Determination of Principal Strains and Their Directions by Moiré Rosette Method*

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The moiré rosette method is a useful means for determining the principal strains and their directions at any point in a strain field.

In this paper, a new method for determining exactly the directions of principal axes is proposed, and new equations for obtaining correctly the principal strains by moiré rosette method are presented. Further, the accuracy of measurement is discussed.

1. Introduction

Nowadays, as a means for determining the principal strains and their directions at any point in a complicated strain field, the method using the electric resistance wire (or foil) strain gage is widely known, but this method is mainly used for measuring small strains in the materials of high elasticity. On the contrary, the moiré method using two concentric circular grids (this will be called moiré rosette method) is useful for obtaining large principal strains in the materials of low elasticity and high ductility, such as, the high-polymer 19. And this method has two advantages that any amount of strain may be measured by properly selecting the amount of mismatch 20 and that the principal axes may be approximately determined by merely observing the fringe shapes.

P. S. Theocaris measured Poisson's ratio of epoxy resin by this method 21 and in his experiment he determined the principal strains by the conventional (moiré) method. 22) When the conventional method is used for obtaining the principal strains, the directions of principal axes must be determined by joining the apices of ellipse-like or hyperbola-like fringes, but it is difficult to locate the apices correctly. And also, this method takes no account of the factor for obtaining correct strain values. Therefore, in the case of complex strain fields, it has a possibility of giving erroneous strain values. Then, it is considered that an exact theory for this method has to be established, in order to determine exactly the principal strains.

In this paper, a general equation for fringes formed by moiré rosette method is derived, and the relations between principal strains and fringe shapes are clarified. Further, a new method for the exact determination of principal axes, that is, a method of rotating a deformed model grid is proposed, and new equations for obtaining correctly the principal strains are presented. Besides, the accuracy of measurement is examined.

2. Fringe shapes formed by moiré rosette method

Let the center of a model grid consisting of equispaced concentric circles with a grid pitch p be placed on the origin of orthogonal axes x, y set up on an unstrained specimen, and the X and Y axes of principal strains ε₁ and ε₂ of the strained specimen be inclined by angles α and (α + π/2) with the x axis, respectively. Taking the coordinates of a point on the model grid before deformation as (X₀, Y₀) and those after deformation as (X, Y), the equations for the undeformed and deformed model grids can be written as follows:

\[ X₀^2 + Y₀^2 = (1p)^2 \]  \hspace{1cm} (1)\n
\[ \frac{X^2}{(1+\varepsilon_x)^2} + \frac{Y^2}{(1+\varepsilon_y)^2} = (1p)^2 \]  \hspace{1cm} (2)\n
where l is a grid line index.

The equation for the master grid consisting of concentric circles with a pitch p₀ can be expressed as

\[ X^2 + Y^2 = (kp₀)^2 \]  \hspace{1cm} (3)\n
where k is a grid line index and \( \lambda \) is a mismatch value.

When the master grid is superposed upon the deformed model grid in such a way that their centers coincide, moiré fringes are formed as the loci of intersected points of master grids with line indices k's and model grids with line indices l's. The equation of moiré fringes is written as

\[ k-l=m = (m=0, \pm 1, \pm 2, \ldots) \]  \hspace{1cm} (4)\n
where m is a fringe index.

To obtain the fringe equation, let the coordinates of intersected points of the deformed model grid and the master grid be (X, Y), and by substituting Eqs. (2) and (3) into Eq. (4), we obtain a general fringe equation as follows:
\[
\frac{(1+\epsilon_1)^2-(1+\lambda)^2}{(1+\epsilon_1)^2(1+\lambda)^2} \cdot X + \frac{(1+\epsilon_2)^2-(1+\lambda)^2}{(1+\epsilon_2)^2(1+\lambda)^2} \cdot Y + 2 \frac{(1+\epsilon_1)^2-(1+\lambda)^2}{(1+\epsilon_1)^2(1+\lambda)^2} \cdot \frac{X^2}{(1+\epsilon_1)^2(1+\lambda)^2} + \frac{(1+\epsilon_2)^2-(1+\lambda)^2}{(1+\epsilon_2)^2(1+\lambda)^2} \cdot \frac{Y^2}{(1+\epsilon_2)^2(1+\lambda)^2}\]

As may be expected from the above equation, the fringe shapes will become complicated biquadratic curves for various values of \(\epsilon_i\) (i = 1 or 2) and \(\lambda\).

For the purpose of simply recognizing the outline of fringe shapes and of easily obtaining the fringe-strain relations on the principal axes, the orthogonal coordinates will be transformed into the polar coordinates \((R, \theta)\). Substituting the relations \(X = R \cos \theta\) and \(Y = R \sin \theta\) into Eq. (5), we obtain the fringe equation in polar coordinates as follows:

\[
R = \frac{m(1+\lambda)p}{\sqrt{1 - \left[\frac{(1+\epsilon_1)^2}{(1+\epsilon_1)^2 + (1+\epsilon_2)^2} \cos^2 \theta + \frac{(1+\lambda)^2}{(1+\epsilon_2)^2 + (1+\lambda)^2} \sin^2 \theta\right]}}.
\]

The fringe patterns drawn for various values of \(\epsilon_i\) and \(\lambda\) are shown in Fig. 1. In the figures, (a) represents the case in which \(\epsilon_i = 0\) and \(\lambda = 0.2\), and (b) to (e) indicate the cases of \(\lambda = 0.3, 0.5, 0.5\) and \(0.6\), respectively, under the condition that \(\epsilon_1 = 0.5\) and \(\epsilon_2 = 0.2\) are maintained. The fringe shapes in the figures may be divided into three kinds, the first is a circle as shown in (a), the second is of an ellipse-like shape as shown in (b), (d) or (e) and the third is similar to a hyperbola accompanied with a straight line of zero fringe order as shown in (c).

By referring to Fig. 1, the fringe

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![Fig. 1 Model fringe patterns produced under various conditions between principal strains and mismatch values](image)

**Table 1** Fringe shapes and signs of fringe index under various conditions between \(\epsilon_i\) and \(\lambda\)

<table>
<thead>
<tr>
<th>Strain condition</th>
<th>(\epsilon_1) &amp; (\lambda)</th>
<th>(\epsilon_1 &gt; \epsilon_2 &gt; \lambda)</th>
<th>(\epsilon_1 &gt; \lambda &gt; \epsilon_2)</th>
<th>(\epsilon_1 &gt; \lambda &gt; \epsilon_2)</th>
<th>(\lambda &gt; \epsilon_1 &gt; \epsilon_2)</th>
<th>(\lambda &gt; \epsilon_1 &gt; \epsilon_2)</th>
<th>(\lambda &gt; \epsilon_1 &gt; \epsilon_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fringe shape</td>
<td>Circle</td>
<td>Ellipse</td>
<td>Hyperbola</td>
<td>Hyperbola</td>
<td>Hyperbola</td>
<td>Ellipse</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Zero order fringe</td>
<td>Whole field</td>
<td>Origin</td>
<td>Origin</td>
<td>Y axis</td>
<td>Y axis</td>
<td>Origin</td>
<td>Y axis</td>
</tr>
<tr>
<td>Sign of fringe</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

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shapes produced by various combinations of $\varepsilon$, and $\lambda$ are shown in the first row of Table 1. The second row in the table shows the shape or position of zero order fringe, and the third row indicates the sign of fringe index $m$ of fringes intersecting the principal axes. As is seen in the table, the fringe shape and the sign of $m$ are varied simply. But, in practical strain fields, the discrimination between X and Y axes and the determination of the sign of $m$ can not easily be performed, since the relation between $\varepsilon_1$ and $\lambda$ is unknown.

3. A method for determining exactly the directions of principal axes

To obtain the correct values of principal strains, it is necessary to determine exactly the directions of principal axes. The principal axes may be determined by joining the apices of ellipse-like or hyperbola-like fringes, but it is difficult to locate the apices correctly.

A method for determining exactly the directions of principal axes is to utilize the crossed straight lines of zero fringe order as shown in Fig. 1(c). In the case that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$, the zero order fringes appear as crossed straight lines passing through the origin which are symmetrical with respect to the principal axes. Then, the directions bisecting these zero order fringes can be concluded to be the directions of principal axes. In all cases except those of $\varepsilon = \lambda$, $\varepsilon \neq \lambda$ and $\varepsilon_1 > \lambda > \varepsilon_3$, the following new method will be proposed. Prepare two sheets of model grids deformed under the same condition, and suppose one grid A upon the other grid B in such a way that their centers coincide. When the grid B is rotated by an angle $\beta$ against the other grid A, the equation for the stationary grid A is expressed by Eq. (2) and the equation for the rotated grid B is written as follows:

$$\frac{x'}{1+\varepsilon_1} + \frac{y'}{1+\varepsilon_2} = (k_0) p.$$  

Between the coordinates $(x,y)$ and $(X,Y)$, the following relation holds:

$$\begin{vmatrix} X^2 + Y^2 - (kp) & 0 \\ -X & Y \end{vmatrix} = 0.$$  

Substituting Eqs. (2), (7) and (8) into Eq. (4) and putting $m = 0$, we obtain the equation for the zero order fringe as follows:

$$(1+\varepsilon_1 X^2 + (1+\varepsilon_2) Y^2) \sin \beta \cos \beta - X \sin \beta \cos \beta = 0.$$  

When the strain condition is $\varepsilon_1 > \varepsilon_2$, no fringes appear in the case of $\beta = 0^\circ$ and the fringe patterns become complicated curves in the case of $0^\circ < \beta < 90^\circ$. And also, in the case of $\beta = 90^\circ$, the zero order fringes become crossed straight lines passing through the origin which are symmetrical with respect to the principal axes, because, by putting $\beta = 90^\circ$ in Eq. (9), we have

$$Y = X.$$  

The fringe patterns for the case of $\beta = 90^\circ$ are shown in Fig. 2, and the directions bisecting zero order fringes passing through the origin can be decided as the directions of principal axes.

**Fig. 2** Fringes produced by the superposition of two model grids deformed in the same condition ($\beta = 90^\circ$)

4. Fringe-principal strain relation

The principal strains have a positive or negative sign, but the information obtained from moiré fringe patterns is only the absolute value of interfringe spacing. Therefore, the correct sign of strain can not be determined by merely observing fringe patterns, especially in the case of complex strain fields. To solve such a problem, we have utilized the fringe-principal strain equation involving the fringe index $m$. Now, in the fringe patterns as shown in Fig. 1, let the interfringe spacing between the origin and any fringe on the principal axis X or Y be $R_x$ or $R_y$, and by putting $\beta = 0^\circ$ and $90^\circ$ in Eq. (6), we obtain

$$R_x \frac{m(1+\varepsilon_1)}{(1+\lambda)p} \varepsilon_1 - \lambda$$  

and

$$R_y \frac{m(1+\varepsilon_2)}{(1+\lambda)p} \varepsilon_2 - \lambda.$$  

The principal strains can be calculated from these equations, but here we will introduce equations similar to the conventional equations. Now, let the fringe indices of any two adjacent fringes on the X or Y axis be $m_x$ and $m_y$, respectively, and the interval between these two adjacent fringes on the X or Y axis be $\delta_x$ or $\delta_y$. Then, from Eqs. (11) and (12) we have

$$\delta_x \frac{(1+\varepsilon_1)}{(1+\varepsilon_2)} \frac{m(m+1)}{(1+\lambda)p} \varepsilon_1 - \lambda$$  

and

$$\delta_y \frac{(1+\varepsilon_2)}{(1+\varepsilon_1)} \frac{m(m+1)}{(1+\lambda)p} \varepsilon_2 - \lambda,$$

where $M$ is the difference in fringe indices between two adjacent fringes and has a value of $\pm 1$. Eqs. (13) and (14) yield the expression of principal strains $\varepsilon$ as

$$\varepsilon_1 = \frac{\delta_x - (1+\lambda)p}{(1+\lambda)p},$$  

where $\delta_x (X = Y)$ corresponds to $\varepsilon_1 (1 = \rho 2)$.  

5. A method for determining the sign of $M$

The sign of $M$ plays an important role in obtaining correct strain values but the determination of this sign from one fringe
pattern is very difficult in practical strain fields. However, this sign can easily be determined by using a method proposed by the authors previously.\(^{47}\) When the master grid superposed upon the deformed model grid is shifted in the direction parallel to the i axis by an amount \(\Delta\), Eqs. (11) and (12) are transformed into

\[
R_i = \frac{e_{i-\lambda}}{e_i - \lambda} + \frac{\Delta}{e_i - \lambda} \tag{16}
\]

The first term in the right hand side of the above equation has a constant value in a given strain field, and the second term shows the displacement of fringes produced by the shift of the master grid by \(\Delta\). The second term is useful for determining the sign of \(M\). Now, let the principal strains \(e_i\) and the mismatch \(\lambda\) be always larger than \(-1\). When the master grid is given a positive shifting \(\Delta\) (when the shifting is given in the direction toward increasing the positive coordinate, the sign of \(\Delta\) is prescribed to be positive), the fringes cutting the i axis are shifted toward the positive or negative direction (the positive or negative direction means that of increasing the absolute value of coordinate on the axis), according as \((e_i - \lambda)\) is larger or smaller than zero, respectively. This direction of the fringe shift can be correlated with the sign of \(M\) as follows. If the fringes cutting the i axis are shifted toward the positive or negative direction, then \(M\) is 1 or -1, respectively.

6. Examination of the accuracy of measurement

To examine the practical applicability of the methods as described in the preceding sections, the master and model grids were drawn by an automatic plotter of E. I. Co. linked to a computer. The grids drawn were a master grid with \(\lambda = 0\) and two kinds of model grids which would give the strain conditions of \(e_1 = 0.05, e_2 = -0.015\) and \(e_1 = 0.1, e_2 = -0.03\). The fringe patterns obtained by superposing these grids are shown in Fig. 3. In the figure, (a1), (b1), and (a2), (b2) correspond to the cases of grid with \(T = 1\) and \(l/9\), respectively, where \(T\) means the ratio of the width of bright part to that of dark part, and it was introduced by D. Post as a method of increasing the fringe sharpness\(^{19}\) by introducing \(T\) into the grid, the sharpening effect, that is, the possibility of exact determination of the center lines of fringes can be expected. To study the effect of \(T\), the analysis was carried out for the case of \(T = 1/3\), too.

Now, when the fringes do not appear as seen on the vertical axis of Fig. 3(a), the principal strain in this direction can not be analysed. In such a case we may proceed as follows. When a few fringes appear on the X axis but not on the Y axis, by putting \(m = 0\) in Eq. (5), we have

\[
y = \frac{1 + e_2}{1 + e_1} \frac{1 + (1 + e_1)^2 - 1}{1 + e_1(1 + e_2)^2 - 1} \tag{17}
\]

As mentioned previously, in the case of \(e_1 > e_2\), the zero order fringes appear as crossed straight lines passing through the origin. Now, the angle between the straight line in the first quadrant and the X axis is denoted as \(\gamma\), and considering the experimental condition that \(\lambda = 0\) (that is, by putting \(\lambda = 0\) in Eq. (17), we obtain

\[
\tan \gamma = \frac{1 + e_2}{1 + e_1} \tag{18}
\]

Therefore, from the above equation we have

\[
e_2 = \frac{(1 + e_1) \tan \gamma - 1}{(1 + e_1)^2 \sec^2 \gamma - 1} \tag{19}
\]

The principal strain \(e_2\) can be calculated by substituting the value \(e_1\) obtained from Eq. (15) and the measured angle \(\gamma\) into Eq. (19). In the reverse case to that mentioned above also, the unknown principal strain may be obtained similarly.

The analytical results obtained in the case of fringe patterns as shown in Fig. 3 are presented in Table 2. In the table, \(e_i^{(1+i)} (Ap)\) is the prescribed (exact) value, \(e_i^{(1+i)} (H)\) is the value calculated according to

<table>
<thead>
<tr>
<th>Table 2 Relative errors</th>
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<tbody>
<tr>
<td>(e_1^{(1+i)} (Ap))</td>
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<tr>
<td>(e_1^{(1+i)} (Ap))</td>
</tr>
<tr>
<td>0.05 (e_1^{(1+i)} (Ap))</td>
</tr>
</tbody>
</table>
Eq. (15) or (19) and the relative error of the principal strain $e_1$ is defined as follows:

$$\frac{\epsilon_1 - \epsilon(Ap)}{\epsilon_1(Ap)} = \epsilon_1, \epsilon_2$$

Further, the angular error $\Delta \theta$ is the absolute value of the difference between the prescribed (exact) angle $\theta(Ap)$ and the measured angle $\theta$. As is seen in the table, the relative errors and angular errors are compared according to the value of $T$, and the accuracy is best in the case of $T=1/9$. From these results, it may be recognized that the sharpening effect depends on the value of $T$. On the other hand, $R_2^2$ is always smaller than $R_2^2$ for these strain conditions. The reason for this may be considered as the multiple effect of the accuracy of measured values, that is, $\epsilon_1$ was calculated from Eq. (15) by substituting the measured value $\delta$ only, but $\epsilon_2$ was calculated from Eq. (19) by inserting both $\epsilon_1$ and the measured value $\gamma$.

7. Conclusions

From the results of this study, the following may be concluded:

(1) To obtain principal strains, it is necessary to determine exactly the directions of principal axes. In the case of hyperbola-like fringes accompanied with straight lines of zero fringe order passing through the origin, the directions of principal axes can be determined by bisecting the straight line fringes. In the case of ellipse-like fringes, the following procedure is proposed. When two model grids deformed in the same condition are prepared, and one grid is superposed upon the other grid so that both centers coincide and one grid is rotated by an angle $90^\circ$ with respect to the other grid, the fringe patterns consisting of straight lines are obtained as shown in Fig. 2. Then, the directions bisecting straight line fringes passing through the origin can be decided as the directions of principal axes.

(2) After the sign of $M$ is discriminated by using the master grid shifting method and the intervals between two adjacent fringes $\delta_i$ are measured, the principal strain $\epsilon_i$ can be obtained from Eq. (15).

(3) When no fringes appear on the $Y$ axis as shown in Fig. 3(a), the principal strain $\epsilon_2$ can not be obtained. But, by selecting properly the mismatch value so that the condition $\epsilon_1 > \lambda > \epsilon_2$ is satisfied, the crossed straight line fringes passing through the origin may be obtained. Then, by measuring the angle $\gamma$ between the straight line fringe in the first quadrant and the $X$ axis and by calculating the principal strain $\epsilon_1$ from Eq. (15), the principal strain $\epsilon_2$ can be obtained with the aid of Eq. (19).

References