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Forced Flexural Vibrations of an Elastic Beam Supported by an Elastic Half-Space*

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This paper investigates analytically the plane stress problem of flexural vibrations of an elastic beam supported by an elastic half-space and subjected to a sinusoidally varying force at the free end of the beam. Since the actual stress and displacement distributions at the interface between the half-space and beam are still unknown, we assume linear normal stress distribution and uniform shear stress distribution at the interface, and apply mean displacement and mean slope of the half-space. The dynamic effects of foundation stiffness and slenderness ratio on the response curves of the beam are presented.

1. Introduction

When the flexural vibrations of an elastic beam, one end of which is restrained, are investigated, the restrained end has been usually considered to be fixed in the classical literature, and its boundary conditions involve only the shape of the beam deflection curve. In the case where the end of a beam is fixed to an elastic material, the support is commonly idealized as a combination of torsional spring and lateral spring support, and these spring constants can be decided by moment equilibrium and force equilibrium at the support. However, in reality, it is difficult to determine these spring constants satisfying the actual state, and it is more practical to suppose that the coupled vibrations will occur between the beam and the support owing to the effect of the support flexibility. When the support is of sufficient width and thickness, it may be considered as a half-space and the problem becomes a study of the flexural vibrations of an elastic beam supported by an elastic half-space. During vibrations, the elastic waves are reflected and refracted at the interface between the half-space and elastic beam, and the dissipation of waves to infinity produces a damping of the motion in the system.

In the static case of an elastically supported cantilever beam or plate, the additional deflection due to the elasticity of a half-space support was investigated both analytically and experimentally by O'Donnell [1], Small [2], Matusz, O'Donnell and Erdlac [3]. Using the results derived from the static deformation problem for an elastic half-space support, MacBain and Jemin [4] examined the effect of support compliance on the natural frequencies of a cantilever beam.

In this paper, the plane stress problem of forced flexural vibrations of an elastic beam supported by an elastic half-space is investigated with non-classical boundary conditions, and the dynamic effects of foundation on the flexural vibrations of an elastic beam are presented.

2. Equations of motion

Consider an elastic beam of length l and cross-sectional area $A_R (=b \times h)$, one end of which is attached to an elastic half-space and the other free end subjected to a harmonic disturbing force $Pe^{i\omega t}$. The origin O is located at the beam center of the interface and the x and y -axes are adopted as shown in Fig. 1. The displacement of a beam in the y direction is taken to be v_R .

Fig.1 Elastic beam supported by elastic half-space.

If we assume that the beam deforms only in the y direction, the corresponding equation for the flexural vibrations of a beam is

$$\frac{\partial^2 v_R}{\partial t^2} + a^2 \frac{\partial^4 v_R}{\partial x^4} = 0 \quad \dots\dots\dots (1)$$

with

$$a^2 = \frac{E_R I_R}{\rho_R A_R} \quad \dots\dots\dots (2)$$

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where ρ_R is the density of the beam, E_R is the elastic modulus, I_R is the moment of inertia and t is the time.

The solution of equation (1) is given in the following form

$$v_R(x, t) = V_R(x) e^{i\omega t} \quad (3)$$

where ω is the forced angular frequency of the system. Substituting equation (3) into equation (1) gives

$$V_R(x) = D_1 \sin\{k(l-x)\} + D_2 \cos\{k(l-x)\} + D_3 \sinh\{k(l-x)\} + D_4 \cosh\{k(l-x)\} \quad (4)$$

where

$$k^4 = \frac{\omega^2}{a^2} \quad (5)$$

and D_1 , D_2 , D_3 and D_4 are the unknown function of ω .

The displacement components of a half-space in the x and y directions are taken to be u and v respectively and we only consider the anti-symmetrical case to the x -axis. Then the displacement components u and v can be expressed as follows:

$$u = \int_0^\infty \{A(\xi) \alpha e^{\alpha x} - B(\xi) \xi e^{\beta x}\} \sin(\xi y) d\xi e^{i\omega t} \quad (6)$$

$$v = \int_0^\infty \{A(\xi) \xi e^{\alpha x} - B(\xi) \beta e^{\beta x}\} \cos(\xi y) d\xi e^{i\omega t} \quad (7)$$

where

$$\alpha^2 = \xi^2 - \frac{\omega^2}{c_1^2}, \quad \beta^2 = \xi^2 - \frac{\omega^2}{c_2^2} \quad (8)$$

$$c_1^2 = \frac{2}{1-\nu} \frac{G}{\rho}, \quad c_2^2 = \frac{G}{\rho} \quad (9)$$

In equation (9), c_1 and c_2 are the velocities of dilatation and distortion waves respectively, and ρ is the density of the half-space, G is the shear modulus and ν is Poisson's ratio.

From the well-known relation between stress and displacement, we have

$$\sigma_x = \frac{2G}{1-\nu} \int_0^\infty \{A(\xi) (\alpha^2 - \nu \xi^2) e^{\alpha x} - B(\xi) \beta \xi (1-\nu) \times e^{\beta x}\} \sin(\xi y) d\xi e^{i\omega t} \quad (10)$$

$$\tau_{xy} = G \int_0^\infty \{2A(\xi) \alpha \xi e^{\alpha x} - B(\xi) (\beta^2 + \xi^2) e^{\beta x}\} \times \cos(\xi y) d\xi e^{i\omega t} \quad (11)$$

3. Boundary conditions

When the harmonic disturbing force $P e^{i\omega t}$ shown in Fig. 1 acts on the free end of the beam, the boundary conditions at $x=l$ are

$$E_R I_R \frac{\partial^3 v_R}{\partial x^3} = -P e^{i\omega t} \quad (12)$$

$$\frac{\partial^2 v_R}{\partial x^2} = 0 \quad (13)$$

At the interface $x=0$, the boundary conditions are for the stress

$$\tau_{xy}(0, y, t) = \tau_{xy,R}(0, y, t) \quad \text{for } |y| \leq \frac{h}{2} \\ = 0 \quad \text{for } |y| > \frac{h}{2} \quad (14)$$

$$\sigma_x(0, y, t) = \sigma_{x,R}(0, y, t) \quad \text{for } |y| \leq \frac{h}{2} \\ = 0 \quad \text{for } |y| > \frac{h}{2} \quad (15)$$

and for the displacement and slope

$$v(0, y, t) = v_R(0, t) \quad \text{for } |y| \leq \frac{h}{2} \quad (16)$$

$$\frac{\partial v(0, y, t)}{\partial x} = \frac{\partial v_R(0, t)}{\partial x} \quad \text{for } |y| \leq \frac{h}{2} \quad (17)$$

where the suffix R indicates the quantity of a beam.

The normal and shear stress distributions in a beam are derived from bending moment and shearing force. Since the actual stress distributions at the interface depend on the combined effect of the half-space and beam, the methods which consider the half-space and beam separately are not suitable for obtaining the actual stress distributions, but it is very difficult to solve the problem satisfying the continuous conditions of stress and displacement at the interface $x=0$. Therefore, in the present work, an approximate solution is obtained by assuming certain stress distributions at the interface. Classical beam theory gives shear stress distributions of parabolic shape. However, the fibers near $y = \pm \frac{h}{2}$ take a larger shear load than those given by parabolic distribution and it seems reasonable to assume uniform shear stress distribution at the interface [1], [2]. Thus, we apply

$$\left. \begin{aligned} \tau_{xy,R}(0, y, t) &= -\frac{E_R I_R}{A_R} \frac{\partial^3 v_R(0, t)}{\partial x^3} \\ \sigma_{x,R}(0, y, t) &= -E_R \gamma \frac{\partial^2 v_R(0, t)}{\partial x^2} \end{aligned} \right\} \quad (18)$$

to the shear and normal stress distributions of the beam at the interface. Then, equations (14) and (15) can be written in the following forms

$$\left. \begin{aligned} \tau_{xy}(0, y, t) &= -\frac{E_R I_R}{A_R} \frac{\partial^3 v_R(0, t)}{\partial x^3} \\ &= 0 \end{aligned} \right\} \quad \begin{aligned} &\text{for } |y| \leq \frac{h}{2} \\ &\text{for } |y| > \frac{h}{2} \end{aligned} \quad (19)$$

$$\left. \begin{aligned} \sigma_x(0, y, t) &= -E_R \gamma \frac{\partial^2 v_R(0, t)}{\partial x^2} \\ &= 0 \end{aligned} \right\} \quad \begin{aligned} &\text{for } |y| \leq \frac{h}{2} \\ &\text{for } |y| > \frac{h}{2} \end{aligned} \quad (20)$$

However, due to mathematical complexity, it is impossible to satisfy all the boundary conditions (16)-(20) simultaneously. Consequently, we will apply such approximate boundary conditions as

$$\overline{v(t)} = v_R(0, t) \quad \text{for } |y| \leq \frac{h}{2} \quad (21)$$

$$\frac{\partial \overline{v(t)}}{\partial x} = \frac{\partial v_R(0, t)}{\partial x} \quad \text{for } |y| \leq \frac{h}{2} \quad (22)$$

where $\overline{v(t)}$ is the mean displacement and $\frac{\partial \overline{v(t)}}{\partial x}$ is the mean slope of the half-space at the interface, and can be expressed as follows:

$$\overline{v(t)} = \frac{2}{h} \int_0^{\frac{h}{2}} v(0, y, t) dy \dots\dots\dots (23)$$

$$\frac{\partial \overline{v(t)}}{\partial x} = \frac{2}{h} \int_0^{\frac{h}{2}} \frac{\partial v(0, y, t)}{\partial x} dy \dots\dots\dots (24)$$

Since the actual displacement and slope are variable with the y - axis at the interface $x = 0$, the above assumption will give more reasonable results than the simpler assumption that the displacement and slope of the specific point (e.g. $y = 0$) at the half-space and beam are equal with each other.

dimensionless quantities,

$$\zeta = \xi l, \quad c_R^2 = \frac{E_R}{\rho_R}, \quad \Omega = \frac{\omega l}{c_R}, \quad \kappa_1^2 = \frac{c_R^2}{c_1^2}, \quad \kappa_2^2 = \frac{c_R^2}{c_2^2} \dots\dots\dots (27)$$

the left sides of equations (21) and (22) become

$$\begin{aligned} \overline{v(t)} = & [D_1 \frac{E_R k^2 l^3}{\pi h G} \{E(\sin(kl) - \sinh(kl)) + \frac{I_R k}{A_R l} \kappa_2^2 \Omega^2 F(\cos(kl) - \cosh(kl))\} \\ & + D_2 \frac{E_R k^2 l^3}{\pi h G} \{E(\cos(kl) - \cosh(kl)) - \frac{I_R k}{A_R l} \kappa_2^2 \Omega^2 F(\sin(kl) + \sinh(kl))\} \\ & - \frac{P l^3}{\pi h G I_R k} \{E \sinh(kl) + \frac{I_R k}{A_R l} \kappa_2^2 \Omega^2 F \cosh(kl)\}] e^{i\omega t} \dots\dots\dots (28) \end{aligned}$$

$$\begin{aligned} \frac{\partial \overline{v(t)}}{\partial x} = & [D_1 \frac{E_R k^2 l^2}{\pi h G} \{\kappa_2^2 \Omega^2 H(\sin(kl) - \sinh(kl)) + \frac{2 I_R k}{A_R l} J(\cos(kl) - \cosh(kl))\} \\ & + D_2 \frac{E_R k^2 l^2}{\pi h G} \{\kappa_2^2 \Omega^2 H(\cos(kl) - \cosh(kl)) - \frac{2 I_R k}{A_R l} J(\sin(kl) + \sinh(kl))\} \\ & - \frac{P l^2}{\pi h G I_R k} \{\kappa_2^2 \Omega^2 H \sinh(kl) + \frac{2 I_R k}{A_R l} J \cosh(kl)\}] e^{i\omega t} \dots\dots\dots (29) \end{aligned}$$

where

$$E = \int_0^\infty \frac{1}{D} (2\zeta^2 - \kappa_2^2 \Omega^2 - 2\sqrt{\zeta^2 - \kappa_1^2 \Omega^2} \sqrt{\zeta^2 - \kappa_2^2 \Omega^2}) \{-\frac{h}{2l} \cos(\frac{h\zeta}{2l}) + \frac{1}{\zeta} \sin(\frac{h\zeta}{2l})\} \frac{1}{\zeta} \sin(\frac{h\zeta}{2l}) d\zeta \dots\dots\dots (30)$$

$$F = \int_0^\infty \frac{1}{D} \sqrt{\zeta^2 - \kappa_2^2 \Omega^2} \{-\frac{1}{\zeta} \sin(\frac{h\zeta}{2l})\}^2 d\zeta \dots\dots\dots (31)$$

$$H = \int_0^\infty \frac{1}{D} \sqrt{\zeta^2 - \kappa_1^2 \Omega^2} \{-\frac{h}{2l} \cos(\frac{h\zeta}{2l}) + \frac{1}{\zeta} \sin(\frac{h\zeta}{2l})\} \frac{1}{\zeta} \sin(\frac{h\zeta}{2l}) d\zeta \dots\dots\dots (32)$$

$$J = \int_0^\infty \frac{1}{D} \sqrt{\zeta^2 - \kappa_2^2 \Omega^2} \{\zeta^2 \sqrt{\zeta^2 - \kappa_1^2 \Omega^2} - (\zeta^2 - \frac{\kappa_2^2}{2} \Omega^2) \sqrt{\zeta^2 - \kappa_2^2 \Omega^2}\} \{-\frac{1}{\zeta} \sin(\frac{h\zeta}{2l})\}^2 d\zeta \dots\dots\dots (33)$$

and

$$D = (\zeta^2 - \frac{\kappa_2^2}{2} \Omega^2)^2 - \zeta^2 \sqrt{\zeta^2 - \kappa_1^2 \Omega^2} \sqrt{\zeta^2 - \kappa_2^2 \Omega^2} \dots\dots\dots (34)$$

Applying equations (25) and (26) to equations (3) and (4), we get

$$v_R(0, t) = [D_1 \{\sin(kl) + \sinh(kl)\} + D_2 \{\cos(kl) + \cosh(kl)\} + \frac{P}{E_R I_R k^3} \sinh(kl)] e^{i\omega t} \dots\dots\dots (35)$$

$$\frac{\partial v_R(0, t)}{\partial x} = k [-D_1 \{\cos(kl) + \cosh(kl)\} + D_2 \{\sin(kl) - \sinh(kl)\} - \frac{P}{E_R I_R k^3} \cosh(kl)] e^{i\omega t} \dots\dots\dots (36)$$

By making equations (28) and (35), and equations (29) and (36) equivalent, the unknown functions D_1 and D_2 can be determined. The amplitude of a beam at $x = l$ is evaluated from equations (3), (4), (25) and (26) as

$$V_R(l) = 2D_2 \dots\dots\dots (37)$$

5. Numerical results

For the numerical examples, we consider three kinds of beam and half-space material combinations, namely;

(i) The beam is plastic and the half-space is aluminium.

4. Analysis

Substituting equations (3) and (4) into boundary conditions (12) and (13) gives

$$D_3 = D_1 + \frac{P}{E_R I_R k^3} \dots\dots\dots (25)$$

$$D_4 = D_2 \dots\dots\dots (26)$$

The unknown functions $A(\xi)$ and $B(\xi)$ can be determined from equations (10), (11), (19) and (20). Substituting $A(\xi)$ and $B(\xi)$ into equation (7), the mean displacement and mean slope can be calculated from equations (23) and (24). Introducing the following

(ii) The beam and the half-space are the same material. In this case, $E_R = E$, $\rho_R = \rho$ and we take Poisson's ratio $\nu = 0.3$.

(iii) The beam is aluminium and the half-space is plastic.

For illustrative purpose, the following physical constants are chosen:

	$E(N/m^2)$	$\rho(kg/m^3)$	ν
Aluminium	7.06×10^{10}	2.7×10^3	0.34
Plastic	3.92×10^9	1.3×10^3	0.36

In carrying out the integrations E , F , H and J , we use the Miller and Pursey treatment [5] and the Gauss approximate method.

The relationships between the amplitude of the free end of the beam at $x = l$ and the forcing frequency, i.e., the response curves, are obtained from equation (37). Figs. 2, 3 and 4 show the response curves for the three different examples (i), (ii) and (iii) and the slenderness ratio $l/h =$

Fig.2 Dynamic response curves of the free end of the beam. Beam is plastic and half-space is aluminium.

Fig.3 Dynamic response curves of the free end of the beam. Beam and half-space are the same material.

Fig.4 Dynamic response curves of the free end of the beam. Beam is aluminium and half-space is plastic.

20, 10, 5. In these figures, the forced angular frequency ω is nondimensionalized with a/l^2 and the amplitude $V_R(l)$ with the static deflection at the end of a beam produced by a load P for the case of a rigid support, i.e., classical boundary conditions. From these figures, we can see that the damping effect increases as the slenderness ratio decreases and the elastic modulus of the half-space becomes smaller than that of the beam. In the case where the beam is plastic and the half-space is aluminium, i.e., the beam is soft and the half-space is hard, the response curves as shown in Fig. 2, almost coincide with that of a cantilever beam with the classical boundary conditions within the first mode region. However, both the second resonant frequency and amplitude values decrease with a decreasing slenderness ratio. In Figs. 2, 3 and 4, the fine solid line curve depicts the case of a rigid half-space, i.e., the response curve of a cantilever beam with the classical boundary conditions. The difference between the first resonant frequency of the cantilever beam with the non-classical boundary conditions and that with the classical boundary conditions can be expressed as

$$\lambda = \frac{\omega_1 - \omega_2}{\omega_1} \dots \dots \dots (38)$$

where ω_2 is the first resonant frequency of the cantilever beam with the non-classical boundary conditions and ω_1 is that with the classical boundary conditions. In Fig. 5, the difference λ is given as a percentage.

Fig.5 Difference between the first resonant frequency of a cantilever beam with the non-classical boundary conditions and that with the classical boundary conditions. (i) Beam is plastic and half-space is aluminium. (ii) Beam and half-space are the same material. (iii) Beam is aluminium and half-space is plastic.

The difference increases when either or both of the slenderness ratio and the half-space elastic modulus over that of the beam

E/E_R decrease. In the case where the beam and the half-space are the same material and the slenderness ratio $l/h = 6$, the first resonant frequency value differs from that of a cantilever beam with the classical boundary conditions by 11.3 %; the experimental results of Rissone and Williams [6] differ from that of a cantilever beam with the classical boundary conditions by 13.3 % while the analytical results of MacBain and Gemin [4] do by 12.5 %.

6. Conclusions

The plane stress problem of forced flexural vibrations of an elastic beam supported by an elastic half-space and subjected to a sinusoidally varying force at the free end was investigated analytically. The dynamic effects of foundation stiffness and slenderness ratio on the response curves of the beam were clarified.

From these results, it can be said that the damping effect increases when either or both of the slenderness ratio and the half-space elastic modulus over that of the beam decrease.

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