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Distortions and Residual Stresses of Gears
Caused by Hardening*
(2nd Report, Through-Hardening of Gear
Made of S45C Carbon Steel)

By Hiroshige FUJIO**, Toshio AIDA***
and Yuji MASUMOTO****

From the hardening test of S45C carbon steel cylinders, it is confirmed that the maximum cooling rate of each part of cylinders is the most important factor influencing the amount of martensite of the part. Then this result and many useful data obtained in the previous report are applied to the calculation of hardening distortion of a standard gear with module $m=3$ mm, number of teeth $z=16$, cutter pressure angle $\alpha_c=20^\circ$ and face width $b=200$ mm.

The results of the numerical calculation are compared with those of the experiment and a fairly good coincidence is observed between them.

1. Introduction

It is very difficult to analyze distortions and residual stresses of gear caused by hardening theoretically, on account of the complexity of gear form. This is the reason why they have been investigated only qualitatively, and there are few quantitative and theoretical studies.

In the 1st report⁽¹⁾, the authors compared calculated results by finite element method with experimental ones by Sachs' method for the residual stresses generated in a cylinder made of S45C by water through-hardening process. Those results showed a good coincidence and, many and necessary data for the calculation of quenched gear deformation were obtained.

In this report, particularly focusing our attention to the deformations of gear tooth form, the distortions of gears were analyzed theoretically in the case of water through-hardening gears made of S45C, which is of the same material used in the 1st report. By comparing the theoretical results with the experimental ones, availability of the analytical method was confirmed and the characteristics of gear distortions by hardening were determined.

2. Equation of temperature distribution

Analytical method adopted in this report is similar to the one used by Berg et al.⁽²⁾, and it is described briefly in this paper.

2.1 Application of finite element method to thermal conduction

When density, specific heat and thermal conductivity are denoted as ρ , c and λ respectively, equation for unsteady thermal conduction is given as follows.

$$\lambda \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - c\rho \frac{\partial \theta}{\partial t} + q = 0 \quad \dots\dots(1)$$

Where θ , t and q denote temperature, time and the rate of heat generation respectively. The functional with respect to Eq.(1) can not be given because of the problem being an unsteady one. Supposing that the unsteady term $-c\rho(\partial\theta/\partial t)$ is constant from a time t during time interval Δt , and that the term is considered to be only a function of x and y , the functional has the following form in consideration of boundary conditions.

$$X_t = \iint \left[\frac{1}{2} \lambda \left\{ \left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 \right\} + c\rho \frac{\partial \theta}{\partial t} \theta - q\theta \right] dx dy + \int_C q_0 ds + \int_C \frac{1}{2} \alpha_0 \theta^2 ds \quad \dots\dots(2)$$

Where s , α_0 and q_0 denote the surface length along the boundary, the heat transfer coefficient and heat flux per unit surface on the boundary respectively. The second and third terms in Eq.(2) are calculated along the boundary C . From the Euler theorem, minimization of Eq.(2) is equivalent to the solution of Eq.(1). So, from those points of view, the finite

* Received 19th June, 1975.

** Assistant, Faculty of Engineering, Kyoto University.

*** Professor, Faculty of Engineering, Kyoto University.

**** Engineer, Ishikawajima-Harima Heavy Industries Co. Ltd.

element method is applied to the unsteady thermal conduction. Temperature θ in the element e is given by linear polynomials with respect to coordinates x and y .

$$\theta = \alpha_1 + \alpha_2 x + \alpha_3 y \quad \dots\dots\dots(3)$$

The temperatures at the nodal points of triangular element ijm are formulated as follows.

$$\left. \begin{aligned} \theta_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ \theta_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ \theta_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m \end{aligned} \right\} \quad \dots\dots\dots(4)$$

Equation (3) is rewritten finally as follows by solving Eq.(4) for α_1 , α_2 and α_3 in terms of the nodal temperatures

$$\theta = \frac{1}{2\Delta} \{ (a_i + b_i x + c_i y) \theta_i + (a_j + b_j x + c_j y) \theta_j + (a_m + b_m x + c_m y) \theta_m \} \quad \dots\dots\dots(5)$$

, in which notation Δ is the area of the triangular element, and a_i , b_i , c_i , ... are given by the coordinates of the nodal points in the triangular element. Equation (5) is represented in matrix form.

$$\theta = [N_i \ N_j \ N_m] \{ \theta \}^T = [N] \{ \theta \}^T \quad \dots\dots\dots(6)$$

Where N_i is defined as follows:

$$N_i = (a_i + b_i x + c_i y) / (2\Delta) \quad \dots\dots\dots(7)$$

X_i is minimized so that the temperature θ in the element is defined identically with the coordinates, and continuously. Therefore, by the differentiation of the functional X_i , summation of differential coefficient for each nodal point is taken as zero. From Eqs.(5), (6) and (7), differential coefficient of the first term X_i of the functional is represented as follows.

$$\begin{aligned} \frac{\partial X_i}{\partial \theta_i} &= \frac{1}{(2\Delta)^2} \left\{ \lambda \left[b_i b_i \int \{ \theta \}^2 dx dy + [c_i \ c_i] \int \{ \theta \}^2 dx dy \right. \right. \\ &\quad \left. \left. + \int c_i p [N] \left\{ \frac{\partial \theta}{\partial t} \right\}^T N_i dx dy - \int q [N] dx dy \right] \right\} \quad \dots\dots\dots(8) \end{aligned}$$

And for the nodal point of the element, differential coefficient is represented in vector as follows:

$$\left\{ \frac{\partial X_i}{\partial \theta} \right\} = \left\{ \frac{\partial X_i}{\partial \theta_i} \ \frac{\partial X_i}{\partial \theta_j} \ \frac{\partial X_i}{\partial \theta_m} \right\}^T = [h] \{ \theta \}^T + [p] \left\{ \frac{\partial \theta}{\partial t} \right\}^T + \{ Q \}_q^T \quad \dots\dots\dots(9)$$

Where matrix $[h]$ depends on b_i , c_i , ..., λ and Δ . And matrices $[p]$, $\{ Q \}_q^T$ are written as follows.

$$\left. \begin{aligned} [p] &= \int \int c_p [N]^T [N] dx dy \\ \{ Q \}_q^T &= - \int \int q [N] dx dy \end{aligned} \right\} \quad \dots\dots\dots(10)$$

Thus matrices $[H]$, $[P]$ and $\{ Q \}_q$ in the global system of assembled elements are

obtained from element matrices $[h]$, $[p]$ and $\{ Q \}_q^T$. And the second and third terms in Eq.(2) are integrated along the boundary C . The differential coefficients of these terms with respect to the coordinate of nodal point are simply added to Eq.(9). For example, assumption that the temperature changes linearly from θ_r to θ_s on the boundary $r-s$, and that q_0 and α_0 are constant, leads them to Eq.(11).

$$\frac{\partial}{\partial \theta_r} \left\{ \int_C q_0 ds + \int_C \frac{1}{2} \alpha_0^2 ds \right\} = \frac{q_0 L}{2} + \frac{\alpha_0}{3} (\theta_r + \frac{1}{2} \theta_s) \quad \dots\dots\dots(11)$$

The matrices $\{ Q \}_q$ and $[A]$ are assembled by the submatrices for the second and third terms of Eq.(2). From Eqs.(2), (9) and (11), the following equation is obtained:

$$[H] \{ \theta \} + [P] \left\{ \frac{\partial \theta}{\partial t} \right\} + \{ Q \}_q + \{ Q \}_q + [A] \{ \theta \} = 0 \quad \dots\dots\dots(12)$$

Assumption that the term $\{ d\theta/dt \}$ changes linearly during time interval Δt leads to the following equation.

$$\{ \theta \}_t = \{ \theta \}_{t-\Delta t} + \left(\left\{ \frac{\partial \theta}{\partial t} \right\}_{t-\Delta t} + \left\{ \frac{\partial \theta}{\partial t} \right\}_t \right) \frac{\Delta t}{2} \quad \dots\dots\dots(13)$$

Also suffixes t and $t-\Delta t$ denote the values of each term in Eq.(12) at the t th and $t-\Delta t$ th stages of time, and the following equation is formulated from Eq.(13).

$$\begin{aligned} \left([H] + \frac{2}{\Delta t} [P] + [A] \right) \{ \theta \}_t &= \left(\frac{2}{\Delta t} [P] - [H] \right. \\ &\quad \left. - [A] \right) \{ \theta \}_{t-\Delta t} - \left(\{ Q \}_{qt} + \{ Q \}_{qt-\Delta t} + \{ Q \}_{qy} + \{ Q \}_{qt-\Delta t} \right) \quad \dots\dots\dots(14) \end{aligned}$$

From this equation, the temperature in the problem of unsteady thermal conduction can be solved one by one.

2.2 Comparison of the solution by finite element method with the theoretical solution

As the accuracy of solution from the finite element method is not always satisfactory, the solution from Eq.(14) is compared with a known solution. Then the solution for a cylinder which has simple shape is chosen. The theoretical solution has been shown already in the first report. For analysis from the finite element method, elements division is considered in the region which is enclosed by two planes containing axis as shown Fig.1, and those two planes are adiabatic boundaries. Cooling conditions are taken as follows. The initial temperatures of the cylinder and the cooling medium are 850°C and 20°C respectively. The density, the specific heat, the thermal conductivity and the heat transfer coefficient are taken as $\rho=7.86 \text{ g/cm}^3$, $c=0.14 \text{ cal/g}^\circ\text{C}$, $\lambda=0.1 \text{ cal/(cm}^2\text{ }^\circ\text{C sec)}$ and $\alpha_0=5000 \text{ kcal/(m}^2\text{ }^\circ\text{C h)}$. Those values are equal to ones in the case of the gear cooling process. For example, Fig.2 indicates the temperature on the cylindrical surface containing with the cooling medium with the elapse of time. In the case of calculated result

by finite element method, the temperature is oscillatory, but this phenomenon can never take place actually. As they agree well with the theoretical results in consideration of the mean value before and after a time, these method were applied to the calculation of gear temperature.

3. Stress calculation

3.1 Calculation of displacement, distortion and stress

Displacements for the gear hardening by the finite element method can be calculated according to the method described in the 1st report. But the contents in the 1st report were in regard to axi-symmetric deformation, so they were analyzed by the cylindrical coordinates. In this report, the problem is analyzed under the x, y and z coordinates system. Therefore, stress-strain matrix is formulated as follows in the same way as Eq.(25) in the 1st report.

$$[D] = [D] - \frac{2G}{S} \begin{bmatrix} \sigma_x'^2 & & & & & \\ \sigma_x' \sigma_y' & \sigma_y'^2 & & & & \\ \sigma_x' \sigma_z' & \sigma_y' \sigma_z' & \sigma_z'^2 & & & \\ \sigma_x' \tau_{yz} & \sigma_y' \tau_{yz} & \sigma_z' \tau_{yz} & \tau_{yz}^2 & & \\ \sigma_x' \tau_{zx} & \sigma_y' \tau_{zx} & \sigma_z' \tau_{zx} & \tau_{yz} \tau_{zx} & \tau_{zx}^2 & \\ \sigma_x' \tau_{xy} & \sigma_y' \tau_{xy} & \sigma_z' \tau_{xy} & \tau_{yz} \tau_{xy} & \tau_{zx} \tau_{xy} & \tau_{xy}^2 \end{bmatrix} \quad \text{symm.} \quad \dots\dots\dots(15)$$

Where notation G is shear modulus, so σ_x' , σ_y' and σ_z' denote the deviatoric stresses in the x, y and z directions respectively. Moreover $[D]$ and $[D]$ indicate stress-strain matrix in the plastic region and the elastic region respectively. Also, S is determined by the shear modulus, strain hardening parameter, equivalent stress and so on.

The elements division on all teeth of gear in the case of finite element method makes the number of elements enormous. Then calculation is impossible under this

Fig.1 Elements division of cylinder

condition. Therefore, by symmetry of gear shape and deformation to center line of gear tooth, the distortions are analyzed in the region between center lines of gear tooth and space. Also, it is assumed that the nodal points on the center lines of gear tooth and space can deform only in the radial direction and are prevented in the circumferential direction. Hence the supporting method of nodal point on the inclined plane⁽³⁾ was applied. The elements division of gear is shown in Fig.3.

3.2 Thermal expansion coefficient and yield stress

As the material of gears is the same as that in the 1st report, the same data as to the thermal expansion coefficient and yield stress are adopted.

Fig.4 indicates the relation between the temperature and the thermal expansion coefficient. The broken line shows an actual thermal expansion coefficient in consideration of A₁ transformation during the heating process. But in this analysis it is assumed that this transformation is negligible and the elongation ratio during the heating process between the initial temperature and heating tempera-

Fig.2 Comparison of solutions by finite element method with theoretical solution

Fig.3 Elements division of gear

ture just before quenching is uniform. Thus, the thermal expansion coefficient shown as a full line was adopted for the calculation.

3.3 Estimation of volumetric fraction of martensite

The volumetric fraction of martensite was determined by measuring the hardness in the 1st report. But when we wish to know the deformation caused by quenching, it is necessary to estimate the volumetric fraction of martensite beforehand. For this purpose, solid test cylinders of the same material as one for test gears, with diameters $\phi 50$, $\phi 40$, $\phi 30$, $\phi 20$ and $\phi 15$, and a larger length than the diameter, were made. After the test pieces were

Fig. 4 Thermal expansion coefficient during heating and quenching process

Fig. 5 Hardness distribution of through-hardening cylinders

Fig. 6 Relation between time and volumetric fraction of martensite

held at 850°C in the furnace, they were quenched in a well-stirred water at 20°C. Fig. 5 shows the hardness distribution on the section perpendicular to the axis of the solid cylinders

(1) Estimating method by the time to reach the standard temperature after quenching or by the cooling rate at Ms point

Fig. 6 is obtained from the arrangement of relation between volumetric fraction of martensite and the time to reach the standard temperature which is higher than the starting temperature of martensite transformation θ_{Ms} . When the standard temperature is taken as 500°C, the volumetric fraction of martensite can not be estimated identically for all sizes of the solid cylinders. Also, when the standard temperature is taken as 550°C, the calculated points for the solid cylinders with diameters $\phi 50 \sim \phi 30$ exist nearly on a curved line. But those for the ones with small diameters are away from the curved line, so this method is unsatisfactory for estimating the volumetric fraction of martensite.

Next, it was tried to estimate the volumetric fraction of martensite from the cooling rate at the starting temperature of martensite transformation, 390°C. The result is shown in Fig. 7. Also, mass effect is observed in this case, so this method is found inapplicable for gears with complex shape.

(2) Method by maximum cooling rate

Fig. 8 is the result obtained assuming that the maximum cooling rate $|d\theta/dt|_{\max}$

Fig. 7 Relation between cooling rate at Ms point and volumetric fraction of martensite

Fig. 8 Relation between maximum cooling rate and volumetric fraction of martensite

during the cooling process will have much influence on the volumetric fraction of martensite. The result shows that, though the calculated points are scattered more or less, a low mass effect in comparison with the other methods gives the measured points on a curved line. As the estimation by this result was recognized to be the most effective method, martensite contents of gear were estimated by Fig.8, calculating the maximum cooling rate from the temperature obtained by the method in Chap.2.

4. Dimensions of test gears, and terms of measuring distortions

4.1 Dimensions of test gears and quenching condition

Test gears have the following dimensions: module=3 mm, pressure angle $\alpha_c=20^\circ$ and number of teeth $z=16$. Their face width $b=200$ mm is greater than the diameter of the tip cylinder, as shown in Fig.9, so that heat diffuses on the plane perpendicular to the gear axis and stresses can be analyzed under the condition of plane strain.

The raw material of test gears was normalized. And, test gears were hobbled and finished by grinding to the required size.

4.2 Terms of measuring distortions

4.2.1 Eccentricity of tip cylinder and runout of gear teeth space

Eccentricity of tip cylinder was measured by an indicator whose minimum scale was $1\text{ }\mu\text{m}$. Runout of gear teeth space was

Fig.9 Size of test gear

Fig.10 Measurement of tooth thickness by over-pins method

measured, according to JIS B 1752, by the transferred distance of pins in the radial direction, which were in contact with both tooth surfaces.

4.2.2 Diameter of tip cylinder

Diameters of tip cylinder were measured for all teeth of gear by the micrometer.

4.2.3 Change of tooth thickness, and calculation of standard pressure angle

Over-pins method⁽⁴⁾ was applied to the measurement of change of tooth thickness. Two roller pins made for roller bearings and having $d_i=\phi 4, \phi 5, \dots, \phi 9$, were used. And outside diameters over the two pins located in diagonally opposite tooth space, were measured by a micrometer⁽⁵⁾.

For the gear with even number of teeth, the following equation is formulated.

$$\left. \begin{aligned} \text{inv}\phi &= \text{inv}\alpha_c + \frac{d_i}{mz\cos\alpha_c} - \frac{\pi}{2z} + \frac{2\tan\alpha_c}{z}x \\ M &= d_i + \frac{mz\cos\alpha_c}{\cos\phi} \end{aligned} \right\} \dots (16)$$

Where notation ϕ is shown in Fig.10. And x is the addendum modification coefficient, but it is assumed to be zero in this calculation. When the outside diameter

Fig.11 Change of tooth thickness before and after quenching

Fig.12 (a) Temperature distribution during cooling process
(b) Isochronic lines for 550°C

over two pins after quenching changes by ΔM and the pressure angle changes, a change of half tooth thickness in the circumferential direction Δs is formulated as follows from Fig.11.

$$\Delta s = \frac{\Delta M}{2} \tan \phi + \frac{d_i}{2} (\sin \phi - \sin \phi_0) \tan \phi + \frac{d_i}{2} (\cos \phi - \cos \phi_0) \dots (17)$$

The coordinate of contact points between pin and tooth surface is obtained in Fig.10.

$$\left. \begin{aligned} x &= \frac{d_i}{2} \cos \phi \\ y &= (M - d_i) \frac{1}{2} - \frac{d_i}{2} \sin \phi \end{aligned} \right\} \dots (18)$$

Next, change of pressure angle is calculated. Standard pressure angle can be established approximately by application of Newton's method by Eq.(16), in which α_c is taken as an unknown.

4.2.4 Tooth height

Tooth heights of all teeth of gear were measured by making the detector of dial indicator contact with tooth crest and bottom land, whose minimum scale is 1/100 mm.

4.2.5 Diameter of root cylinder

Diameter of root cylinder is calculated by the diameter of tip cylinder and tooth height.

4.2.6 Tooth profile error

Tooth profile and lead error were measured before and after quenching. As the axes of test gears are bent by quenching and those gears have eccentricity, tooth profile error is measured by omitting the influence of the eccentricity from the result of runout.

5. Experimental and calculated results

5.1 Temperature distribution during the cooling process

Fig.12 shows temperature distribution calculated during the cooling process, in which the test gears uniformly heated at 850°C were quenched into the water at 20°C. Fig.12(a) shows the temperature distribution with the elapse of time and Fig.12(b) shows the isochronic lines for 550°C. In the calculation of temperature, it is assumed that the heat transfer coefficient of water is kept constant at

Fig.13 Maximum cooling rate distribution during cooling process

Fig.14 (a) Hardness distribution
(b) Comparison of estimated martensite contents distribution

Fig.15 Micro structures along the tooth center line

5000 kcal/(m²°C h). But it is considered that the value at the tooth crest is somewhat different from that in the vicinity of the bottom land. Because of the difficulty to obtain it experimentally, in this calculation, the above mentioned constant value was adopted regardless of the tooth parts. In Fig.12(a), the isothermal lines are crossing the tooth profile at somewhat larger angle. This may depend on the size of the finite elements. Therefore if the elements are meshed finely, the isothermal lines will run along the tooth profile.

5.2 Martensite distribution

Fig.13 shows the distribution of the maximum cooling rates during the quenching process. The maximum cooling rate $|\theta|_{\max}$ in the most part of tooth is over 200°C/s, which is enough for the martensite transformation.

Fig.14(a) shows the hardness distribution in the quenched gear. And Fig.14(b) is the martensite distribution induced from Fig.13, Fig.14(a) and the relation between the volumetric fraction of martensite and hardness (Fig.8 in the 1st report). Also, in this figure, the distribution of volumetric fractions of martensite, which is estimated from Fig.6 and Fig.8, is described.

Comparing both results, it is clear that the calculated values by Fig.8 agree better with the measured values. This fact shows that the method of estimation from $|\theta|_{\max}$ is effective.

Fig.15 shows the micro structures at the tooth crest and at the parts distant 1, 2, 3, 4 and 5 modules from the tooth crest. From this result, it can be ob-

served that the most parts of gear tooth have a characteristic needle type structure of martensite. The black part in the microscopic photograph at the part distant 2 modules from the tooth crest is a troostite structure, moreover a pearlite structure is observed at the parts distant 4 and 5 modules.

5.3 Elastic region and plastic region

Fig.16 shows the variation of elastic region and plastic one with the elapse of time. The hatched parts represent an assembly of plastic elements. Just after quenching the gear into water, parts of tooth crest have a large temperature gradient, and a low yield stress. So, in these parts there occurs a plastic region at an earlier period. The plastic region expands along the tooth profile with the elapse of time, and the tooth is covered with the plastic region at 0.1 sec after quenching. But it is estimated that the plastic region may recover to the elastic region as the time goes on. The reason is as follows: The temperature in a part of tooth drops to the martensite transformation temperature at 1 sec ~ 3 sec after quenching, as shown in Fig.12(a). Also the martensite transformation at those times, in consideration of the maximum cooling rate distribution in Fig.13 and the result from Fig.8, brings about an increase in yield stress.

Fig.16 Elastic region and plastic region during cooling process

Fig.17 Changes of tooth height and diameter of tip cylinder

Fig.18 Change of tooth profile before and after quenching

5.4 Change of dimensions before and after quenching

The gear axis is bent by quenching. But changes of the dimensions are shown as follows by omitting its effect.

(1) Changes of tooth height and tip cylinder

Fig.17 shows a change of tooth height before and after quenching. The tooth height shrinks by quenching, but it does not diminish so much as the calculated values. The change of diameter of tip cylinder, as described in Fig.17, shows a similar tendency to that of the tooth height.

(2) Change of tooth profile

Fig.18 shows the tooth profile in the middle of face width at every 90° around the gear axis. The tooth profile increases by quenching. The reason lies in the pressure angle error.

(3) Changes of tooth thickness and pressure angle

Tooth thickness measured in the middle of face width after quenching, is shown in Fig.19. Also the analytical calculated tooth profile is described in this figure. The quenched tooth thickness at the region 0~20 mm far from the side surfaces have different values under the influence of cooling effect of both sides, but those at the central region except both sides have uniform values. As it is difficult to distinguish the change of tooth thickness shown in Fig.19 by the values obtained from over-pins method and theoretical analysis, those results are compared as a change of standard pressure angle. The changes of standard pressure angles from the values obtained by over-pins method and tooth profile error curve (Fig.18) are 0.58° and 0.5° respectively. And the result obtained by the theoretical analysis indicates 0.67°.

Fig.19 Change of tooth thickness before and after quenching

so it agrees fairly well with the measured values.

6. Conclusions

The distortions of gears caused by through-quenching, which have the complex shape, can be obtained analytically and the availability of calculated method is confirmed. Particularly as to the pressure angle change which influences the grinding time after quenching, the calculated and experimental values are in good coincidence qualitatively and quantitatively. But, the diameter change of tip cylinder and change of tooth height before and after quenching obtained from calculation do not always agree well with the measured value. It can be considered that the disagreement in those results may be due to the influence of a work-hardened layer caused by gear machining. And the assumption that the constants in the calculation process are uniform for the temperature, may have an undesirable influence on the calculated results.

It was proved that the utilization of the relation between the maximum cooling rate $|d\theta/dt|_{\max}$ and the hardness does not show any great mass effect and is found to be the most effective method for the estimation of metallic structure caused by through-quenching.

In this report, only the case of through-quenching process was dealt with, but it may be necessary to investigate into the availability of the calculating method and to ascertain the validity of the authors' approach to the case of induction hardening and case-hardening.

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