and Morley [4] commented on the accuracy of Donnell's equations using Flügge's equations as the standard of comparison that as the characteristic circumferential wavelength increases so does the error in Donnell's equations.

And Kempner [6] showed that, for short cylindrical shells with small regions of their surface loaded, the simple Donnell's equations are quite adequate for the determination of the displacements and stresses.

For a concrete example, Yuan [7],[8] had used Donnell's and Flügge's equations to determine the radial displacement of a cylindrical shell subjected to equal and opposite radial concentrated loads and discovered that the solutions from Donnell's equations give a result some 25 percent erroneous for the

case of infinite length.

On the other hand, concerning the dynamic problem, Yao [9] presented an analytical study of non-axisymmetrical responses of simply-supported cylindrical shells to locally distributed impulsive loads on the basis of Morley's equations including the effect of radial inertia force.

Similarly, Liao [10] investigated the dynamic responses of simply-supported cylindrical shells to general three-dimensional non-axisymmetric loads, which are arbitrary functions of time, by the use of Donnell's and Flügge's equations including the effects of axial, circumferential and radial inertia forces and showed that Donnell's equations are satisfactorily accurate as compared with Flügge's equations for the case of thin-walled and short cylindrical shells.

Waingarten [11] also treated the dynamic responses of simply-supported cylindrical shells within the framework of the three-dimensional theory of elasticity and Flügge's shell theory and showed that the accurate shell theory is an excellent approximation in the specific numerical examples considered.

In these mentioned papers, dynamic displacements and stresses are represented by eigenfunctions with trigonometric expression in the axial and circumferential directions, so that the method of solution is restricted to the analysis of circular cylindrical shells whose ends are simply supported.

Concerning the static problems, Hoff [12] and Ting [14] proposed a method for obtaining displacements and stresses in cylindrical shells under arbitrary boundary conditions.

However, in these papers, numerical results are not depicted.

Previously, the authors [13] analyzed the dynamic displacements and stresses in an infinite circular cylindrical shell, which is suddenly subjected to equal and opposite radial loads, on the basis of Flügge's and Donnell's equations and found

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** Faculty of Engineering,
Tokyo Institute of Technology,
2-12-1, Ookayama, Meguro-ku, Tokyo.

*** Ebara Factory Co. Ltd.

that the inertia forces in the axial and circumferential directions of the shell are not effective on the results and, for predicting the transient responses of the shell, Donnell's equations are satisfactorily accurate as compared with Flügge's equations.

The purposes of the present paper are to analyze the dynamic displacements and stresses in circular cylindrical shells, having finite lengths with both ends free and both ends clamped, which are suddenly subjected to equal and opposite radial concentrated loads by the use of Donnell's equations and to propose a method for analyzing dynamic stresses in cylindrical shells of finite lengths under arbitrary loading and boundary conditions and demonstrate the effect of the lengths of shells on the results.

2. Fundamental Equations

Stress resultants in a cylindrical shell are defined as shown in Fig.1 and (u, v, w) are the displacement components corresponding to the cylindrical coordinates (x, θ, z) for the midsurface of the shell. And R is the mean radius, h is the thickness, E is Young's modulus, ν is Poisson's ratio and γ is weight per unit volume of the cylindrical shell and g is the acceleration of gravity and t is time.

Then, the fundamental equations for a cylindrical shell employed in this paper are written in the following way.

For the analysis of axisymmetric motions, a single differential equation for the radial displacement including the effect of radial inertia force is given by

$$\frac{\partial^4 w}{\partial \xi^4} + J^2 w + \frac{1}{\kappa^2} \frac{\partial^2 w}{\partial \tau^2} = 0 \quad \dots (1)$$

where, $\xi = x/R$, $\tau = ct/R$, $J^2 = 12(1-\nu^2)R^2/h^2$, $c^2 = gE/\gamma(1-\nu^2)$, $\kappa^2 = h^2/12R^2$.

For the analysis of non-axisymmetric motions, Donnell's equations including the effect of radial inertia force are given in the following single form for the radial displacement.

$$\nabla^8 w + J^2 \frac{\partial^2 w}{\partial \xi^4} + \frac{1}{\kappa^2} \frac{\partial^2}{\partial \tau^2} \nabla^4 w = 0 \quad \dots (2)$$

Fig.1 Stress Resultants In a Cylindrical Shell

where, $\nabla^2 = \partial^2 / \partial \xi^2 + \partial^2 / \partial \theta^2$ and the relations between the three displacement components are written in the form

$$\left. \begin{aligned} \nabla^4 u &= \nu \frac{\partial^3 w}{\partial \xi^3} - \frac{\partial^3 w}{\partial \xi \partial \theta^2} \\ \nabla^4 v &= (2+\nu) \frac{\partial^3 w}{\partial \xi^2 \partial \theta} + \frac{\partial^3 w}{\partial \theta^3} \end{aligned} \right\} \dots (3)$$

3. Method of Solution

As shown schematically in Fig.2, consider a circular cylindrical shell of finite length $2l$ which is suddenly subjected to equal and opposite radial concentrated loads at the center of the surface.

Suppose the boundary conditions at both ends are equal to each other, in consideration of symmetry of the problem, then the boundary conditions may be prescribed at the planes of $x=0$ and $x=l$. Thus, the boundary conditions at the plane of $x=0$ in this problem are given by

$$\left. \begin{aligned} u &= \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0 \\ R T_x &= - \frac{F_0}{2} \{ \delta(\theta) + \delta(\theta-\pi) \} H(t) \end{aligned} \right\} \dots (4)$$

where, T_x denotes the transverse effective shearing force at the edge perpendicular to the axial direction, $\delta(\theta)$ is Dirac's delta function and $H(t)$ is Heaviside's step function. Expanding Eqs.(4) into infinite Fourier series, the second equation becomes

$$T_x = - \frac{q_0}{2} \left\{ \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=2,4,\dots}^{\infty} \cos(n\theta) \right\} H(t) \dots (5)$$

where, $q_0 = F_0/R$.

As Eq. (5) consists of a constant term and cosine terms, the concentrated loads in Fig.2 are represented as a superposition of a uniformly distributed line load and line loads varying in the form of trigonometric functions along the circumferential direc-

Fig.2 Equal and Opposite Radial Concentrated Loads Acting On a Finite Cylindrical Shell

tion as shown in Fig.3.

On the other hand, for the boundary conditions at the end of $x=l$, the following two kinds of conditions are employed.

(I) Both Ends Free

$$M_x = N_x = T_x = S_x = 0 \quad \dots\dots\dots (6)$$

(II) Both Ends Clamped

$$u = \text{const.}, \quad v = w = \frac{\partial w}{\partial x} = 0 \quad \dots (7)$$

where, $T_x = Q_x - \partial M_{x\theta} / R \partial \theta$, $S_x = N_{x\theta} + M_{x\theta} / R$

3.1 Analysis for Line Loads Varying in the Form of Trigonometric Functions

As shown in Fig.3(b), a radial line load circumferentially varying in the form of trigonometric function, whose magnitude is $q_0 \cos(n\theta)$, is suddenly applied to the meridian through the center of a circular cylindrical shell having finite length $2l$ with both ends free. The boundary conditions at the planes of $x=0$ and $x=l$ can be written in the following way.

(i) at $x=0$

$$u = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0, \quad T_x = -\frac{q_0}{2} \cos(n\theta) H(t) \dots(8)$$

(ii) at $x=l$

$$M_x = N_x = T_x = S_x = 0 \quad \dots\dots\dots (9)$$

The solutions of separation of variables for displacement components may be sought in the following form.

$$u = U_n \cos(n\theta), \quad v = V_n \sin(n\theta), \quad w = W_n \cos(n\theta) \quad \dots\dots\dots (10)$$

where, (U_n, V_n, W_n) are functions of axial coordinate x and time t . The shell is at rest and in an unstrained state, then the initial conditions are given by

$$(W_n)_{t=0} = (\partial W_n / \partial t)_{t=0} = 0 \quad \dots\dots (11)$$

Substituting the third equation of Eqs.(10) into Eq.(2) and employing the Laplace transformation, the following ordinary differential equation is derived.

$$\frac{d^8 \bar{W}_n}{d\xi^8} - 4n^2 \frac{d^6 \bar{W}_n}{d\xi^6} + (6n^4 + J^2 + \frac{\lambda^2}{\kappa^2}) \frac{d^4 \bar{W}_n}{d\xi^4} - (4n^6 + 2n^2 \frac{\lambda^2}{\kappa^2}) \frac{d^2 \bar{W}_n}{d\xi^2} + (n^8 + n^4 \frac{\lambda^2}{\kappa^2}) \bar{W}_n = 0 \quad \dots (12)$$

$$\text{where, } \bar{f} = \int_0^\infty f e^{-pt} dt, \quad \lambda = Rp/c$$

The general solutions of the Laplace transformations of the functions (U_n, V_n, W_n) can be introduced as follows

$$\bar{U}_n = \sum_{j=1}^8 A_j e^{\alpha_j \xi}, \quad \bar{V}_n = \sum_{j=1}^8 B_j e^{\alpha_j \xi}, \quad \bar{W}_n = \sum_{j=1}^8 C_j e^{\alpha_j \xi} \quad \dots\dots\dots (13)$$

where, α_j is the root of the following characteristic equation of eighth degree.

$$\alpha^8 - 4n^2 \alpha^6 + (6n^4 + J^2 + \lambda^2/\kappa^2) \alpha^4 - (4n^6 + 2n^2 \lambda^2/\kappa^2) \alpha^2 + n^8 + n^4 \lambda^2/\kappa^2 = 0 \quad \dots\dots (14)$$

Three coefficients (A_j, B_j, C_j) are determined from the mentioned boundary conditions, whereas those are not independent of one another. Substituting Eqs.(10) and (13) into Eqs.(3), the relations between the three coefficients are derived as follows

$$\left. \begin{aligned} (\alpha_j^2 - n^2)^2 A_j &= (v \alpha_j^3 + n^2 \alpha_j) C_j \\ (\alpha_j^2 - n^2)^2 B_j &= \{ n^3 - (2+v) n \alpha_j^2 \} C_j \end{aligned} \right\} \dots(15)$$

Substituting the relations between the stress resultants and the displacement components[3] into the boundary conditions, i.e., Eqs.(8) and (9), the Laplace transformations of the boundary conditions are obtained in the following way.

(i) at $x=0$

Fig.3 Line Loads Acting On a Finite Cylindrical Shell

(a) Uniformly Distributed Line Load

(b) Line Load Varying in the Form of Trigonometric Function

$$\left. \begin{aligned} \bar{U}_n &= \frac{d\bar{V}_n}{d\xi} = \frac{d\bar{W}_n}{d\xi} = 0 \\ \frac{d^3\bar{W}_n}{d\xi^3} &= \frac{q_0}{p} \frac{R^3}{2D} \end{aligned} \right\} \dots\dots\dots (16)$$

(ii) at $x = \ell$

$$\left. \begin{aligned} \frac{d^2\bar{W}_n}{d\xi^2} - \nu n^2 \bar{W}_n &= 0 \\ \frac{d\bar{U}_n}{d\xi} + \nu n \bar{V}_n - \nu \bar{W}_n &= 0 \\ \frac{d^3\bar{W}_n}{d\xi^3} - n^2(2-\nu) \frac{d\bar{W}_n}{d\xi} &= 0 \\ n\bar{U}_n - \frac{d\bar{V}_n}{d\xi} + 2\kappa^2 n \frac{d\bar{W}_n}{d\xi} &= 0 \end{aligned} \right\} \dots\dots\dots (17)$$

where, $D = Eh^3/12(1-\nu^2)$.

Substituting Eqs.(13) and (15) into Eqs.(16) and (17), the set of eight linear algebraic equations in the eight undetermined coefficients C_j ($j=1,\dots,8$) is given by

$$\begin{pmatrix} a_{11} & \dots & a_{18} \\ \vdots & & \vdots \\ a_{81} & \dots & a_{88} \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_8 \end{pmatrix} = \frac{q_0}{p} \frac{R^3}{2D} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \dots\dots\dots (18)$$

where, the element a_{ij} is defined in the following form and the symbol $[A]$ denotes the matrix consisting of this element a_{ij} .

$$\begin{aligned} a_{1j} &= (\alpha_j^2 - \nu n^2) e^{\alpha_j L} \\ a_{2j} &= \{\alpha_j^2 - n^2(2-\nu)\} \alpha_j e^{\alpha_j L} \\ a_{3j} &= \alpha_j^2 e^{\alpha_j L} / (\alpha_j^2 - n^2)^2 \\ a_{4j} &= (1+\nu) \alpha_j^3 e^{\alpha_j L} / (\alpha_j^2 - n^2)^2 + \kappa^2 \alpha_j e^{\alpha_j L} \\ a_{5j} &= \alpha_j / (\alpha_j^2 - n^2)^2, \quad a_{6j} = \alpha_j^3 / (\alpha_j^2 - n^2)^2 \\ a_{7j} &= \alpha_j, \quad a_{8j} = \alpha_j^3, \quad L = \ell/R \end{aligned}$$

Solving Eq.(18), the solutions of the Laplace transformations of displacement components ($\bar{U}_n, \bar{V}_n, \bar{W}_n$) are easily obtained.

The Laplace inverse transformations of the solutions ($\bar{U}_n, \bar{V}_n, \bar{W}_n$) are determined as the sum of all the residues of simple poles at the origin and on the imaginary axis. Thus, putting the poles as 0 and $\pm i p_m$ ($0 < p_0 < p_1 < p_2 < p_3 < \dots$), for example, the inverse transformation of $\bar{W}_n$ is

given by the equation

$$\frac{2D}{q_0 R^3} \bar{W}_n = \frac{\sum_{j=1}^8 A_{8j}(\alpha_j^*) e^{\alpha_j \xi}}{|A(\alpha^*)|} - \frac{\cos(\kappa n^2 \tau)}{2Ln^4} - 4\kappa^2 \sum_{m=1}^{\infty} \frac{\cos(\lambda_m \tau)}{\lambda_m^2} \frac{\sum_{j=1}^8 A_{8j}(\alpha_m^*) e^{\alpha_m j \xi}}{\sum_{j=1}^8 |B_j(\alpha_m^*)|} \dots\dots\dots (19)$$

where, A_{8j} denotes the cofactor of the element a_{ij} in the determinant $|A|$ of the matrix $[A]$ and the determinant $|B_j|$ is defined as follows

$$|B_j(\alpha_m^*)| = \begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_{1j} & a_{1,j+1} & \dots & a_{18} \\ a_{21} & \dots & a_{2,j-1} & b_{2j} & a_{2,j+1} & \dots & a_{28} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{71} & \dots & a_{7,j-1} & b_{7j} & a_{7,j+1} & \dots & a_{78} \\ a_{81} & \dots & a_{8,j-1} & b_{8j} & a_{8,j+1} & \dots & a_{88} \end{vmatrix}$$

where,

$$\begin{aligned} b_{1j} &= (\nu Ln^2 - 2\alpha_{mj} - L\alpha_{mj}^2) e^{\alpha_{mj} L} \beta_m \\ b_{2j} &= \{n^2(2-\nu)(1+L\alpha_{mj}) - 3\alpha_{mj}^2 - L\alpha_{mj}^3\} e^{\alpha_{mj} L} \beta_m \\ b_{3j} &= (2n^2 + Ln^2 \alpha_{mj} + 2\alpha_{mj}^2 - L\alpha_{mj}^3) \alpha_{mj} e^{\alpha_{mj} L} \beta_m \\ &\quad / (\alpha_{mj}^2 - n^2)^3 \\ b_{4j} &= (1+\nu)(3n^2 + Ln^2 \alpha_{mj} + \alpha_{mj}^2 - L\alpha_{mj}^3) \alpha_{mj}^2 e^{\alpha_{mj} L} \beta_m \\ &\quad / (\alpha_{mj}^2 - n^2)^3 - \kappa^2 (1+L\alpha_{mj}) e^{\alpha_{mj} L} \beta_m \\ b_{5j} &= (n^2 + 3\alpha_{mj}^2) \beta_m / (\alpha_{mj}^2 - n^2)^3 \\ b_{6j} &= (3n^2 + \alpha_{mj}^2) \alpha_{mj}^2 \beta_m / (\alpha_{mj}^2 - n^2)^3, \\ b_{7j} &= -\beta_m, \quad b_{8j} = -3\alpha_{mj}^2 \beta_m \\ \beta_m &= (\alpha_{mj}^2 - n^2)^2 / \alpha_{mj} \{2(\alpha_{mj}^2 - n^2)^3 + J^2 \alpha_{mj}^2 \\ &\quad - \lambda_m^2 (\alpha_{mj}^2 - n^2) / \kappa^2\} \end{aligned}$$

The values of the pole p_m are determined from the roots of the transcendental equation $|A| = 0$ and coincide with the natural frequencies of a cylindrical shell having finite length 2ℓ with both ends free, and α_{mj} is the root of the following characteristic equation of eighth degree.

$$\alpha^8 - 4n^2 \alpha^6 + (6n^4 + J^2 - \lambda_m^2 / \kappa^2) \alpha^4 - (4n^6 - 2n^2 \lambda_m^2 / \kappa^2) \alpha^2 + n^8 - n^4 \lambda_m^2 / \kappa^2 = 0 \dots\dots\dots (20)$$

where, $\lambda_m = R p_m / c$.

α_j denotes the root in the case that $\lambda_m = 0$ and α_m^* means a function of α_{mj} ($j = 1, \dots, 8$). The first term of Eq.(19) is derived from the residue at the origin and coincides with the solutions of the corresponding static problem. The second and third terms are derived from the residues respectively at the pole $\lambda = \pm i \lambda_0 = \pm i \kappa n^2$, i.e., in the case that $\alpha = 0$, and at the other poles $\lambda = \pm i \lambda_m$ ($m=1, 2, 3, \dots$).

In the results, the solution is expressed as a superposition of the static solution and the solutions for all the modes of the natural vibrations of the shell. The other solutions for U_n and V_n are readily derived by the use of the relations of Eqs.(15) and the solutions for stress resultants are easily obtained from the representation of the differentiated formulas of displacement components.

Next, in the case that both ends are clamped, the boundary conditions at the plane of $x=l$, i.e., Eqs.(9), are replaced by the equations

(ii) at $x=l$

$$u = v = w = \partial w / \partial x = 0 \quad \dots \quad (21)$$

Applying the method of solution mentioned in the former analysis, the solution for the radial displacement in this case becomes

$$\frac{2D}{q_0 R^3} W_n = \frac{\sum_{j=1}^8 A_{8j}(\alpha^*) e^{\alpha_j \xi}}{|A(\alpha^*)|} - 4\kappa^2 \sum_{m=1}^{\infty} \frac{\cos(\lambda_m \tau)}{\lambda_m^2} \frac{\sum_{j=1}^8 A_{8j}(\alpha_m^*) e^{\alpha_{mj} \xi}}{\sum_{j=1}^8 |B_j(\alpha_m^*)|} \quad \dots \quad (22)$$

where, the definitions of notations are the same as those in Eq.(19) and the elements a_{ij} and b_{ij} in the determinants $|A|$ and $|B_j|$ are defined in the following way.

$$\begin{aligned} a_{1j} &= e^{\alpha_{mj} L}, \quad a_{2j} = \alpha_{mj} e^{\alpha_{mj} L} \\ a_{3j} &= (n^2 + \nu \alpha_{mj}^2) \alpha_{mj} e^{\alpha_{mj} L} / (\alpha_{mj}^2 - n^2)^2 \\ a_{4j} &= \{n^2 - (2 + \nu) \alpha_{mj}^2\} e^{\alpha_{mj} L} / (\alpha_{mj}^2 - n^2)^2 \\ b_{1j} &= -L e^{\alpha_{mj} L} \beta_m, \quad b_{2j} = -(1 + L \alpha_{mj}) e^{\alpha_{mj} L} \beta_m \\ b_{3j} &= [n^4 (1 + L \alpha_{mj}) + \{3(1 + \nu) - (1 - \nu) L \alpha_{mj}\} n^2 \alpha_{mj}^2 \\ &\quad + \nu (1 - L \alpha_{mj}) \alpha_{mj}^4] e^{\alpha_{mj} L} \beta_m / (\alpha_{mj}^2 - n^2)^2 \\ b_{4j} &= [L n^4 - \{2\nu + (3 + \nu) L \alpha_{mj}\} n^2 \alpha_{mj}^2 \\ &\quad - 2(2 + \nu)(1 - L \alpha_{mj}) \alpha_{mj}^3] e^{\alpha_{mj} L} \beta_m / (\alpha_{mj}^2 - n^2)^2 \end{aligned}$$

and the others are the same as those in the former solution.

3.2 Analysis for a Uniformly Distributed Line Load

As shown in Fig.3(a), a radial line load uniformly distributed along the circumferential direction, whose magnitude is q_0 , is suddenly applied to the meridian through the center of a circular cylindrical shell having finite length $2l$ with both ends free. The boundary conditions at the planes of $x=0$ and $x=l$ can be written in the following way.

(i) at $x=0$

$$\frac{\partial w}{\partial x} = 0, \quad Q_x = -\frac{q_0}{2} H(t) \quad \dots \quad (23)$$

(ii) at $x=l$

$$M_x = Q_x = 0 \quad \dots \quad (24)$$

On the basis of the fundamental equation (1), applying the method of solution mentioned in the previous article, the solution for the radial displacement in this problem is derived as follows

$$\frac{2D}{q_0 R^3} w = \frac{1}{4a^3 M_0} \frac{g(\xi)}{L J^2} - \frac{\cos(\kappa J \tau)}{L J^2} - \frac{\kappa^2}{L} \sum_{m=1}^{\infty} \frac{f_1 \cosh(\alpha_m \xi) + f_2 \cos(\alpha_m \xi)}{\lambda_m^2 \cosh(\alpha_m L) \cos(\alpha_m L)} \cos(\lambda_m \tau) \quad \dots \quad (25)$$

$$\begin{aligned} \text{where, } g(\xi) &= \cosh^2(aL) (\cosh(a\xi) \cos(a\xi) \\ &\quad - \sinh(a\xi) \sin(a\xi)) + \cos^2(aL) (\cosh(a\xi) \cos(a\xi) \\ &\quad + \sinh(a\xi) \sin(a\xi)) + M_0 (\cosh(a\xi) \sin(a\xi) \\ &\quad - \sinh(a\xi) \cos(a\xi)), \quad a = \sqrt{J/2} \\ M_0 &= \cosh(aL) \sinh(aL) + \cos(aL) \sin(aL) \\ f_1 &= 1 + \cosh(\alpha_m L) \cos(\alpha_m L) + \sinh(\alpha_m L) \sin(\alpha_m L) \\ f_2 &= 1 + \cosh(\alpha_m L) \cos(\alpha_m L) - \sinh(\alpha_m L) \sin(\alpha_m L) \end{aligned}$$

The values of α_m are determined from the roots of the following transcendental equation

$$\tanh(\alpha L) + \tan(\alpha L) = 0 \quad \dots \quad (26)$$

and the relation between λ_m and α_m is given by

$$\lambda_m = \kappa \sqrt{\alpha_m^4 + J^2} \quad \dots \quad (27)$$

Next, in the case that both ends are clamped, the boundary conditions at the plane of $x=l$, i.e., Eqs.(24), are replaced by the equations

(ii) at $x=l$

$$w = \partial w / \partial x = 0 \quad \dots \quad (28)$$

Then, the solution for the radial displacement in this case becomes

$$\frac{2D}{q_0 R^3} w = \frac{1}{4a^3} \frac{g(\xi)}{M_0} - \frac{\kappa^2}{L} \sum_{m=1}^{\infty} \frac{f_1 \cosh(\alpha_m \xi) - f_2 \cos(\alpha_m \xi)}{\lambda_m^2 \cosh(\alpha_m L) \cos(\alpha_m L)} \cos(\lambda_m \tau) \dots (29)$$

where, the functions f_1 , f_2 and $g(\xi)$ are defined in the following way.

$$\begin{aligned} g(\xi) &= \sin^2(aL) (\cosh(a\xi) \cos(a\xi) - \sinh(a\xi) \times \sin(a\xi)) + \sinh^2(aL) (\cosh(a\xi) \cos(a\xi) + \sinh(a\xi) \sin(a\xi)) - M_0 (\cosh(a\xi) \sin(a\xi) + \sinh(a\xi) \cos(a\xi)) \\ f_1 &= -1 + \cosh(\alpha_m L) \cos(\alpha_m L) + \sinh(\alpha_m L) \sin(\alpha_m L) \\ f_2 &= -1 + \cosh(\alpha_m L) \cos(\alpha_m L) - \sinh(\alpha_m L) \sin(\alpha_m L) \end{aligned}$$

and the meanings of the other notations are the same as those in Eq.(25).

The first terms of Eqs.(25) and (29) are derived from the residues at the origin and coincide respectively with the solutions of the corresponding static problems. The second term of Eq.(25) is derived from the residue at the pole $\lambda = \pm i \kappa J$, i.e., in the case that $\alpha = 0$, and the third term of Eq.(25) and the second term of Eq.(29) are derived from the residues at the other poles $\lambda = \pm i \lambda_m$ ($m=1,2,3,\dots$) given from the roots of the transcendental equation (26) except that $\alpha = 0$.

3.3 Solutions for Equal and Opposite Radial Concentrated Loads

For the problem of equal and opposite radial concentrated loads, whose magnitude is F_0 , as shown in Fig.2, the solutions for the displacement components are obtained by the use of superposition of the results given in 3.1 and 3.2 according to the following equation and by equating Rq_0 to F_0 .

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} u_0 \\ 0 \\ w_0 \end{pmatrix} + \frac{2}{\pi} \sum_{n=2,4,\dots}^{\infty} \begin{pmatrix} U_n \cos(n\theta) \\ V_n \sin(n\theta) \\ W_n \cos(n\theta) \end{pmatrix} \dots (30)$$

where, u_0 and w_0 denote the results for the uniformly distributed line load given in 3.2.

4. Numerical Results

In the evaluations, the data $\nu = 0.3$ and $h/R = 0.05$ are used and the effects of the lengths and end conditions of the cylindrical shells on the displacement and stress resultants are examined.

4.1 Results for Static Loads

First, static distributions of displacement and stress resultants are evaluated on the basis of the first terms of

Eqs.(19), (22), (25) and (29).

In Figs. 4, 5 and 6, the static distributions of the radial displacement w and bending moments M_x and M_θ along the generator through the loaded point of the cylindrical shells with both ends free and both ends clamped are shown for the data $L/R = 1, 2, 4$ and 8 . As the shell becomes long, the values of the radial displacement and bending moments decrease in the case of free ends, while they increase in the case of clamped ends and come close to the results for an infinite cylindrical shell. At the loaded point, the radial displacement possesses finite values but the values of the bending moments are equal to infinity and decrease steeply as the situation goes away a little from the loaded point. In the case of free ends, the value of M_θ is always greater than that of M_x regardless of the length of the

Fig.4 Distribution of Deflections Along the Generator Through the Loaded Point ($L=\ell/R, \theta=0$)

Fig.5 Distribution of Bending Moments Along the Generator Through the Loaded Point ($L=\ell/R, \theta=0$)

Fig.6 Distribution of Bending Moments Along the Generator Through the Loaded Point ($L=\ell/R, \theta=0$)

shell and remains considerably great at the free ends. In the case of clamped ends, the value of M_x increases near the clamped end and becomes greater than that of M_θ . Similarly, the static dis-

tributions along the meridian through the loaded point of the cylindrical shells are shown in Fig.7. In the case of free ends, the values of the radial displacement along the meridian are much influenced by the length of the shell and those distributions have similar patterns.

In the case of clamped ends, the value of the radial displacement at the point $\theta = \pi/2$ is small as far as the length is short. Meanwhile the values and distributions of the bending moments along the meridian are not so much influenced by the lengths of the shells.

Fig.7 Distribution of Deflections and Bending Moments Along the Meridian Through the Loaded Point ($L = l/R$, $x = 0$)

4.2 Results for Impulsive Loads

The convergence of the infinite series with respect to n , i.e., Eq.(30), is comparable to that in the case of the static loads. On the other hand, putting $\tau=0$, the absolute values of infinite series with respect to m of the parts including time τ in Eqs.(19), (22), (25) and (29) agree with the values of the first terms in those equations, represented in closed forms, for the static loads.

Therefore, the convergences of those infinite series are verified at the time $\tau = 0$. In Figs. 8 and 9, the radial displacement variations in time at the loaded point for the data $l/R=1, 2$ and 4 are shown respectively in the cases that both ends are free and both ends are clamped.

In the case of free ends, those oscillations are mainly affected by the period of the lowest natural frequency, which is independent of the length of the shell and whose mode has no knot in the axial direction, and the amplitudes tend to be small accompanied with the static solutions as the length of the shell becomes small.

Contrarily, in the case of clamped ends, those oscillations are profoundly

affected by the length of the shell and become complicated as the length becomes short owing to the periods of higher natural frequencies. In Fig.10, variations of the bending moments M_x and M_θ in time

at the point of $\xi=0.5$ on the generator through the loaded point for the data $l/R=2$ are shown in the case that both ends are

Fig.8 Deflection Variations at the Loaded Point in Time (Both Ends Free, Chain Lines Denote Static Results)

Fig.9 Deflection Variations at the Loaded Point in Time (Both Ends Clamped, Chain Lines Denote Static Results)

Fig.10 Bending Moment Variations in Time (Both Ends Free, $l/R=2$, $x/R=0.5$, $\theta=0$)

Fig.11 Bending Moment Variations in Time (Both Ends Clamped, $l/R=1$, $x/R=0.5$, $\theta=0$)

free. Those are mainly affected by the same period of frequency as that in the radial displacement variations in time in Fig.8.

In the case that both ends are clamped, the bending moments variations in time at two different points of $\xi=0.5$ and 1.0 on the generator through the loaded point for the data $l/R=1$ are shown in Figs .11 and 12. These oscillations are also affected by the periods of higher natural frequencies and are complicated as compared with those in the shell with free ends.

The effect of length of the cylindrical shells with both ends free and both ends clamped on the results is summarily shown in Fig.13.

As the shells become long, the static radial displacements in the two kinds of end conditions at the loaded point, i.e., w_{st} , gradually approach one another and approximately agree with result for an infinitely long cylindrical shell, i.e., w_{∞} , in the vicinity of $l/R=8$. The values of the dynamic factor k at the loaded point, defined by the ratio of the dynamic maximum radial displacement to the static one, gradually decrease as the shells become long, but do not coincide with the result for an infinite cylindrical shell unless the length is sufficiently great and are always greater than the result. The value of the lowest natural frequency in the case of free ends remains unchanged as the shell becomes long and coincides with that of an infinite cylindrical shell. Meanwhile the value in the case of clamped ends becomes greater than that in the case of an infinite length as the length becomes

short.

5. Conclusions

In this paper, the problem of circular cylindrical shells having finite length with both ends free and both ends clamped, which are suddenly subjected to equal and opposite radial concentrated loads, is analytically investigated on the basis of Donnell's shell equations including the effect of radial inertia force and the following results are obtained.

(1) Applying the method of solution proposed in this paper, the dynamic responses of finite circular cylindrical shells may be analyzed under arbitrary loading and boundary conditions. (2) In the case of concentrated loads, the displacement components possess finite values but the values of the bending moments are equal to infinity at the loaded point independent of the shell geometries. (3) All the results for static loads approximately agree with those for an infinite cylindrical shell if the shells are four times as long as the diameter of shells. (4) In the case of free ends, the radial displacement and bending moments variations in time are mainly influenced by the period of the lowest natural frequency, which is independent of the length of the shell and whose mode has no knot in the axial direction. (5) In the case of clamped ends, the radial displacement and bending moments variations in time are profoundly affected by the length of the shell and become complicated owing to the period of higher natural frequencies as the length becomes short. (6) The values of the dynamic factors at the loaded point gradually decrease as the shells become long, but do not agree with the result for an infinite cylindrical shell unless the length is sufficiently great and are always greater than the result.

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Fig.12 Bending Moment Variations in Time (Both Ends Clamped, $l/R = x/R = 1$, $\theta = 0$)

Fig.13 The Effect of Lengths of the Shells on the Results