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Bauchinger Effect of Polycrystals and Mutual Restriction Between Grains*

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Relation between mutual restriction of grains in polycrystals and Bauchinger effect is studied analytically. Polycrystals are assumed to be composed of grains with different work-hardening rates. Discussion is made both on the linear and on the exponential stress-strain relations of grains. The mutual restriction of grains is represented quantitatively with constraint ratio. It follows that, owing to the non-uniform deformation of polycrystals, the absolute value of flow stress reduces when the direction of loading is reversed even if the flow stress of each grain is assumed to be dependent only on total strain. Discussion is made on factors affecting the decrease in flow stress of polycrystals. The amount of decrease in flow stress increases with an increase in the difference in stress-strain relations of grains.

1. Introduction

When metallic materials are plastically deformed under uniaxial stress and then the applied load is reversed after unloading, the absolute values of yield stress and flow stress in the reverse loading are lower than those of re-loading in the initial direction, which is known as Bauchinger effect. Although Bauchinger effect was first established in 1881, it is still an important problem in strength of materials as well as in plastic working of metals.

Many investigations have been done on Bauchinger effect and characteristic features of Bauchinger effect have been revealed by experiments⁽¹⁾⁻⁽⁶⁾. Also, theoretical studies have been done on the mechanism of Bauchinger effect. It was considered first that the strain is different in each grain in polycrystals after deformation, for the easiness of deformation is not equal, and as a result the internal stress is different in each grain (the Heyn stress)⁽⁷⁾, which causes Bauchinger effect. This idea was evolved into an analytical model of Bauchinger effect of polycrystalline metals based on slips in each grain⁽⁸⁾⁽²⁴⁾. Bauchinger effect, however, was recognized even in single crystal⁽⁹⁾. Hence, work-hardening theories based on crystal dislocation have been applied to explain Bauchinger effect⁽¹⁾⁽³⁾⁽⁴⁾ and correspondingly many experimental studies have been done from a metallurgical viewpoint⁽⁴⁾. On the other hand, the mathematical plasticity or solid mechanics has made marked progress in recent years and there has been much effort to include Bauchinger effect in the analysis

as supplementary terms with coefficients determined by experiments.

Bauchinger effect is thought to be caused essentially by inhomogeneity of plastic deformation of materials⁽²³⁾⁽²⁵⁾⁽²⁶⁾. There are usually inhomogeneity of various scales in materials, that is, from macroscopic to microscopic dimensions, for instance, from grains in polycrystals to dislocations or lattice defects in crystals.

Meanwhile, one of the authors proposed⁽¹⁰⁾⁽¹¹⁾ a conception of constraint ratio which represents quantitatively the mutual restriction in deformation between grains in polycrystals, which have different hardening rates owing to elastic and plastic anisotropy of crystal. The conception has been applied to various aspects of deformation behaviour of polycrystals and inhomogeneous materials. It is the purpose of the present study to apply the conception of constraint ratio to the case of reverse loading and analyse the quantitative relation between Bauchinger effect and deformation behaviour of polycrystals.

2. Model

In order to discuss Bauchinger effect analytically, a model is introduced in the following, based on the known experimental results⁽⁴⁾.

Let us first consider a stress-strain diagram and denote the uniaxial flow stress of polycrystals by σ_F . If the reversed load is applied after an arbitrary pre-strain a , the absolute flow stress is smaller than σ_F , which will be denoted by σ_R . Now, we consider the flow stress σ_R' which has the same absolute value as σ_R and an opposite sign.

$$\sigma_R' = -\sigma_R \dots\dots\dots(1)$$

When both σ_F and σ_R' are defined as functions of total strain, that is, the sum of absolute value of strain in each loading, the decrease σ_B of the flow stress is defined by

$$\sigma_B = \sigma_F - |\sigma_R| = \sigma_F - \sigma_R' \dots\dots\dots(2)$$

The softening after reverse loading is

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usually divided into two parts, that is, the transient softening and the permanent softening⁽¹¹⁾⁽¹²⁾. In the latter portion σ_B is almost constant with respect to strain. (σ_B in Eq. (2) is called Bauehinger stress in the transient softening region. In the following analysis, however, Eq. (2) will be used for both transient and permanent softening.)

The permanent softening has not drawn much attention theoretically, compared with the transient softening, though many experimental results showed a considerable decrease in flow stress even long after the load reversing. Discussion is also made on the permanent softening in the following.

Constraint ratio η which represents the restriction between grains is defined by⁽¹⁰⁾

$$\eta = \Delta\epsilon / (\Delta\epsilon)_s \dots \dots \dots (3)$$

where $\Delta\epsilon = \epsilon - \bar{\epsilon}$ (ϵ ; strain in a grain, $\bar{\epsilon}$; the mean strain for polycrystals) and $(\Delta\epsilon)_s$ is the value of $\Delta\epsilon$ under constant stress condition. ($\Delta\epsilon$ and $(\Delta\epsilon)_s$ are represented approximately with $\overline{AW}$ and $\overline{AV}$ in Fig. 1(a), respectively. Similar definition can be made for nonlinear stress-strain relation⁽¹¹⁾.) It is apparent that $\eta=0$ and $\eta=1$ correspond to the deformation under constant strain and under constant stress, respectively. It was shown in the previous papers⁽¹⁰⁾⁽¹²⁾ that in most cases the mode of deformation of polycrystals lies between the constant strain and the constant stress conditions.

Polycrystals considered hereafter is assumed to be composed of two kinds of grains, i.e. I and J. The following assumptions are made for the analysis.

(1) When the applied tensile load is removed the path of unloading of grains is represented with a vertical line AB in Fig. 1. The elastic strain is negligible compared with plastic strain^(*).

(2) When the load is applied again after the process (1), the path of grains rises along the unloading path BA and comes back to the original loading path. That is, there is no elastic hysteresis of the material.

(3) When polycrystals is compressed after the process (1), the path of grains follows the line which is drawn by 180-degree rotation of the re-loading path around the point B. Namely, the stress-strain curve of each grain does not show a decrease in flow stress due to Bauehinger effect.

(4) Constraint ratio η is constant during deformation throughout the polycrystals.

(5) The volume and the shape of the two kinds of grains are equal.

Under the above assumptions, the analysis can be reduced to a simple geometrical discussion on stress-strain diagram, which will be shown in the following.

3. Linear stress-strain relation

In order to study basic characteristics of deformation behaviour of polycrystals under reversed stress it is assumed in this chapter that each grain has linear stress-strain relation (The stress-strain relation considered is not elastic but plastic. c.f.

Fig. 5). Discussion will be extended to the case of non-linear stress-strain relation in the next chapter. The hardening rates of the grains I and J are denoted by K_i and K_j .

Fig. 1 illustrates the deformation process of the model of polycrystals. The two kinds of thin solid lines are the stress-strain paths of the grains I and J, while the thick line is the averaged behaviour of the model composed of two grains. Namely, the thick line is given as the trajectory of the middle point of the points P and Q, which are the cross points of the thin lines

(a) Physical diagram

(b) Rotated diagram

Fig. 1 Deformation behaviour of grains and polycrystals having linear stress-strain relation

(*) The effect of elasticity on the calculated result is neglected in the present analysis, for it is estimated at less than 0.2 %.

and a line with constant inclination β ($\beta \leq 0$). $\bar{K}$ in the figure is the mean value of K for polycrystals.

The portion $\sigma < 0$ in Fig. 1(a) is rotated 180-degree around the point B and is shown in Fig. 1(b). The pre-strain of polycrystals is denoted by a . The yield stress after inverse loading is assumed to be given in the model by the point C'. The portions BC'D' and D'E' in Fig. 1(b) represent a region of transient softening and one of permanent softening, respectively.

3. 1. Permanent softening

At first, the degree of softening in the region of permanent softening is considered. The positions of the points P and Q in Fig. 1 are written as $P[a+\Delta a, K_i(a+\Delta a)]$ and $Q[a-\Delta a, K_j(a-\Delta a)]$, respectively, and the inclination angle of PQ as β . The relation between β and Δa is given by

$$\beta = \frac{K_i(a+\Delta a) - K_j(a-\Delta a)}{(a+\Delta a) - (a-\Delta a)} = \frac{a(K_i - K_j)}{2\Delta a} + \frac{K_i + K_j}{2}$$

Hence,

$$\Delta a = a(K_j - K_i) / (K_i + K_j - 2\beta) \quad (4)$$

The stress-strain relations after reloading (reverse loading) in the grains I and J in Fig. 1(b), which are rotated 180-degree around the point B(a, 0), are expressed respectively by

$$\sigma = K_i(\varepsilon + 2\Delta a) \quad (\varepsilon \geq a - \Delta a) \quad (5)$$

$$\sigma' = K_j(\varepsilon - 2\Delta a) \quad (\varepsilon \geq a + \Delta a) \quad (6)$$

The deformation curve of the model polycrystals is the trace of the middle point between the cross points of the lines given by Eqs. (5) and (6) and the straight line with the inclination β . The equation of a straight line with the inclination β through the point $R[\varepsilon_0, K_j(\varepsilon_0 - 2\Delta a)]$ which is on the line expressed by Eq. (6) is given by

$$\sigma = \beta(\varepsilon - \varepsilon_0) + K_j(\varepsilon_0 - 2\Delta a) \quad (7)$$

The cross point $S(\varepsilon_1, \sigma_1)$ of the lines given by Eqs. (5) and (7) is given by

$$\varepsilon_1 = \frac{1}{K_i - \beta} \{ \varepsilon_0(K_j - \beta) - 2\Delta a(K_i + K_j) \}$$

$$\sigma_1 = \frac{K_i}{K_i - \beta} \{ \varepsilon_0(K_j - \beta) - 2\Delta a(K_i + K_j) \}$$

The coordinates of the middle point of the line segments RS are given by

$$\varepsilon' = \frac{1}{2(K_i - \beta)} \{ \varepsilon_0(K_i + K_j + 2\beta) - 2\Delta a(K_i + K_j) \}$$

$$\sigma' = \frac{1}{2(K_i - \beta)} \{ \varepsilon_0[2K_i K_j - \beta(K_i + K_j)] - 2\Delta a[2K_i K_j + \beta(K_i - K_j)] \}$$

Eliminating ε_0 from the above two equations and substituting Δa given by Eq. (4), we have

$$\sigma_R' = \frac{2K_i K_j - \beta(K_i + K_j)}{K_i + K_j - 2\beta} \varepsilon' + \frac{2a\beta(K_j - K_i)^2}{(K_i + K_j - 2\beta)^2} \quad (8)$$

Eq. (8) expresses the line D'E' in Fig. 1(b). Meanwhile, the original line OA in Fig. 1 is given as follows. As the point A

is given by $[a, \{K_i(a+\Delta a) + K_j(a-\Delta a)\}/2]$, we have from Eq. (4)

$$\sigma_F = \frac{K_i(a+\Delta a) + K_j(a-\Delta a)}{2a} \varepsilon = \frac{2K_i K_j - \beta(K_i + K_j)}{K_i + K_j - 2\beta} \varepsilon \quad (9)$$

Consequently, the softening after the reverse loading is given from Eqs. (2), (8) and (9).

$$\sigma_B = -2a\beta(K_j - K_i)^2 / (K_i + K_j - 2\beta) > 0$$

If the definitions

$$\kappa \equiv K_j / K_i, \quad \xi \equiv -2\beta / (K_i + K_j) \quad (10)$$

are introduced, and both sides of the above equation are divided by σ_F to get dimensionless expression of the softening, we have

$$\frac{\sigma_B}{\sigma_F} = \frac{2\xi(\kappa - 1)^2}{(1 + \xi)(4\kappa + \xi(1 + \kappa)^2)} \frac{a}{\varepsilon} \quad (11)$$

Figs. 2(a) and (b) represent Eq. (11) for various values of κ and ξ , where the case $\kappa \geq 1$ is considered representatively, as the case $0 \leq \kappa < 1$ is given just as the reverse of the case $\kappa \geq 1$.

3. 2. Decrease in yield stress

Next, Bauchinger stress, that is, the decrease in the yield stress after reverse loading is considered. In the present model the yield point during reverse loading is given with the point C' in Fig. 1(b).

The position of the point C' is obtained as follows. The equation of the straight

Fig. 2 Relation between ratio $(\sigma_B/\sigma_F)(\varepsilon/a)$ of permanent softening and (a) ratio κ of hardening rates, and (b) ξ

line with the inclination β through the turning point $T[a-\Delta a, K_i(a+\Delta a)]$ on the reversed stress-strain relation of the I grain is given by

$$\sigma = \beta(\epsilon - a + \Delta a) + K_i(a + \Delta a) \quad (12)$$

The cross point U of the above line with the vertical line $\epsilon = a + \Delta a$ is given by U $[a + \Delta a, 2\beta\Delta a + K_i(a + \Delta a)]$. The middle point C' of the line segment TU is expressed by C' $[a, \beta\Delta a + K_i(a + \Delta a)]$. Therefore, the yield stress σ_{R0}' in the reverse loading is given from Eq. (4) by

$$\sigma_{R0}' = \frac{\beta(K_i + 3K_j) + 2K_iK_j}{K_i + K_j - 2\beta} a \quad (13)$$

Meanwhile, the yield stress σ_{F0} in the simple reloading process is given by the point A. Substituting $\epsilon = a$ in Eq. (9), σ_{F0} is given by

$$\sigma_{F0} = \frac{2K_iK_j - \beta(K_i + K_j)}{K_i + K_j - 2\beta} a \quad (14)$$

Consequently, Bauehinger stress σ_{B0} is given from Eqs. (13) and (14) by

$$\sigma_{B0} = \sigma_{F0} - \sigma_{R0}' = \frac{2a\beta(K_i - K_j)}{K_i + K_j - 2\beta} > 0$$

Both sides of the above equation are divided by σ_{F0} to obtain non-dimensional expression. From Eqs. (10) and (14) we have

$$\frac{\sigma_{B0}}{\sigma_{F0}} = \frac{2\xi(\kappa^2 - 1)}{4\kappa - \xi(1 + \kappa)^2} \quad (15)$$

Figs. 3(a) and (b) show Eq. (15) for various values of κ and ξ in the case $\kappa \geq 1$.

Now, let us consider the relation between the inclination β and the constraint ratio η . The point V in Fig. 1(a) is expressed by

Fig. 3 Relation between ratio σ_{B0}/σ_{F0} of decrease in yield stress and (a) ratio κ of hardening rates, and (b) ξ

$$V \left(\frac{K_i(a + \Delta a) + K_j(a - \Delta a)}{2K_i}, \frac{K_i(a + \Delta a) + K_j(a - \Delta a)}{2} \right)$$

From the definition of constraint ratio η (Eq. (3)) and Eq. (10), we have

$$\eta = \frac{\overline{AW}}{\overline{AV}} = \frac{K_i}{K_i - \beta} = \frac{2}{2 + \xi(1 + \kappa)} \quad (16)$$

Hence, if K_i and K_j are given, it is possible to replace β or ξ by η . If Eq. (16) is solved with respect to ξ , we have

$$\xi = 2(1 - \eta)/(\eta(1 + \kappa)) \quad (17)$$

Substituting Eq. (17) into Eqs. (11) and (15), the ratio of permanent softening and decrease in yield stress are expressed with η respectively as follows.

$$\frac{\sigma_B}{\sigma_F} = \frac{2\eta(1 - \eta)(\kappa - 1)^2}{\{\eta(\kappa - 1) + 2\}\{\eta(\kappa - 1) + \kappa + 1\}} \frac{a}{\epsilon} \quad (18)$$

$$\frac{\sigma_{B0}}{\sigma_{F0}} = \frac{2(1 - \eta)(\kappa - 1)}{\eta(\kappa - 1) + \kappa + 1} \quad (19)$$

Some examples obtained from Eqs. (18) and (19) are shown in Figs. 4(a) and (b), respectively.

The ratio of the permanent softening increases almost linearly with an increase in the ratio κ of hardening rates as is shown in Fig. 2(a). The ratio of permanent softening takes its maximum value near $\xi = 1$ as shown in Fig. 2(b), or at $\eta = 0.3 \sim 0.5$ as

Fig. 4 Relation between (a) ratio $(\sigma_B/\sigma_F)(\epsilon/a)$ of permanent softening and (b) ratio σ_{B0}/σ_{F0} of decrease in yield stress, and constraint ratio η

shown in Fig. 4(a). No softening occurs in the cases $\eta=0$ (constant strain) and $\eta=1$ (constant stress).

As shown in Fig. 3, the ratio of decrease in yield stress increases monotonously with the values of κ and ξ . The ratio of decrease in yield stress is comparatively larger than that of the permanent softening. In Fig. 3, the decrease ratio exceeds 100% when κ and ξ are large, which means that the turning point C is in the region $\sigma>0$ in Fig. 1. Namely, the reverse yielding starts during unloading, that is, before the stress drops to zero. This phenomenon has been observed experimentally⁽¹³⁾. As pointed out by R. Hill⁽¹⁴⁾, it happens that macroscopic plastic work is occasionally negative in this case, though microscopic plastic work is always positive.

As shown in Fig. 4(b), there is no decrease in the yield stress for the case $\eta=1$ (constant stress). The ratio increases as η reduces, and takes the maximum value near $\eta=0$, though as is seen from Fig. 1 it is zero exactly at $\eta=0$ (constant strain). That is, permanent softening and decrease in yield stress do not occur under the condition of constant stress, while the reversed yield stress fairly drops under the condition near constant strain, though there are no permanent softening and no yield stress drop exactly at constant strain condition. When the value of η lies between 0 and 1, both the permanent softening and the decrease in yield stress appear, which are largely dependent on the ratio κ of hardening rates.

The above analysis can be applied to the material having rigid-linear stress-strain relation such as shown in Fig. 5, where the total flow stress σ_{total} is represented by

$$\sigma_{\text{total}} = \sigma_0 + K\varepsilon \quad (20)$$

The ratio of the permanent softening is given by the following relation instead of Eq. (11).

$$\frac{\sigma_B}{(\sigma_F)_{\text{total}}} = \frac{\sigma_B}{\sigma_F + \sigma_0} = \frac{\sigma_B}{\sigma_F} \frac{\sigma_F}{\sigma_F + \sigma_0} \quad (21)$$

Namely, the factor $\sigma_F/(\sigma_F + \sigma_0) < 1$ has to be multiplied on the result shown in Figs. 2 and 4(a), where σ_F is given by Eq. (9) corresponding to the case shown in Fig. 1. Similarly, the factor $\sigma_{F0}/(\sigma_{F0} + \sigma_0)$ has to be multiplied on the result of the yield stress shown in Figs. 3 and 4(b), where σ_{F0} is given by Eq. (14).

Fig. 5 Rigid-linear stress-strain relation

4. Non-linear stress-strain relation

The stress-strain curves of many real metals are known to be expressed with an exponential function. The stress-strain curve of each grain is assumed to be expressed with an exponential function $\sigma = k\varepsilon^n$ in this chapter ($0 < n < 1$), as shown in Fig. 6. Analytic solution such as obtained in the case of linear stress-strain relation is not always available in the present case, and numerical calculation is inevitable.

The basic idea of the analysis is the same as that of the previous chapter. The following points are assumed in the present case in addition to the model in Chapter 2. (1) The difference in stress-strain curves of grains is expressed with the factor k , while the exponent n is the same for every grain.

(2) Two kinds of grains having the work-hardening factors k_i and k_j are considered.

4.1. Permanent softening

If pre-strain is denoted with a as shown in Fig. 6, the positions of the points P and Q are given by $P[a + \Delta a, k_i(a + \Delta a)^n]$ and $Q[a - \Delta a, k_j(a - \Delta a)^n]$, respectively. Then, denoting the inclination of $\overline{PQ}$ by β ($\beta \leq 0$), we have

$$k_i(a + \Delta a)^n - k_j(a - \Delta a)^n - 2\beta \Delta a = 0 \quad (22)$$

Substituting the values of a , n , k_i , k_j , and β into Eq. (22), Δa can be calculated numerically (Eq. (22) is composed of such terms as $(\Delta a)^n$ etc. However, the solution Δa with necessary accuracy can be obtained by repeated approximation, considering the facts that the probable range of the solution is relatively narrow (e.g. $0 < \Delta a < a$) and that the left side of Eq. (22) changes its sign on both sides of the solution).

The equations of the curves obtained from 180-degree rotation of the path of

Fig. 6 Deformation behaviour of grains and polycrystals having exponential stress-strain relation

compressive loading for respective grains around the point B (the thin broken lines in Fig. 6) are given by

$$\sigma = k_i(\varepsilon + 2\Delta a)^n \quad (\varepsilon \geq a - \Delta a) \dots\dots\dots(23)$$

$$\sigma = k_j(\varepsilon - 2\Delta a)^n \quad (\varepsilon \geq a + \Delta a) \dots\dots\dots(24)$$

The path of compressive (reversed) loading for the model polycrystals is given by the trace of the middle point of the two points R and S in Fig. 6 which are the cross points of Eqs. (23) and (24) and the straight line with the inclination β . Considering the point $R[\varepsilon_0, k_j(\varepsilon_0 - 2\Delta a)^n]$ on the curve given by Eq. (24) with the abscissa ε_0 and a straight line with the inclination β through R, the abscissa ε_1 of the cross point $S[\varepsilon_1, k_i(\varepsilon_1 + 2\Delta a)^n]$ of this line and the curve given by Eq. (23) satisfy the following equation.

$$\beta(\varepsilon_1 - \varepsilon_0) + k_j(\varepsilon_0 - 2\Delta a)^n - k_i(\varepsilon_1 + 2\Delta a)^n = 0 \dots\dots\dots(25)$$

If the strain of the middle point of the line segment RS is denoted by $\bar{\varepsilon}$, the strain ε_1 of the point S is given by $\varepsilon_1 = 2\bar{\varepsilon} - \varepsilon_0$. Substituting this into Eq. (25), $\bar{\varepsilon}$ is given as a root of the following equation,

$$2\beta(\bar{\varepsilon} - \varepsilon_0) + k_j(\varepsilon_0 - 2\Delta a)^n - k_i(2\bar{\varepsilon} - \varepsilon_0 + 2\Delta a)^n = 0 \dots\dots\dots(26)$$

where Δa is given by Eq. (22). Thus, $\bar{\varepsilon}$ can be obtained for respective values of ε_0 by numerical calculation, similarly to the case of Eq. (22). The stress $\sigma_{R'}$ of the middle point of RS is given by

$$\sigma_{R'} = \{k_j(\varepsilon_0 - 2\Delta a)^n + k_i(2\bar{\varepsilon} - \varepsilon_0 + 2\Delta a)^n\} / 2 \dots\dots\dots(27)$$

The trace of the middle point ($\bar{\varepsilon}, \sigma_{R'}$) obtained from Eq. (27) as a function of ε_0 represents the path of reverse loading of polycrystals.

Meanwhile, the original stress-strain

curve for the model polycrystals, i.e. OA in Fig. 6 is given as follows. The stress-strain relations of the two grains are given by

$$\sigma = k_i \varepsilon^n \dots\dots\dots(28)$$

$$\sigma = k_j \varepsilon^n \dots\dots\dots(29)$$

If the position of the cross point S_0 of Eq. (28) and the straight line passing through the point $R_0(\varepsilon_0, k_j \varepsilon_0^n)$ on the line given by Eq. (29) and having inclination β is denoted by $S_0(\varepsilon_1, k_i \varepsilon_1^n)$, ε_1 satisfies the following equation.

$$\beta(\varepsilon_1 - \varepsilon_0) + k_j \varepsilon_0^n - k_i \varepsilon_1^n = 0 \dots\dots\dots(30)$$

If the strain of the middle point of $\overline{R_0 S_0}$ is equal to the strain $\bar{\varepsilon}$ in Eq. (27), ε_1 is given by $\varepsilon_1 = 2\bar{\varepsilon} - \varepsilon_0$. Then, substituting this into Eq. (30), we have

$$2\beta(\bar{\varepsilon} - \varepsilon_0) + k_j \varepsilon_0^n - k_i(2\bar{\varepsilon} - \varepsilon_0)^n = 0 \dots\dots\dots(31)$$

ε_0 can be calculated numerically from Eq. (31) for respective values of $\bar{\varepsilon}$ (in the program for computer, Eqs. (22), (26) and (31) can be solved with the same subroutine). The stress σ_F of the middle point of $\overline{R_0 S_0}$ is given by

$$\sigma_F = \{k_j \varepsilon_0^n + k_i(2\bar{\varepsilon} - \varepsilon_0)^n\} / 2 \dots\dots\dots(32)$$

Consequently, the stress $\sigma_{R'}$ in the reverse loading is obtained from Eqs. (22), (26) and (27) for the strain $\bar{\varepsilon}$, while the stress σ_F of the simple loading is obtained from Eqs. (31) and (32) for the same $\bar{\varepsilon}$. Thus, if the values of a , n , k_i , k_j and β are given the decrease σ_B of flow stress or the ratio σ_B/σ_F in the permanent softening range can be calculated for respective values of $\bar{\varepsilon}$.

Figs. 7(a)-(d) show a change in the ratio σ_B/σ_F of permanent softening with respect to the total strain. Numerical calculation is made for eight different combina-

Fig. 7 Ratio σ_B/σ_F of permanent softening versus strain $\bar{\varepsilon}$ for various values of n , a and β

tions of the parameters, that is, for the hardening exponent $n = 0.3$ and 0.5 , the pre-strain $a = 0.05$ and 0.1 , and the ratio of hardening rates $\kappa = 1.5$ ($k_i = 10, k_j = 15$) and 2.125 ($k_i = 8, k_j = 17$). The inclination β is varied from -4 to -30 . The corresponding values of constraint ratio η are also shown in the figure. The upper curve in each figure represents a case where the softening is maximum. In the case $n = 0.3$ shown in Fig. 7(a), the maximum value of the softening ratio is about 4.3%. The softening ratio is greater in the case $n = 0.3$ than in $n = 0.5$ shown in Fig. 7(b). It is noteworthy that the maximum softening ratio for $\kappa = 2.125$ (Fig. 7(d)) is nearly 2.5 times that for $\kappa = 1.5$ (Fig. 7(a)) and it reaches almost 10% when strain is small. Namely, the softening ratio σ_B/σ_F is largely affected by the ratio κ of hardening rates, similar to the previous case of linear hardening. It is also said that the softening ratio takes its maximum value at a certain value of β (, that is ξ or η). It is seen that if κ is constant the softening ratio takes its maximum value at a certain value of η , similarly to the case of linear hardening. The value of η where the softening ratio is maximum lies between 0.37 and 0.59.

4. 2. Decrease in yield stress

Consider a straight line with an inclination β which runs through the turning point $T[a - \Delta a, k_i(a + \Delta a)^n]$ in Fig. 6. If the cross point of this line and the vertical line $\epsilon = a + \Delta a$ is denoted by U , its position is given by $U[a + \Delta a, 2\beta\Delta a + k_i(a + \Delta a)^n]$. The coordinates of the middle point C' of the line segment TU is given by $C'[a, \beta\Delta a + k_i(a + \Delta a)^n]$. The yield stress σ_{R0}' in the reverse loading is given by

$$\sigma_{R0}' = \beta\Delta a + k_i(a + \Delta a)^n \quad (33)$$

Meanwhile, the coordinates of the point A which is the middle point of $P[a + \Delta a, k_i(a + \Delta a)^n]$ and $Q[a - \Delta a, k_j(a - \Delta a)^n]$ is given by

$$A[a, (k_i(a + \Delta a)^n + k_j(a - \Delta a)^n)/2]$$

Thus, the yield stress σ_{F0} in reloading is given by

$$\sigma_{F0} = (k_i(a + \Delta a)^n + k_j(a - \Delta a)^n)/2 \quad (34)$$

Therefore, if we denote κ and ξ by

$$\kappa = k_j/k_i, \quad \xi = -2\beta/(k_i + k_j) \quad (35)$$

the ratio σ_{B0}/σ_{F0} of decreases in the yield stress is given from Eqs. (33)-(35) as follows.

$$\frac{\sigma_{B0}}{\sigma_{F0}} = \frac{\kappa(a - \Delta a)^n - (a + \Delta a)^n + a\xi(1 + \kappa)}{\kappa(a - \Delta a)^n + (a + \Delta a)^n} \quad (36)$$

where Δa is obtainable from Eq. (22).

Now, let us find the relation between ξ and the constraint ratio η . Consider a horizontal line through the point A in Fig. 6 (c.f. Fig. 1(a), the line is not shown in Fig. 6). We denote the coordinates of the point A by $A(a, t)$, where t is the stress at A and given by

$$t = (k_i(a + \Delta a)^n + k_j(a - \Delta a)^n)/2$$

Considering the point $W(a + \Delta a, t)$ on the line and the cross point $V(\sqrt{t/k_i}, t)$ of the horizontal line and the curve given by Eq. (28) (c.f. Fig. 1(a)), the constraint ratio η is given from the definition (10)(11) and Eq. (36) as follows.

$$\eta = \overline{AW}/\overline{AV} = \Delta a / [\sqrt{((a + \Delta a)^n + \kappa(a - \Delta a)^n)/2} - a] \quad (37)$$

where Δa is given by Eq. (22). Therefore, if the values of a, n, k_i, k_j and β are given, η is obtained from Eq. (37).

Fig. 8 shows the decrease ratio of yield stress versus ξ . It is seen that the decrease ratio is also largely affected by κ . It is, however, little affected by the hardening exponent n or the pre-strain

Fig. 8 Relation between ratio σ_{B0}/σ_{F0} of decrease in yield stress and ξ

Fig. 9 Relation between ratio σ_{B0}/σ_{F0} of decrease in yield stress and constraint ratio η

α . Fig. 9 shows the relation between the decrease ratio σ_{B0}/σ_{F0} of yield stress and the constraint ratio η . There is no decrease in yield stress at $\eta=1$ (constant stress) and the ratio σ_{B0}/σ_{F0} increases as η decreases, though the ratio is zero exactly at $\eta=0$, similarly to the case of linear hardening.

5. Discussion

The above discussion is based on the model of polycrystals composed of two kinds of grains having different stress-strain relations. In order to apply the above results to polycrystalline metals composed of many grains, it is necessary to estimate the equivalent value of κ for polycrystalline metals. If we denote the equivalent value by $\bar{\kappa}$, this is supposed to be given in the first approximation as follows.

$$\bar{\kappa} = \int \kappa dQ' \dots\dots\dots (38)$$

where κ is the ratio of hardening for a pair of grains (two grains) in polycrystalline metals which show symmetric hardening with respect to the averaged stress-strain relation for polycrystalline metals. $\int dQ'$ implies the normalized integral for all possible pairs of grains in polycrystalline metals ($\int dQ' = 1$).

Furthermore, in order to apply the present analysis to real polycrystalline metals, it is necessary to know the stress-strain relation of each grain quantitatively. Although the stress-strain curves of single crystals with various orientations are known experimentally, those of grains in polycrystals are not well known. In the former case, the difference in stress-strain curve due to orientation is relatively large. For instance, when the tensile flow stress σ of single crystal is considered, in which slip occurs on a single slip plane according to Schmid law⁽¹⁵⁾⁽¹⁶⁾, that is, the critical resolved shear stress is constant independent of crystal orientation, the ratio $\sigma_{\max}/\sigma_{\min}$ (the maximum value of κ) of the maximum and the minimum flow stress is 1.59 for bcc crystals⁽¹⁷⁾ and 1.78 for fcc crystal⁽¹⁸⁾ (for hcp crystal the ratio is more than 5⁽¹⁵⁾⁽¹⁹⁾). Although the ratio may be large when the work hardening due to interaction between slips is considered in single crystal⁽²⁰⁾, the difference is expected to be less in the case of grains in polycrystals, for their deformation becomes more uniform owing to the restriction between grains during deformation. Hence, it may be expected that the maximum value of κ lies between 1 and 2 in most of polycrystalline metals, and the value of $\bar{\kappa}$ in Eq. (38) is smaller than that.

Now, we consider the correlation between the present analysis and fatigue or repeated loading. If the applied load is reversed again in the region D'E' in Fig. 1(b), a similar softening to the first reversal of the applied load is expected to occur⁽²¹⁾ (this is not expected, however, if the load is reversed during C'D'). It may be said that softening accumulates in every reversing of load. Therefore, when the applied stress is reversed many times, a fairly large result-

ant sum of softening may be expected even if softening in each stage is very small. This fact may be important in the initial stage of fatigue damage together with the trend that strain concentrates in those grains which have small Young's modulus and small grain size⁽²¹⁾.

Polycrystals, in which each grain has different stress-strain relation, are mainly considered in the present analysis. However, this assumption is not essential to the present analysis. Namely, any kind of inhomogeneity of materials can be replaced by the present model of grains in polycrystals. For instance, if there is a slip zone in crystal which is easier to deform than the other regions in the crystal, it may be possible to apply similar analysis to the slip zone in crystal.

It is characteristic of the result of the present analysis that it could give a quantitative estimate of the softening caused by inhomogeneity of materials so far as κ and η are known. It is expected, in the case of composite materials or materials composed of several phases, that the softening caused by the mutual restriction between elements or phases in the materials which have different hardening rates may play an essential role in Bauehinger effect of the materials, as is shown in the present analysis, for the values of κ for those materials are expected to be quite large in many cases. For instance, when a fiber reinforced composite is subjected to the uniaxial stress in its fiber direction, the deformation mode of fiber and matrix is known to be close to that under constant strain. Hence, it is expected from the above analysis that yield stress fairly decreases in this case, though the permanent softening is small.

Kishi and Tanabe⁽²²⁾ proposed based on their experiments that the Bauehinger stress σ_B be expressed by

$$\sigma_B = k' a^m \dots\dots\dots (39)$$

where a is plastic pre-strain and k' and m are constants. In order to compare the present result with Eq. (39), the relation between the Bauehinger yield stress σ_{B0} and a is calculated from Eqs. (33) and (34). The result is shown in Fig. 10 for various

Fig. 10 Relation between σ_{B0} and a for various values of k_i , k_j , n , and β

values of k_1 , k_2 , n and β . The relation between σ_{B0} and a is nearly straight, suggesting Eq. (39) is approximately deduced from the present analysis. The inclination of each line, which corresponds to the value of m in Eq. (39), is estimated from the inclination between $a = 0.05$ and 0.5 as follows; (1) $m = 0.68$, (2) 0.78 , (3) 0.81 , (4) 0.56 , (5) 0.72 . Thus, it is expected that the value of m increases with an increase in n , with a decrease in the absolute value of β , or with an increase in the hardening ratio κ .

Finally, it may be said from the present analysis that Bauehinger effect has close connection with heterogeneity of materials, such as polycrystals, and that the conception of constraint ratio offers a method of quantitative analysis of the problem.

6. Conclusions

Bauehinger effect of inhomogeneous materials, especially polycrystals composed of grains having various stress-strain relations is studied analytically with the aid of constraint ratio which represents quantitatively the condition of mutual restriction between grains. The results may be summarized as follows.

- (1) Stress-strain curve for each grain is assumed to be linear at first and next to be exponential ($\sigma = k\epsilon^n$). The ratio of softening in permanent softening region and the ratio of decrease in yield stress are calculated for both cases.
- (2) The permanent softening and the decrease in yield stress are most affected by the difference in the hardening rate or the ratio κ in both cases.
- (3) In the case of exponential stress-strain relation, the ratio of softening is not largely affected by pre-strain a , while it increases as hardening exponent n comes close to 0 . Meanwhile, the ratio of decrease in yield stress is hardly affected by a or n .
- (4) When the stress-strain relation is linear, no permanent softening occurs either at constant stress ($\eta=1$) or at constant strain condition ($\eta=0$). The permanent softening takes its maximum value between

them ($\eta \approx 0.3 \sim 0.5$). On the other hand, the decrease in yield stress is large near constant strain, though no decrease occurs exactly at both constant strain and constant stress. Similar trends are also seen in the case of exponential stress-strain relation.

References

- (1) Gokyu, Y. and Kishi, T., J. of Plasticity, Vol. 10, No. 107 (1969), p. 863 (in Japanese).
- (2) Kumakura, S., J. Japan Soc. Mech. Eng., Vol. 70, No. 584 (1967), p. 1287 (in Japanese).
- (3) Orowan, E., Internal Stress and Fatigue in Metals, Ed. by Rassweiler, G. M. and Grube, W. L., (1959), p. 59, Elsevier.
- (4) Udoguchi, T. and Okamura, H., The 5th Japan Cong. Test. Mat., (1962), p. 86.
- (5) Abel, A. and Ham, R. K., Acta Met., Vol. 14 (1966), p. 1489.
- (6) Taira, S., Hayashi, K. and Kimura, T., Preprint of 10th X-ray Symp. on Strength of Mat., (1972), p. 109, Soc. Mat. Sci., Japan (in Japanese).
- (7) Heyn, E., Met. und Erz., Vol. 15 (1918), p. 411.
- (8) Hutchinson, J. W., J. Mech. Phys. Solids, Vol. 12 (1964), p. 11; p. 25.
- (9) Sachs, G. and Shoji, H., Z. Physik, Vol. 45 (1927), p. 776.
- (10) Abe, T., Bulletin of JSME, Vol. 12, No. 50 (1969), p. 165.
- (11) Abe, T., Bulletin of JSME, Vol. 14, No. 78 (1971), p. 1263.
- (12) Abe, T., Bulletin of JSME, Vol. 16, No. 93 (1973), p. 474.
- (13) Feltner, C. E. and Laird, C., Acta Met., Vol. 15 (1967), p. 1621.
- (14) Hill, R., Prik. Mat. Mekh., Vol. 35 (1971), p. 31.
- (15) Schmid, E. and Boas, W., Plasticity of Crystals, (1950), F. A. Hughes & Co.
- (16) Jaoul, B., Étude de la plasticité et application aux métaux, (1964), Dunod.
- (17) Calnan, E. A. and Clews, J. B., Phil. Mag., Vol. 42 (1951), p. 616.
- (18) Sachs, G., VDI-Z, Vol. 72 (1928), p. 734.
- (19) Taira, S. and Abe, T., Bulletin of JSME, Vol. 11, No. 45 (1968), p. 419.
- (20) Hosford, Jr., W. F., Acta Met., Vol. 14 (1966), p. 1085.
- (21) Abe, T., Bulletin of JSME, Vol. 20, No. 147 (1977), p. 1076.
- (22) Kishi, T. and Tanabe, T., J. Mech. Phys. Solids, Vol. 21 (1973), p. 303.
- (23) Hencky, H., ZAMM, Vol. 4 (1924), p. 323.
- (24) Hutchinson, J. W., Proc. Roy. Soc. Lond., Vol. A319 (1970), p. 247.
- (25) Drucker, D. C., Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids, Ed. by Parkus, H. and Sedov, L. I., (1968), p. 77, Springer.
- (26) Kesten, J., ibid., p. 177.