Vibrations of Elliptic Arc Bar Perpendicular to its Plane*

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We have researched the vibrations of elliptic arc bar perpendicular to its plane. We obtain the Lagrangian of curved bars expressed by arc length, and then by using a new variable we reconstruct the Lagrangian, from which we obtain the equations of motion and the boundary conditions. We solve the equations of motion in the case of elliptic bar and show in graphs the frequencies and the modes of vibrations of symmetric elliptic arc bars. Their boundary conditions are built-in, rolled and supported at both ends. This method is also useful to research other curved bars as cosine, catenary and parabolic bars.

1. Introduction

We have studied the vibrations of an uniform bar, of which the center line is an arc of ellipse and which vibrates perpendicularly to the plane of center line. The method used in this paper is similarly available to study vibrations of curved bars of which center lines are the other quadratic curves, cosine curve and catenary, etc., as well as the vibration of the elliptic arc bar.

Neglecting the rotatory inertia and the deformation due to shear, we obtain the Lagrangian of an uniform curved bar, where the independent variable is the length of arc of center line, and we obtain the equations of motion and the boundary conditions. Then we introduce a new independent variable⁽¹⁾, which is obtained by integrating the curvature along center line, to simplify these equations and boundary conditions. The solution of equations is of form of power series and within the converging range of series it is rigorous.

As numerical examples of out-of-plane vibrations of elliptic arc bar, we have calculated frequencies and modes of symmetric elliptic arc bar with both built-in ends and frequencies of elliptic ring. In the former case, we have also made experiments and calculations by Ritz method, and made certain that both results agree with the present theoretical one.

As shown in the previous paper⁽¹⁾ in which in-plane vibrations of curved bars were treated, this method has demerits: if the elliptic arc bar is longer than semiellipse, the series of solution is divergent, and if this method is applied to curved bars with complex configuration of center lines, it is sometimes difficult to obtain the new variables mentioned above.

There are a few researches about out-of-plane vibrations of curved bars except the circular arc bar. By Ritz method, Volterra

and Morell have researched out-of-plane vibrations of symmetric curved bars, of which center lines are cycloid, catenary and parabola, and of which both ends are built-in⁽²⁾ or supported⁽³⁾, and they have shown the first frequencies. Chang-Volterra⁽⁴⁾ have analyzed the vibrations of the same bars as mentioned above by Lehmann-Machly method and have obtained the frequencies up to the fourth mode of vibration in the case that both ends are built-in.

2. Lagrangian of vibration

We treat vibrations of uniform curved bars, of which center lines are plane curves and which vibrate perpendicularly to planes of center lines. Let l denote the representative length: s , the length along center line; $1/\rho$, the curvature of center line and $v \sin pt$ and $\phi \sin pt$, the lateral and torsional displacements respectively, where p and t are the circular frequency and the time. Let δ/g , S and I_p denote the density, the cross-sectional area and the polar moment of inertia of bar; EI and C , the bending rigidity and the torsional one and M , M_t and F , the bending moment, the twisting moment and the shearing force, respectively. From Love⁽⁵⁾ we obtain

$$M = EI(\phi/\rho - d^2v/ds^2) \sin pt$$

and

$$M_t = C[d\phi/ds + (1/\rho)dv/ds] \sin pt$$

We use the following notations.

$$\begin{aligned} \alpha^4 &= \gamma S p^2 l^4 / EI g, \quad \beta^2 = \gamma I_p p^2 l^2 / C g, \quad \nu = C / EI \\ \bar{X} &= d^2v/ds^2 - \phi/\rho, \quad \bar{Z} = (1/\rho)dv/ds + d\phi/ds \end{aligned} \quad \dots\dots\dots (1)$$

These are, by turns, two nondimensional frequencies, the ratio of both rigidities and quantities proportional to bending and twisting moments. We put the slenderness ratio as $(Sl^2/I_p)^2$ and from the relation $\alpha^4/\beta^2 = \nu Sl^2/I_p$, we are able to obtain the ratio of both nondimensional frequencies if the ratio of both rigidities and the slenderness ratio are given.

Denoting the values of s at both ends

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of bar by s_1 and s_2 , we obtain the Lagrangian of vibration of the bar in a period,

$$L = (1/2) \int_{s_1}^{s_2} [\gamma S p^2 v^2 / g + \gamma I_p p^2 \phi^2 / g - E I \bar{X}^2 - C \bar{Z}^2] ds \dots\dots\dots (2)$$

Assuming that γ/g , S , I_p , EI and C are independent on s , we obtain the following equation by partial integral,

$$L = (EI/2) \left[\int_{s_1}^{s_2} \{ (\alpha/l)^4 v - d^2 \bar{X} / ds^2 + \nu d(\bar{Z}/\rho) / ds \} v ds + \int_{s_1}^{s_2} \{ \nu (\beta/l)^2 \phi + \bar{X}/\rho + \nu d\bar{Z}/ds \} \phi ds - (\bar{X} dv / ds - (d\bar{X}/ds - \nu \bar{Z}/\rho) v + \nu \bar{Z} \phi) \Big|_{s_1}^{s_2} \right] \dots\dots\dots (3)$$

Putting two integrands in the right side of the above equation into zero, we obtain two Euler equations (equations of motion) and putting the third term into zero, we obtain the boundary conditions.

We consider a nondimensional curvature $\bar{\Psi}$ and use a new independent variable x for s , which is obtained from the following relation,

$$dx/ds = 1/\rho \equiv G \bar{\Psi} \dots\dots\dots (4)$$

where G is a constant and $\bar{\Psi}$ is the function of x , and the dimension of G is of reciprocal of length. We put as follows:

$$\left. \begin{aligned} \bar{\Psi} &= G \bar{\Psi} \phi, \\ X &= \bar{\Psi}^2 d^2 v / dx^2 + (1/2) (d\bar{\Psi}^2 / dx) dv / dx - \bar{\Psi}^2 \phi \\ Z &= \bar{\Psi}^2 dv / dx + \bar{\Psi}^2 d\phi / dx + (1/2) (d\bar{\Psi}^2 / dx) \phi \end{aligned} \right\} \dots\dots\dots (5)$$

From equ.(4) we obtain

$$\begin{aligned} d/ds &= G \bar{\Psi} d/dx \\ d^2/ds^2 &= G^2 [\bar{\Psi}^2 d^2/dx^2 + (1/2) (d\bar{\Psi}^2/dx) d/dx] \end{aligned}$$

and so $\bar{X} = G^2 X$ and $\bar{Z} = G^2 Z$. Denoting the values of x at $s=s_1$ and s_2 by x_1 and x_2 respectively, we obtain from equ.(3)

$$L = (EIG^3/2) \left[\int_{x_1}^{x_2} \{ (\alpha/GI)^4 v - (\bar{\Psi}^2 d^2 X / dx^2 + (1/2) (d\bar{\Psi}^2 / dx) dX / dx) + \nu [\bar{\Psi}^2 dZ / dx + (1/2) (d\bar{\Psi}^2 / dx) Z] \} (v/\bar{\Psi}) dx + \int_{x_1}^{x_2} \{ \nu (\beta/GI)^2 \phi + X + \nu dZ / dx \} \bar{\Psi} \phi dx - [\bar{\Psi} \{ X dv / dx - (dX / dx - \nu Z) v + \nu Z \phi \}] \Big|_{x_1}^{x_2} \right] \dots\dots\dots (6)$$

Similarly as mentioned above, we obtain equ-

$$\left. \begin{aligned} [(1-\mu^2)\alpha]v - [\bar{\Psi}^2 d^2 X / dx^2 + (1/2) (d\bar{\Psi}^2 / dx) dX / dx] + \nu [\bar{\Psi}^2 dZ / dx + (1/2) (d\bar{\Psi}^2 / dx) Z] &= 0 \\ \nu [(1-\mu^2)\beta]^2 \phi + X + \nu dZ / dx &= 0 \end{aligned} \right\} \dots\dots\dots (10)$$

uations of motion by putting two integrands in the right side of above equation equal zero, and also obtain the boundary conditions by putting the third term in brackets equal zero. The product of the third term and $EIG^3/2$ consists with works at boundary, that is, half a product of the bending moment and the inclination, of the twisting moment and the twisting angle and of the shearing force and the deflection, respectively.

We can also obtain equ.(3) by calculus of variation. After the variation of equ.(2) is obtained, the equation omitted the variational sign, coincides with equ.(3). as equ.(2) is a quadratic form of v and ϕ . That is, the equation obtained by putting equ.(3) equal zero is the stationary condition of work integral in dynamics. (See Appendix)

3. Vibration of elliptic arc bar perpendicular to its plane

Restricting ourselves to the vibration of such a curved bar of which the center line is a part of arc of ellipse, we consider the ellipse $\xi^2/a^2 + \eta^2/b^2 = 1$, where ξ and η are cartesian coordinates. As the representative length and a quantity denoting the flatness of ellipse, we use the following notations,

$$l^2 = (a^2 + b^2)/2, \mu = (a^2 - b^2)/(a^2 + b^2) \dots\dots\dots (7)$$

Taking origin of x on ξ -axis, we obtain the curvature of ellipse as

$$1/\rho = (1 + \mu \cos 2x)^{3/2} / [(1 - \mu^2) \{(a^2 + b^2)/2\}^{1/2}]$$

and so we may put as follows,

$$\left. \begin{aligned} 1/G &= (1 - \mu^2) \{(a^2 + b^2)/2\}^{1/2} \\ \bar{\Psi} &= (1 + \mu \cos 2x)^{3/2} \end{aligned} \right\} \dots\dots\dots (8)$$

From the expansion of $\bar{\Psi}^2$, in a power series, we can put

$$\bar{\Psi}^2 = (1 + \mu)^3 \sum_{n=0}^{\infty} \zeta_n x^{2n} \dots\dots\dots (9)$$

where

$$\begin{aligned} \zeta_0 &= 1 \\ \zeta_n &= (-1)^n [2^{2n} \{ 3\mu(1 + \mu^2/4) \} + 4^{2n} (3\mu^2/2) + 6^{2n} (\mu^3/4)] / [(1 + \mu)^3 (2n)!] \end{aligned}$$

For other quadratic curves, cosine curve and catenary, etc., we may take the position of origin of x as their curvatures become even functions, and we are able to use equ.(9) as the expansion of $\bar{\Psi}^2$ and easily calculate the coefficient ζ_n . For ellipse, $\zeta_0 = 1$, but for the other center lines of bars mentioned above, it is not unity and so we leave it as ξ_0 .

The following equations of motion hold.

There are two kinds of solutions: one is the case that both ν and Φ are odd functions of x , and the other is that they are even.

(i) In the case that ν and Φ are odd functions, they become

$$\nu = \sum_{n=0}^{\infty} P_n x^{2n+1}, \quad \Phi = \sum_{n=0}^{\infty} Q_n x^{2n+1} \dots\dots\dots (11)$$

and by substituting eqs.(11) and (9) into equ.(10), we obtain

$$\left. \begin{aligned} & \sum_{n=0}^{\infty} \left[(2n+2)(2n+3)\nu\zeta_0 Q_{n+1} + 2(n+1)\nu \sum_{p=0}^n (n+p+2)\zeta_{n-p+1} Q_p - \sum_{p=0}^n \zeta_{n-p} Q_p \right. \\ & \quad \left. + (1-\mu)^2 \nu \beta^2 Q_n / (1+\mu) + \sum_{p=0}^{n+1} (2p+1) \{ 2(n+1)\nu + (n+p+1) \} \zeta_{n-p+1} P_p \right] x^{2n+1} = 0 \\ & \sum_{n=1}^{\infty} \left[-2n(2n+1)(2n+2)(2n+3)\zeta_0^2 P_{n+1} + (1-\mu)^4 \alpha^4 P_{n-1} / (1+\mu)^2 + \sum_{p=0}^n \left\{ \nu \sum_{q=0}^p (2n-q)\zeta_{p-q}\zeta_q \right\} \right. \\ & \quad \left. - \sum_{q=0}^{p+1} (2n-p-q+1)(2n-q)(2n-2q+1)\zeta_{p-q+1}\zeta_q \right\} (2n-2p+1) P_{n-p} \\ & \quad \left. + \sum_{p=0}^n \left\{ Q_{n-p} \sum_{q=0}^p (2n-q) [\nu(2n-p-q+1) + (2n-2q+1)] \zeta_{p-q}\zeta_q \right\} \right] x^{2n-1} = 0 \end{aligned} \right\} \dots\dots\dots (12)$$

From the above equations, we obtain, by turns, Q_1 , P_2 , Q_2 , P_3 , ..., as follows,

$$\begin{aligned} -6\nu\zeta_0 Q_1 &= [4\nu\zeta_1 - \zeta_0 + (1-\mu)^2 \nu \beta^2 / (1+\mu)] Q_0 + 6\zeta_0(1+\nu)P_1 + \zeta_1(1+2\nu)P_0 \\ 120\zeta_0^2 P_2 &= (-60\zeta_0\zeta_1 + 6\nu\zeta_0^2)P_1 + \{-12\zeta_0\zeta_2 - \zeta_1^2 + 3\nu\zeta_0\zeta_1 + (1-\mu)^4 \alpha^4 / (1+\mu)^2\} P_0 \\ &+ 6\zeta_0^2(1+\nu)Q_1 + \zeta_0\zeta_1(7+5\nu)\zeta_0\zeta_1 Q_0, \dots \end{aligned}$$

where P_0 , Q_0 and P_1 are left as undetermined.

(ii) In the case that ν and Φ are even functions,

$$\nu = \sum_{n=0}^{\infty} P_n x^{2n}, \quad \Phi = \sum_{n=0}^{\infty} Q_n x^{2n} \dots\dots\dots (13)$$

and equations of motion become as follows, and similarly as mentioned above, we can obtain, by turns, Q_1 , P_2 , Q_2 , ..., where P_0 , Q_0 and P_1 are left as undetermined.

$$\left. \begin{aligned} & \sum_{n=0}^{\infty} \left[(2n+1)(2n+2)\nu\zeta_0 Q_{n+1} + (2n+1)\nu \sum_{p=0}^n (n+p+1)\zeta_{n-p+1} Q_p - \sum_{p=0}^n \zeta_{n-p} Q_p \right. \\ & \quad \left. + (1-\mu)^2 \nu \beta^2 Q_n / (1+\mu) + \sum_{p=1}^{n+1} 2p \{ (2n+1)\nu + (n+p) \} \zeta_{n-p+1} P_p \right] x^{2n} = 0 \dots\dots\dots (14) \\ & \sum_{n=1}^{\infty} \left[-(2n-1)(2n)(2n+1)(2n+2)\zeta_0^2 P_{n+1} + (1-\mu)^4 \alpha^4 P_{n-1} / (1+\mu)^2 \right. \\ & \quad \left. + \sum_{p=1}^n 2(n-p+1)P_{n-p+1} \left\{ \nu \sum_{q=0}^{p-1} (2n-q-1)\zeta_{p-q-1}\zeta_q - \sum_{q=0}^p (2n-p-q+1)(2n-2q) \right. \right. \\ & \quad \left. \left. \times (2n-q-1)\zeta_{p-q}\zeta_q \right\} + \sum_{p=0}^n \left\{ Q_{n-p} \sum_{q=0}^p (2n-q-1) [\nu(2n-p-q) + (2n-2q)] \zeta_{p-q}\zeta_q \right\} \right] x^{2n-2} = 0 \end{aligned} \right\} \dots\dots\dots (15)$$

Thus we obtain six mutually independent solutions from the relations in (i) and (ii).

In numerical calculations, assuming values of α , ν , μ and the slenderness ratio, and putting one of P_0 , Q_0 and P_1 equal 1 and the others equal zero, we obtain six solutions and we search the values of x which satisfy boundary conditions. Forty terms of each P_n and Q_n with ten digits are used in calculations from the limitation of computer's capacity. In the case that $x < 1.4$, all quantities used are accurate up to 7~8 digits, while in the case that $x = 1.7$, the accuracies of dv/dx and

$d\Phi/dx$ are 3 or 4 digits and it is impossible to obtain the frequency, because some digits drop down in the calculation of determinant. The convergency of all quantities used in calculation is best when the elliptic arc bar is built-in at both ends, but even in this case, the range of convergency is $x < 1.6$ (about semiellipse). This reason obliges us to calculate the bending moment and the shearing force by integrating equations of motion. From the boundary conditions in equ.(6), the bending moment, the twisting moment and the shearing force are

$$\left. \begin{aligned} M &= -EIG^2 X, \quad M_t = \nu EIG^2 Z \\ F &= -EIG^3 \bar{\Psi} (dX/dx - \nu Z) \end{aligned} \right\} \dots\dots\dots (16)$$

As the following relations hold,

$$\begin{aligned} (d/dx)[\bar{\Psi}(dX/dx - \nu Z)] &= \frac{1}{\bar{\Psi}} [\bar{\Psi}^2 d^2 X/dx^2 \\ &+ (1/2)(d\bar{\Psi}^2/dx)dX/dx - \nu \bar{\Psi}^2 dZ/dx \\ &+ (1/2)(d\bar{\Psi}^2/dx)Z] \end{aligned}$$

we obtain the following bending moment and shearing force: by integrating the first equation in equ.(10)

$$\left. \begin{aligned} -F/EIG^3 &= K_1 + [(1-\mu^2)\alpha]^4 \int_0^x (\nu/\bar{\Psi}) dx \\ -M/EIG^2 &= K_2 + K_1 \int_0^x (dx/\bar{\Psi}) + \nu \int_0^x Z dx \\ &+ [(1-\mu^2)\alpha]^4 \int_0^x [(1/\bar{\Psi}) \int_0^x (\nu dx/\bar{\Psi})] dx \end{aligned} \right\} \dots\dots\dots (17)$$

Using the above equations, we get M and F with about same extent of convergency as $d\nu/dx$, $d\phi/dx$ and M_t . The convergency of $\bar{\Psi}^2$ is very good not only for ellipse but also for the other quadratic curves, cosine curve and catenary etc... In the above equations, K_1 and K_2 are constants and (i) in the case that ν and ϕ are odd functions

$$\left. \begin{aligned} K_1 &= (1+\mu)^{3/2} [6\zeta_0 P_1 + (\zeta_1 - \nu \zeta_0) P_0] \\ &- (1+\nu)\zeta_0 Q_0 \\ K_2 &= 0 \end{aligned} \right\} \dots\dots\dots (18)$$

(ii) in the case that ν and ϕ are even functions

$$K_1 = 0, \quad K_2 = (1+\mu)^3 (2\zeta_0 P_1 - \zeta_0 Q_0) \dots\dots\dots (19)$$

In the case $|x| < \pi/2$, the values of μ , ν and the slenderness ratio have little effect upon the convergency of ν and ϕ . In the same range of x , the convergency becomes worse as the value of α becomes larger.

4. Numerical examples

As numerical examples, we have carried out the calculations of out-of-plane vibrations of symmetric elliptic arc bars with both built-in ends to $x \approx 1.6$ where the value of x at both ends is $\pm x_0$, and vibrations of elliptic rings. For the former case, we have also calculated by Ritz method and made experiments, and we will show that these results agree with those obtained by the method in this paper. The ratio of both rigidities ν used in the following calcu-

lations is $1/1.3$ which corresponds to a value of bar of Poisson's ratio 0.3 with a circular section. As mentioned above, we can obtain ν and ϕ when the values of α , μ , ν and the slenderness ratio are given. For ellipse, $\bar{\Psi} \neq 0$ always holds.

If we assume the elliptic arc bar is symmetric about ξ -axis, the vibration modes of the elliptic arc bar with both built-in ends can be separated into symmetric and antisymmetric ones about ξ -axis, taking the antisymmetric vibration as an example, we obtain the following ν and ϕ by linear combinations of solutions of equs.(11) and (12)

$$\left. \begin{aligned} \nu &= C_1 \nu_1 + C_2 \nu_2 + C_3 \nu_3 \\ \phi &= C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 \end{aligned} \right\} \dots\dots\dots (20)$$

where C_1 , C_2 and C_3 are arbitrary constants. From equ.(6), the boundary condition of both built-in ends is given by

$$\nu = d\nu/dx = \phi = 0$$

From this condition and equ.(20), we obtain the frequency equation, that is, a determinant of the third order, and we search for zero points of the determinant to get x_0 , C_2/C_1 and C_3/C_1 . In the calculation of mode, we assume that $C_1 = 1$. In figures of frequencies for antisymmetric vibrations those of the slenderness ratio 10 are also shown. Because of neglecting the rotatory inertia and the shearing deformation, these figures show only the effect of the slenderness ratio on frequencies.

Denoting the eccentric angle of ellipse by θ , the equation of ellipse is $\xi = a \cos \theta$ and $\eta = b \sin \theta$. It is convenient to consider the eccentric angle than the arc length of the elliptic center line, we show the relation between θ and x in Fig.1. It is

$$\tan \theta = \sqrt{\frac{1-\mu}{1+\mu}} \tan x \dots\dots\dots (21)$$

and at $\theta = 0$ and $\pi/2$, $\theta = x$ and in the midway, $x > \theta$ when $\mu > 0$. In the case that $\theta = \pi/4$, for an example, x is about 12% 27% and 41% larger than θ for $\mu = 0.2$, 0.4 and 0.6 (ratio of major and minor axes is 0.5), respectively.

In Figs. 2~4, the antisymmetric vibrations of both built-in elliptic arc bar, which is symmetric about ξ -axis, are shown. Fig.2 shows the relation between the nondimensional frequency α and the coordinate x_0 , where SR (slenderness ratio) = 2.5, $\mu = 0.2$, 0.4 and 0.6, and SR=10, $\mu = 0.4$. For each value of μ , the curves of SR=100 are very close to the curves of SR=25 from the right, so that the curves for SR=100 are neglected.

The extent of x_0 in the figures is taken from zero to about $\pi/2$. When the slenderness ratios are equal each other, as μ becomes larger, the curves go to right where α is large. But in neighborhood of

$x_0 = \pi/2$, there are little difference of α due to μ .

These tendencies mentioned above, are valid not only to the antisymmetric vibration of the bar with both built-in ends but also to the symmetric one and also symmetric and antisymmetric vibrations of elliptic arc bars of which both ends are supported or rolled.

The first and the second modes in the case of $\mu=0.4$ and of $SR=25$, are shown in Figs.3 and 4, and for other values of μ and SR , the modes are similar to these shown in figures.

In Figs.5 and 6, the symmetric vibrations of symmetric elliptic arc bars with both built-in ends are shown. In Fig.5, $\alpha - x_0$ curves are shown when $SR=25$ and $\mu=0.2, 0.4$ and 0.6 . Each curve of the first mode in Fig.2 is almost in the midst of each curve in Fig.5. In Fig.6 the first and the second modes of $\mu=0.4$ and $SR=25$ are shown.

We consider the vibration of an elliptic arc bar which is symmetric about ξ -axis and which is rolled or supported at both ends. The boundary conditions are, respectively,

$$dv/dx = F = Z = 0, \quad v = M = \Phi = 0$$

From equs.(11) and (12), we have obtained the solution of the antisymmetric vibration and are shown $\alpha - x_0$ curves in Fig.7. It is the case that $\mu=0.2$ and $SR=25$. The vertical dotted line on the right is the line of $x_0 = \pi/2$ and the intersections of this line and the $\alpha - x_0$ curves are the frequencies of elliptic rings. The semielliptic bar with both rolled ends is the same as the elliptic ring vibrating symmetrically about η -axis, and one with both supported ends is the same as elliptic ring vibrating antisymmetrically. We use S and A as abbreviations of the symmetric and the antisymmetric vibrations and make a rule that the first abbreviation is the mode about ξ -axis and the second one is that about η -axis. The numbers shown on the right side in Fig.7 are frequencies of elliptic ring for the first mode of AS vibration, for the first mode of AA, for the second mode of AS and for the second mode of AA, by turns from the bottom. These frequencies are obtained by interpolation. The value of α in the first mode of AS vibration is zero.

Fig.1 Relation between eccentric angle θ and x .

Fig.3 First mode for vibration in Fig.2
Notation x_0 is the value at built-in end.

Fig.2 Symmetric elliptic arc bar
(both ends built-in, antisymmetric vibration) Notation SR denotes the slenderness ratio.

$SR=25, \mu=0.4$

Fig.4 Second mode for vibration in Fig.2

Fig.5 First and second frequencies of the symmetric elliptic arc bar (both ends built-in, symmetric vibration)

Numerical symbols I, II denote the modes for the first and the second vibrations, respectively.

Fig.6 Mode for vibration in Fig.5

The full line denotes the frequency curve of the bar with both rolled ends and the dotted line does the one with both ends supported. The value of α at $x_0 = \pi/2$ corresponds to the frequency of an elliptic ring

Fig.7 First and second frequencies of the symmetric arc bar (antisymmetric vibration, $\mu=0.2, SR=25$)

Fig.8 Comparison of exact solution with experiment and Ritz method.

The present elliptic arc bar is of $\mu = 0.2$ and its center line is somewhat similar to circle. So the first and the second frequencies of symmetric vibrations of this bar with both supported ends and with both rolled ends, are approximately similar to those of both rolled ends and of both supported ends in Fig.7, respectively, and their curves are slightly left along the curves in Fig.7. The values of α of an elliptic ring are 0, 1.61, 2.72 and 3.78 in the cases of the first mode of SA vibration, of that of SS, of the second mode of SA and of that of SS, respectively, and the difference from the antisymmetric vibration is only 1 unit at the second decimal place.

In the vibrations of symmetric elliptic arc bars of other values of μ , the frequencies differ each other in the case of large α , as mentioned above, but in the case of $x_0 = \pi/2$ (it is equivalent to an elliptic ring), the frequencies differ little from the values shown in Fig.7. For examples, the values of the elliptic ring for $\mu = 0.4$ and $SR = 25$ are 1.65, 2.78 and 3.88 in the first mode of AA vibration, in the second mode of AS and in the second mode of AA, respectively.

We have made experiments and calculations by Ritz method on the vibrations of elliptic arc bars which are symmetric about ξ -axis and are built-in at both ends. A comparison between these results and one obtained from the rigorous solution is shown in Fig.8, where $\mu = 0.4$ and $SR = 100$ in the rigorous solution. These results almost agree with values of α in the case that $\mu = 0.4$ and $SR = 25$ shown in Figs.2 and 5. Elliptic arc bars used in experiment are made from cold finished steel bar with the circular section of diameter 0.6 cm and their values of μ and SR are 0.4 and 250 (their major and minor axes are 62.7 and 41.1 cm), respectively. They were vibrated by an exciting magnet and α was determined from the frequency in which the amplitude became largest.

In Ritz method, we have calculated equ. (2), where the independent variable is the arc lengths. Approximating the curvature

by a powerseries with six terms and assuming that trial functions of v and ϕ are also power series with six terms, respectively, we have calculated the integration on desk and used a computer to obtain α from 12×12 determinant. The symmetric and antisymmetric vibrations are calculated separately and the values of μ and SR are 0.4 and 100 respectively. Nondimensional frequencies obtained from experiments and Ritz method are in good agreement with the rigorous one.

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Appendix

We put

$$H = H(v, v_s, v_{ss}, \phi, \phi_s).$$

where

$$v_s = dv/ds,$$

Considering the first variation of

$$L = (EI/2) \int_{s_1}^{s_2} H ds$$

we obtain

$$\delta L = (EI/2) \int_{s_1}^{s_2} [H_v \delta v + H_{v_s} \delta v_s + H_{v_{ss}} \delta v_{ss} + H_\phi \delta \phi + H_{\phi_s} \delta \phi_s] ds.$$

and from partial integration of the above equation, we obtain

$$\delta L = (EI/2) \left[\int_{s_1}^{s_2} \{ H_v - dH_{v_s}/ds + d^2 H_{v_{ss}}/ds^2 \} \delta v ds + \int_{s_1}^{s_2} \{ H_\phi - dH_{\phi_s}/ds \} \delta \phi ds + [H_{v_s} \delta v_s + (H_{v_{ss}} - dH_{v_{ss}}/ds) \delta v + H_{\phi_s} \delta \phi]_{s_1}^{s_2} \right]$$

where

$$H_v = \partial H / \partial v$$

In equ. (2),

$$H = (\alpha/l)^4 v^2 + \nu (\beta/l)^2 \phi^2 - \bar{X}^2 - \nu \bar{Z}^2$$

and so, from

$$\delta L = 0$$

we obtain

$$\int_{s_1}^{s_2} \{ (\alpha/l)^4 v - d^2 \bar{X}/ds^2 + \nu d(\bar{Z}/\rho)/ds \} \delta v ds + \int_{s_1}^{s_2} \{ \nu (\beta/l)^2 \phi + \bar{X}/\rho + \nu d\bar{Z}/ds \} \delta \phi ds - [\bar{X} \delta v_s - (d\bar{X}/ds - \nu \bar{Z}/\rho) \delta v + \nu \bar{Z} \delta \phi]_{s_1}^{s_2} = 0$$

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