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Problem of Rotor Passing Through Critical Speed with Gyroscopic Effect*

(Analysis by Asymptotic Method and Experiments)

By Kenzou NONAMI** and Masakazu MIYASHITA***

The nonstationary vibration of a rotor with a gyroscopic effect in passing through its critical speed is studied by using the asymptotic method. In particular, the influence of a gyroscopic effect in passing through a critical speed is studied in detail on two cases, i.e., one of a constant acceleration and the other considering the interaction between the driving source and the vibration system. The effect of an internal damping on the stability is also studied. The experiments support the analysis.

1. Introduction

The studies of a rotating shaft system in passing through its critical speed have been executed by many authors. The approaches to this problem are divided into two categories. One is the approach assuming that an energy source is not receiving any reaction from the vibration system, i.e., the assumption of constant acceleration. The other is the case that the characteristics of the energy source must be considered because of an appreciable interaction between the energy source and the vibration system. The former has been tried by many authors since Lewis⁽¹⁾, Goloskokow, Filippow⁽²⁾ and Yanabe⁽³⁾ have analyzed completely the dynamic characteristics of a linear vibration system with one degree of freedom by using the error function in complex region. Meuser, Weible⁽⁴⁾ and Mitropol'skii⁽⁵⁾ studied a nonlinear system with one degree of freedom. The latter was tried with a linear and a nonlinear system by Kononenko⁽⁶⁾. Matsuura⁽⁷⁾ studied the linear system with one degree of freedom and reported on the case of a rotating shaft not being able to pass through its critical speed. Kawai, Iwatsubo, Kanki⁽⁸⁾ reported that the amount of the shaft asymmetry is relative to the decrease of the damping. Kodera⁽⁹⁾ studied a flexible rotor system.

These investigations didn't take account of the gyroscopic effect and the effect of the rotor inclination of the rotating shaft. Mitropol'skii studied a vibration system with a gyroscopic effect as well as a nonstationary vibration problem systematically. However, the gyroscopic effect on the nonstationary vibration system has not been treated. Taniguchi, Sakata and Takenouchi⁽¹⁰⁾ investigated a system with the gyroscopic effect,

too. But its conclusion is not clear. The undamped system with a gyroscopic effect was studied by Aiba.⁽¹¹⁾

In this paper, the influence of the gyroscopic effect on the transition response through its critical speed is studied on two cases, i.e., one of a constant acceleration and the other with the interaction between the driving source and the vibration system. The analysis is made using the asymptotic method of Mitropol'skii. The experiments are done on three cases of the gyroscopic moment being positive, zero, and negative. The system with an internal damping are also studied using the method of Kononenko.

2. Notations

- x : x co-ordinate displacement of gravity center of the rotor
- y : y co-ordinate displacement of gravity center of the rotor
- θ_x : Inclination of the rotor in x direction
- θ_y : Inclination of the rotor in y direction
- α, γ, δ : Stiffness of the shaft
- I_x : Moment of inertia of the rotor in axial direction
- I : Moment of inertia of the rotor in diametrical direction
- m : Mass of the rotor
- φ : Revolution angle of the shaft
- e : Static unbalance of the rotor
- r : Dynamic unbalance of the rotor
- β : Phase angle between e and r
- c, c_e : Coefficient of the external damping
- c_i : Coefficient of the internal damping
- ω_n : Critical speed without a gyroscopic effect
- ω_{ng} : Critical speed with a gyroscopic effect
- $M-R\dot{\varphi}$: Driving torque
- $S\dot{\varphi}$: Resistant torque of driving source

3. Equation of motion

Fig.1 shows the co-ordinates system. z -O is a center line of both bearings. a^* is an amplitude and ψ is a difference of phase. For the sake of simplicity, the following assumption is set in the analysis, i.e., the rotor is rigid and has a

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** Postgraduate Course Student, Faculty of Engineering, Tokyo Metropolitan University, Setagaya-ku, Tokyo.

*** Professor, Faculty of Engineering, Tokyo Metropolitan University,

weightless shaft. Moreover, the external damping forces are assumed to be proportional to the velocity of the center of the rotor, while the internal damping forces assumed to be proportional to the velocity of the bending deformation of the shaft and the rotor. Then, the equation of motion for the center of the gravity of the rotor with a gyroscopic effect is expressed as follows:

$$\begin{aligned} m\ddot{x} + \alpha\dot{x} + \gamma\theta_x &= F_1 + \alpha e \cos \varphi, & m\ddot{y} + \alpha\dot{y} + \gamma\theta_y &= F_2 + \alpha e \sin \varphi \\ I\ddot{\theta}_x + I_p\dot{\theta}_x + \delta\theta_x + \gamma x &= F_3 + f_1 \cos \varphi + f_2 \sin \varphi \\ I\ddot{\theta}_y - I_p\dot{\theta}_y + \delta\theta_y + \gamma y &= F_4 + f_1 \sin \varphi - f_2 \cos \varphi \\ I_p\ddot{\psi} &= M - R\dot{\psi} - S\dot{\psi} + \tau(I_p - I)(\dot{\theta}_x \sin(\varphi + \beta) + \dot{\theta}_y \cos(\varphi + \beta)) \\ &+ \frac{1}{2} I_p (\dot{\theta}_x \dot{\theta}_y - \dot{\theta}_y \dot{\theta}_x) + m e \{ (\ddot{x} + e\dot{\psi} \sin \varphi + e\dot{\psi}^2 \cos \varphi) \sin \varphi \\ &- (\ddot{y} - e\dot{\psi} \cos \varphi + e\dot{\psi}^2 \sin \varphi) \cos \varphi \} - \alpha c_1 y (\dot{x} + \dot{\psi} y) - \\ &- \delta c_1 \theta_x (\dot{\theta}_x + \dot{\psi} \theta_y) + \alpha c_1 x (\dot{y} - \dot{\psi} x) + \delta c_1 \theta_y (\dot{\theta}_y - \dot{\psi} \theta_x) \\ &- \gamma c_1 (\dot{x} \theta_y + y \dot{\theta}_x + 2\dot{\psi} y \theta_y) + \gamma c_1 (\dot{y} \theta_x + x \dot{\theta}_y - 2\dot{\psi} x \theta_x) \end{aligned} \quad (1)$$

where

$$\begin{aligned} F_1 &= -c_1 \ddot{x} - \alpha c_1 (\dot{x} + \dot{\psi} y) - \gamma c_1 (\dot{\theta}_x + \dot{\psi} \theta_y) \\ F_2 &= -c_1 \ddot{y} - \alpha c_1 (\dot{y} - \dot{\psi} x) - \gamma c_1 (\dot{\theta}_y - \dot{\psi} \theta_x) \\ F_3 &= -\frac{1}{2} I_p \dot{\psi} \dot{\theta}_y - c_1 \dot{\theta}_x - \delta c_1 (\dot{\theta}_x + \dot{\psi} \theta_y) - \gamma c_1 (\dot{x} + \dot{\psi} y) \\ F_4 &= \frac{1}{2} I_p \dot{\psi} \dot{\theta}_x - c_1 \dot{\theta}_y - \delta c_1 (\dot{\theta}_y - \dot{\psi} \theta_x) - \gamma c_1 (\dot{y} - \dot{\psi} x) \\ f_1 &= \gamma e + \tau \dot{\psi} (I_p - I) \sin \beta + \tau \dot{\psi}^2 (I_p - I) \cos \beta \\ f_2 &= \tau \dot{\psi} (I_p - I) \cos \beta - \tau \dot{\psi}^2 (I_p - I) \sin \beta \end{aligned} \quad (2)$$

To transform the non-dimensional equations (1), (2) the following non-dimensional quantities are introduced:

$$\begin{aligned} x' &= x\sqrt{m/I}, & y' &= y\sqrt{m/I}, & e' &= e\sqrt{m/I}, & c_1' &= c_1/\sqrt{m\alpha}, \\ c_1' &= c_1\sqrt{\alpha/m}, & i_p &= I_p/I, & \dot{\psi}' &= \dot{\psi}\sqrt{m/\alpha}, & R' &= R/I\sqrt{\alpha/m}, \\ i' &= I\sqrt{\alpha/m}, & \gamma' &= \gamma\sqrt{m/I}/\alpha, & \delta' &= m\delta/\alpha I, & c_1^* &= c_1\sqrt{m/\alpha}/I, \\ X &= x'/e', & Y &= y'/e', & \theta_x &= \theta_x/e', & \theta_y &= \theta_y/e', & M' &= Mm/\alpha I, \\ S' &= S/I\sqrt{\alpha/m}, & M^* &= M'/e'^2, & R^* &= R'/e'^2, & S^* &= S'/e'^2, \\ \dot{\psi}' &= \dot{\psi}m/\alpha, & \mu &= \tau/e', & \rho &= i_p/e'^2, & \omega_{n0}' &= \omega_{n0}\sqrt{m/\alpha} \end{aligned} \quad (3)$$

Substituting these expressions into Eqs. (1), (2) the non-dimensional equations of motion are obtained.

$$\begin{aligned} \ddot{X} + X + \gamma'\theta_x &= F_1' + \cos \varphi \\ \ddot{Y} + Y + \gamma'\theta_y &= F_2' + \sin \varphi \\ \ddot{\theta}_x + i_p\dot{\theta}_x + \delta'\theta_x + \gamma'X &= F_3' + f_1' \cos \varphi + f_2' \sin \varphi \\ \ddot{\theta}_y - i_p\dot{\theta}_y + \delta'\theta_y + \gamma'Y &= F_4' + f_1' \sin \varphi - f_2' \cos \varphi \\ \rho\ddot{\psi}' &= M^* - R^*\dot{\psi}' - S^*\dot{\psi}' + \mu(i_p - 1)(\dot{\theta}_x \sin(\varphi + \beta) + \dot{\theta}_y \cos(\varphi + \beta)) \\ &+ \frac{1}{2} i_p (\dot{\theta}_x \dot{\theta}_y - \dot{\theta}_y \dot{\theta}_x) + (\ddot{X} + \dot{\psi}' \sin \varphi + \dot{\psi}'^2 \cos \varphi) \sin \varphi \\ &- (\ddot{Y} - \dot{\psi}' \cos \varphi + \dot{\psi}'^2 \sin \varphi) \cos \varphi - c_1' Y (\dot{X} + \dot{\psi}' Y) \\ &- c_1' \delta' \theta_y (\dot{\theta}_x + \dot{\psi}' \theta_y) + c_1' X (\dot{Y} - \dot{\psi}' X) + c_1' \delta' \theta_x (\dot{\theta}_y - \dot{\psi}' \theta_x) \\ &- \gamma' c_1' (\dot{X} \theta_y + Y \dot{\theta}_x + 2\dot{\psi}' Y \theta_y) + \gamma' c_1' (\dot{Y} \theta_x + X \dot{\theta}_y - 2\dot{\psi}' X \theta_x) \end{aligned} \quad (4)$$

where

Fig.1 co-ordinates system

$$\begin{aligned} F_1' &= -c_1' \ddot{X} - c_1' (\dot{X} + \dot{\psi}' Y) - \gamma' c_1' (\dot{\theta}_x + \dot{\psi}' \theta_y) \\ F_2' &= -c_1' \ddot{Y} - c_1' (\dot{Y} - \dot{\psi}' X) - \gamma' c_1' (\dot{\theta}_y - \dot{\psi}' \theta_x) \\ F_3' &= -\frac{1}{2} i_p \dot{\psi}' \dot{\theta}_y - c_1^* \dot{\theta}_x - \delta' c_1' (\dot{\theta}_x + \dot{\psi}' \theta_y) - \gamma' c_1' (\dot{X} + \dot{\psi}' Y) \\ F_4' &= \frac{1}{2} i_p \dot{\psi}' \dot{\theta}_x - c_1^* \dot{\theta}_y - \delta' c_1' (\dot{\theta}_y - \dot{\psi}' \theta_x) - \gamma' c_1' (\dot{Y} - \dot{\psi}' X) \\ f_1' &= \gamma' + \mu \dot{\psi}' (i_p - 1) \sin \beta + \mu \dot{\psi}'^2 (i_p - 1) \cos \beta \\ f_2' &= \mu \dot{\psi}' (i_p - 1) \cos \beta - \mu \dot{\psi}'^2 (i_p - 1) \sin \beta \end{aligned} \quad (5)$$

4. Approximate equation and numerical solution

4.1 Case of constant angular acceleration

In investigating the influence of a gyroscopic effect on the transition through a critical speed, the condition of a constant angular acceleration can be used, i.e., instead of the fifth equation of Eq.(4), it is assumed that $\dot{\psi}' = \text{constant}$. In this paper, the asymptotic method of Mitropol'skii can be used in place of the direct numerical integration of Eq.(4), (5). Now, using complex variables $Z_1 = X + iY$, $Z_2 = \theta_x + i\theta_y$, Eq.(4), (5) are transformed to

$$\begin{aligned} \ddot{Z}_1 + Z_1 + \gamma' Z_2 &= \bar{F}_1 + \exp(i\varphi) \\ \ddot{Z}_2 - i i_p \dot{Z}_2 + \delta' Z_2 + \gamma' Z_1 &= \bar{F}_2 + f_1' \exp(i\varphi) - i f_2' \exp(i\varphi) \end{aligned} \quad (6)$$

where

$$\begin{aligned} \bar{F}_1 &= -c_1' \ddot{Z}_1 - c_1' (\dot{Z}_1 - \dot{\psi}' Z_2) - \gamma' c_1' (\dot{Z}_2 - \dot{\psi}' Z_1) \\ \bar{F}_2 &= i \frac{1}{2} i_p \dot{\psi}' \dot{Z}_2 - c_1^* \dot{Z}_2 - \delta' c_1' (\dot{Z}_2 - i \dot{\psi}' Z_1) - \gamma' c_1' (\dot{Z}_1 - i \dot{\psi}' Z_2) \end{aligned} \quad (7)$$

In the above, it is assumed that the right hand sides of Eq.(6) are smaller than the left hand sides of it. In expression (6), considering an unperturbation system the following equations are obtained.

$$\ddot{Z}_1 + Z_1 + \gamma' Z_2 = 0, \quad \ddot{Z}_2 - i i_p \dot{Z}_2 + \delta' Z_2 + \gamma' Z_1 = 0 \quad (8)$$

In order to find out the solution of Eq.(8), let the solution of natural frequency be expressed in the form

$$Z_j = a \varphi_j \exp[i(\omega_{n0}'(K) t' + \phi)] \quad (j=1,2) \quad (9)$$

where K means the critical speed of the K th, a and ϕ are constants. Substituting Eq.(9) into Eq.(8), the normal functions are expressed as follows:

$$\varphi_1 = 1.0, \quad \varphi_2 = -(1 - (\omega_{n0}'(K))^2) / \gamma' \quad (10)$$

On the other hand, for the nonstationary vibrations of the transition through its critical speed $\omega_{n0}'(K)$ the solution can be written as

$$Z_j = a(t') \varphi_j \exp[i\{\varphi(t') + \phi(t')\}] \quad (j=1,2) \quad (11)$$

where $a(t')$ and $\phi(t')$ are decided by the following first approximate equations

$$\frac{da}{dt'} = \varepsilon A_1(t', a, \phi), \quad \frac{d\phi}{dt'} = \omega_{n0}'(K) - \dot{\psi}' + \varepsilon B_1(t', a, \phi) \quad (12)$$

Assuming the variation of $\dot{\psi}'$ is small, the first approximate equations can be written as

$$\frac{da}{dt'} = -\frac{c_s' \omega_{ns}'(K) + c_s'' \omega_{ns}'(K) (\varphi_2)^2 - (1/2) i_p \dot{\varphi}' (\varphi_2)^2}{2m_1 \omega_{ns}'(K) + m_2} a - \frac{(1+f_1' \varphi_2) \sin \phi + f_2' \varphi_2 \cos \phi}{m_1 (\omega_{ns}'(K) + \dot{\varphi}') + m_2}$$

$$\frac{d\phi}{dt'} = \omega_{ns}'(K) - \dot{\varphi}' - \frac{(1+f_1' \varphi_2) \cos \phi - f_2' \varphi_2 \sin \phi}{a [m_1 (\omega_{ns}'(K) + \dot{\varphi}') + m_2]} \quad (13)$$

where

$$m_1 = \{\varphi_1\}^2 + \{\varphi_2\}^2, \quad m_2 = -i_p \dot{\varphi}' (\varphi_2)^2 \quad (14)$$

Thus, Eqs.(6),(7) get the solution of Eq.(11), and the variables $a(t')$ and $\phi(t')$ are derived from Eq.(13). Since the approximate equation(13) cannot be solved analytically, they are solved numerically. Considering the stationary vibration for Eqs.(11) and (13), the following stationary solutions are obtained

$$a = \frac{\sqrt{K_1^2 + K_2^2}}{\sqrt{(\omega_{ns}'(K) - \dot{\varphi}')^2 + \delta_s^2}}$$

$$\phi = \text{Tan}^{-1}\left(-\frac{\delta_s}{\omega_{ns}'(K) - \dot{\varphi}'}\right) + \text{Tan}^{-1}\left(-\frac{K_2}{K_1}\right) \quad (15)$$

where

$$\delta_s = \frac{c_s' \omega_{ns}'(K) + c_s'' \omega_{ns}'(K) (\varphi_2)^2 + c_1' (\omega_{ns}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2)}{2m_1 \omega_{ns}'(K) + m_2}$$

$$K_1 = \frac{1+f_1' \varphi_2}{m_1 (\omega_{ns}'(K) + \dot{\varphi}') + m_2}, \quad K_2 = \frac{f_2' \varphi_2}{m_1 (\omega_{ns}'(K) + \dot{\varphi}') + m_2} \quad (16)$$

Coincidence of the stationary solution of Eqs.(15),(16) obtained by the asymptotic method with the strict solution of the stationary motion of Eqs.(6),(7) shows the first approximation of Eq.(13) is sufficient.

On the other hand, disregarding the gyroscopic effect in Eqs.(6),(7), the first approximate equation of the asymptotic solution is written with an internal damping omitted for simplicity as

$$\frac{da}{dt'} = -\frac{c_s'}{2} a - \frac{(\omega_{ns}')^2}{\omega_{ns}' + \dot{\varphi}'} \sin \phi$$

$$\frac{d\phi}{dt'} = \omega_{ns}' - \dot{\varphi}' - \frac{(\omega_{ns}')^2}{a (\omega_{ns}' + \dot{\varphi}')} \cos \phi \quad (17)$$

where natural frequency ω_{ns}' equals $\omega_n \sqrt{m/\alpha}$. These equations are the conventional approximation by the asymptotic method. The stationary solutions of the forced vibrations are expressed as follows:

Fig.2 Influence of β

$$a = \frac{(\omega_{ns}')^2}{\sqrt{[(\omega_{ns}')^2 - (\dot{\varphi}')^2]^2 + [c_s'/2(\omega_{ns}' + \dot{\varphi}')]^2}}$$

$$\phi = \text{Tan}^{-1}\left(-\frac{c_s'/2}{\omega_{ns}' - \dot{\varphi}'}\right) \quad (18)$$

In the neighborhood of the critical speed $\omega_{ns}' + \dot{\varphi}' = 2\omega_{ns}'$ and then Eq.(18) coincides with the exact solutions for the case without a gyroscopic effect.

4.4 Numerical calculations

The approximate Eq.(13) is calculated by the Runge-Kutta method for the initial condition of Eq.(15). The steady state at 0.9 times the critical speed is chosen as the initial condition.

Fig.2 shows the influence of β . The maximum amplitude in the transition through a critical speed is the largest at $\beta=0$ for $i_p > 1$ and $\beta=\pi$ for $i_p < 1$. It is caused by the gyroscopic moment and coincides with the characteristics of the stationary motion.

Fig.3 shows the influence of i_p . The approximate Eq.(13) has higher accuracy in the neighborhood of the resonance and has lower accuracy at the zone far from it. However, the characteristics of the phase difference in Fig.3 is similar to the curve of the direct numerical integration for the equation of motion qualitatively. Fig.3 shows that the decrease of i_p causes a decrease of the maximum amplitude but an increase of the phase difference. The whirling frequency almost coincides with the natural frequency as far as the phase difference increases and converges to the rotating speed of the shaft after the phase difference becomes constant. As for the damping of the amplitude, it is difficult to estimate it from Fig.3 directly. Taking account of the non-dimensional time t' required for getting $a/a_{max}=0.1$, $t'=989.7$ for $i_p=0.09$ and $t'=704.6$ for $i_p=2$, and then the small i_p gives a poor damping.

Fig.3 Influence of i_p

Fig.4 Comparison Eq.(13) and Eq.(17)

Fig.4 shows the comparison of Eq.(13) with the gyroscopic effect with Eq.(17) without it under the same condition. For $i_p=1$, they agree, because the gyroscopic effect doesn't happen. For $i_p=0.09$, a gyroscopic effect gives a smaller maximum amplitude, a larger phase difference and a poor damping. This tendency is remarkable for the greater acceleration. For $i_p=2$, the above tendency is reversed, but the effect is not so great. Considering the initial conditions do not give much influence on the maximum amplitude, the above phenomena are peculiar to the nonstationary vibration with a gyroscopic effect.

Fig.5 shows the influence of a gyroscopic effect on the maximum amplitude for the nonstationary vibrations. In the case of $\dot{\psi}'=0$, the maximum amplitude of the rotating shafts agrees between the cases with and without a gyroscopic effect. And the influence of acceleration $\dot{\psi}'$ causes great influence of the gyroscopic effect on nonstationary vibrations.

4.3 Case with interaction between driving source and vibration systems

Practically, the driving torque depends on the speed and there exists an interaction between the driving source and the vibration system. In this case, the angular acceleration of the rotor is decided by the fifth equation of Eq.(4). Using the normal function, this is transformed to

$$\ddot{\psi}' = \frac{1}{\rho} \left[\begin{aligned} &M^* - R^* \dot{\psi}' - S^* \dot{\psi}' - a(\omega_{ng}'(K))^2 \varphi_2 \mu (i_p - 1) (\cos 2\varphi \sin(\varphi + \beta)) \\ &+ \sin 2\varphi \cos(\varphi + \beta) - i_p a^2 (\omega_{ng}'(K))^2 (\varphi_2)^2 \sin 4\varphi \\ &+ a(\omega_{ng}'(K))^2 \sin \phi + c_1' a^2 (\omega_{ng}'(K) - \dot{\psi}') (1 + 2\gamma' \varphi_2 + \delta' (\varphi_2)^2) \end{aligned} \right] \quad (19)$$

Introducing the first approximate equation, it can be written as

$$\frac{da'}{dt'} = - \frac{c_1' \omega_{ng}'(K) + c_2' \omega_{ng}'(K) (\varphi_2)^2 + c_1' (\omega_{ng}'(K) - \dot{\psi}') (1 + 2\gamma' \varphi_2 + \delta' (\varphi_2)^2)}{2 m_1 \omega_{ng}'(K) + m_2} a' \quad (20)$$

Fig.5 Influence of gyroscopic effect

Fig. 6 Case of interaction

$$\begin{aligned} & - \frac{c_1' [(1 + f_1' \varphi_2) \sin \phi + f_2' \varphi_2 \cos \phi]}{m_1 (\omega_{ng}'(K) + \dot{\psi}') + m_2} \\ & \frac{d\phi}{dt'} = \omega_{ng}'(K) - \dot{\psi}' - \frac{e' [(1 + f_1' \varphi_2) \cos \phi - f_2' \varphi_2 \sin \phi]}{a' [m_1 (\omega_{ng}'(K) + \dot{\psi}') + m_2]} \\ & \frac{d\dot{\psi}'}{dt'} = \frac{1}{i_p} [M' - R' \dot{\psi}' - S' \dot{\psi}' + a' e' (\omega_{ng}'(K))^2 \sin \phi \\ & \quad + c_1' (a')^2 (\omega_{ng}'(K) - \dot{\psi}') (1 + 2\gamma' \varphi_2 + \delta' (\varphi_2)^2)] \end{aligned} \quad (20)$$

where $a' = ae'$.

4.4 Numerical calculations

In the case with an acceleration under the condition of the constant driving torque, the acceleration increases and the maximum amplitude tends to be smaller for smaller i_p . Fig.6 shows the characteristics of the amplitude, the phase difference and the rotating speed of the shaft for $i_p=0.09$ under the condition $(M' - R' \dot{\psi}' - S' \dot{\psi}')/i_p = \text{constant}$. Due to the interaction between the driving torque and the vibration systems, the speed characteristics varies. The acceleration becomes smaller in the transition to the maximum amplitude and becomes larger in the period of becoming minimum. These relations are more remarkable in with larger unbalance, larger R' and S' , and smaller M' . The typical examples are shown in Fig.7 and Fig.8. Fig.8 shows the examples of being able to pass through a critical speed and Fig.7 shows the examples of not being able to pass through it. Fig.9 shows the permissible unbalance enabling passage through a critical speed. According to Fig.9, the permissible unbalance e' increases for larger i_p .

5. Stability of rotating shaft with gyroscopic effects

5.1 Stability for cases disregarding driving source

Here, the influence of a gyroscopic effect on the stability is investigated under the condition of disregarding the

Fig.7

Fig.8

driving source. Using the first approximation of the asymptotic solution, the condition of the stability is introduced by the method of Bogoljubov⁽¹³⁾ etc. From the variation of Eq.(13), the characteristic equations are expressed as follows:

$$\lambda^2 - (A+B)\lambda + (AB-CD) = 0 \quad (21)$$

where

$$A = -\frac{c_1' \omega_{n2}'(K) + c_2' \omega_{n2}'(K) (\varphi_2)^2}{2m_1 \omega_{n2}'(K) - m_2} + \frac{c_1' (\omega_{n2}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2)}{2m_1 \omega_{n2}'(K) - m_2}$$

$$B = \frac{(1+f_1' \varphi_2) \sin \psi + f_2' \varphi_2 \cos \psi}{a[m_1 (\omega_{n2}'(K) + \dot{\varphi}') + m_2]}$$

$$C = \frac{(1+f_1' \varphi_2) \cos \psi - f_2' \varphi_2 \sin \psi}{a^2 [m_1 (\omega_{n2}'(K) + \dot{\varphi}') + m_2]}$$

$$D = -\frac{(1+f_1' \varphi_2) \cos \psi - f_2' \varphi_2 \sin \psi}{m_1 (\omega_{n2}'(K) + \dot{\varphi}') + m_2} \quad (22)$$

So, from Eq.(21), the conditions of stability are as follows:

$$\frac{c_1' \omega_{n2}'(K) + c_2' \omega_{n2}'(K) (\varphi_2)^2}{2m_1 \omega_{n2}'(K) - m_2} + \frac{c_1' (\omega_{n2}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2)}{2m_1 \omega_{n2}'(K) - m_2} > 0 \quad (23)$$

where in the ordinary speed range, it is satisfied that $2m_1 \omega_{n2}'(K) + m_2 > 0$, and considering $c_1' \omega_{n2}'(K) (\varphi_2)^2 \ll c_1' \omega_{n2}'(K)$, $c_2' \omega_{n2}'(K) (\varphi_2)^2$ is reduced and Eq.(23) is written as

$$c_1' \omega_{n2}'(K) + c_1' (\omega_{n2}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2) > 0 \quad (24)$$

Now, $1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2$ is a function of the natural frequency $\omega_{n2}'(K)$ and this is put as $f[\omega_{n2}'(K)]$

$$\frac{df[\omega_{n2}'(K)]}{d\omega_{n2}'(K)} = 4\omega_{n2}'(K) \left[1 - \frac{\delta' (1 - (\omega_{n2}'(K))^2)}{\gamma'^2} \right]$$

$$\text{for } \frac{df[\omega_{n2}'(K)]}{d\omega_{n2}'(K)} = 0, \quad (\omega_{n2}'(K))^2 = \delta' - \gamma'^2 = \frac{\alpha - \gamma'^2}{m}$$

$$\frac{d^2 f[\omega_{n2}'(K)]}{[d\omega_{n2}'(K)]^2} > 0, \quad f[\omega_{n2}'(K)] = 1 - \frac{\gamma'^2}{\delta'} > 0$$

therefore, $f[\omega_{n2}'(K)]$ has the minimum in $(\omega_{n2}'(K))^2 = (\alpha - \gamma'^2)/m$ and satisfies the condition of $f[\omega_{n2}'(K)] > 0$ for the whole $\omega_{n2}'(K)$. That is the conditions of stability are obtained as follows

$$(i) \text{ for } \omega_{n2}'(K) > \dot{\varphi}', \quad \frac{c_1'}{c_2'} > -\frac{\omega_{n2}'(K)}{(\omega_{n2}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2)}$$

$$(ii) \text{ for } \omega_{n2}'(K) < \dot{\varphi}', \quad \frac{c_1'}{c_2'} < -\frac{\omega_{n2}'(K)}{(\omega_{n2}'(K) - \dot{\varphi}') (1+2\gamma' \varphi_2 + \delta' (\varphi_2)^2)} \quad (25)$$

In the case with no internal damping, the system are always stable. Fig.10 shows the stable speed boundaries become lower too. They coincide with the stability of the systems with one degree of freedom without a gyroscopic effect qualitatively.

Fig.9 Relation of e' and i ,

Fig.10 Stability for the disregarding the driving source

5.2 Stability of case considering interaction between driving source and vibration system

Next, in the case considering the driving system, the stability of the stationary motion is investigated. Using the method of Kononenko, Eq.(20) is written as

$$\frac{da'}{dt'} = \Phi_1(a', \psi, Q), \quad \frac{d\psi}{dt'} = \Phi_2(a', \psi, Q), \quad \frac{dQ}{dt} = \Phi_3(a', \psi, Q) \quad (26)$$

where

$$a' = a_0 + \xi, \quad \psi = \psi_0 + \eta, \quad Q = Q_0 + \zeta \quad (27)$$

and a_0, ψ_0, Q_0 are values of a, ψ, Q for the stationary motion, i.e., roots of the following equation

$$\Phi_i(a_0, \psi_0, Q_0) = 0 \quad (i=1,2,3) \quad (28)$$

ξ, η, ζ are small perturbations of the quantities a_0, ψ_0, Q_0 from their values for the stationary motion. Substituting Eq.(27) in Eq.(26), expanding it into a power series of the small quantities ξ, η, ζ and taking only the linear terms of the expansion, the matrix equations are obtained as follows.

$$\frac{d}{dt'} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad b_{ik} = \begin{bmatrix} \frac{\partial}{\partial a'} \Phi_1 & \frac{\partial}{\partial \psi} \Phi_1 & \frac{\partial}{\partial Q} \Phi_1 \\ \frac{\partial}{\partial a'} \Phi_2 & \frac{\partial}{\partial \psi} \Phi_2 & \frac{\partial}{\partial Q} \Phi_2 \\ \frac{\partial}{\partial a'} \Phi_3 & \frac{\partial}{\partial \psi} \Phi_3 & \frac{\partial}{\partial Q} \Phi_3 \end{bmatrix} \quad (29)$$

Then, the characteristic equation of the system is

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \quad (30)$$

where

$$B_1 = -(b_{11} + b_{22} + b_{33})$$

$$B_2 = b_{11}b_{22} + b_{22}b_{33} + b_{33}b_{11} - b_{12}b_{21} - b_{23}b_{32} - b_{31}b_{13}$$

$$B_3 = b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} + b_{13}b_{31}b_{22} - b_{11}b_{22}b_{33} - b_{12}b_{23}b_{31} - b_{13}b_{32}b_{21} \quad (31)$$

On the other hand, the matrix coefficients b_{ik} are expressed as follows.

$$b_{11} = -\frac{A_1 + A_2 (\omega_{n2}'(K) - Q)}{2m_1 \omega_{n2}'(K) - i_p (\varphi_2)^2 Q}, \quad b_{12} = \frac{Q^2 A_3 \sin \psi - (A_3 + Q^2 A_4) \cos \psi}{m_1 (\omega_{n2}'(K) + Q) - i_p (\varphi_2)^2 Q}$$

$$b_{13} = \frac{A_2 a' (2m_1 \omega_{n2}'(K) - i_p (\varphi_2)^2 Q) - a' (A_1 + A_2 (\omega_{n2}'(K) - Q)) i_p (\varphi_2)^2}{(2m_1 \omega_{n2}'(K) - i_p (\varphi_2)^2 Q)^2}$$

$$+ \frac{((m_1 - i_p (\varphi_2)^2) ((A_3 + Q^2 A_4) \sin \psi + Q^2 A_5 \cos \psi) - 2Q (A_4 \sin \psi + A_5 \cos \psi) [m_1 (\omega_{n2}'(K) + Q) - i_p (\varphi_2)^2 Q])}{[m_1 (\omega_{n2}'(K) + Q) - i_p (\varphi_2)^2 Q]^2}$$

$$b_{21} = \frac{(A_3 + Q^2 A_4) \cos \psi - Q^2 A_5 \sin \psi}{(a')^2 [m_1 (\omega_{n2}'(K) + Q) - i_p (\varphi_2)^2 Q]}$$

$$b_{22} = \frac{(A_3 + Q^2 A_4) \sin \psi + Q^2 A_5 \cos \psi}{a' [m_1 (\omega_{n2}'(K) + Q) - i_p (\varphi_2)^2 Q]}$$

Fig.11 Stability chart for $i_p=0.09$

$$b_{23} = -1 - \frac{(2Q(A_4 \cos \phi - A_5 \sin \phi)[m_1(\omega_{ng}'(K) + Q) - i_p Q(\varphi_2)^2] - (A_3 + Q^2 A_4) \cos \phi - Q^2 A_5 \sin \phi)[m_1 - i_p(\varphi_2)^2]}{a'[m_1(\omega_{ng}'(K) + Q) - i_p(\varphi_2)^2 Q]^2}$$

$$b_{31} = \frac{1}{i_p} [e' \{ \omega_{ng}'(K) \}^2 \sin \phi + 2a' A_2(\omega_{ng}'(K) - Q)]$$

$$b_{32} = \frac{1}{i_p} [a' e' \{ \omega_{ng}'(K) \}^2 \cos \phi], \quad b_{33} = \frac{1}{i_p} [-R' - S' - A_2(a')^2]$$

$$A_1 = c_s' \omega_{ng}'(K) + c_s' \omega_{ng}'(K) (\varphi_2)^2, \quad A_2 = c_s' (1 + 2\gamma' \varphi_2 + \delta' (\varphi_2)^2)$$

$$A_3 = e' + \gamma' e' \varphi_2, \quad A_4 = \tau \varphi_2 (i_p - 1) \cos \beta, \quad A_5 = -\tau \varphi_2 (i_p - 1) \sin \beta$$

From the characteristic equation(30), the discriminations for the stability are as follows. Fig.11 shows the stability of $e'=0.00021, 0.00084, 0.00147$ and the case disregarding the driving source. The unstable regions exist in the neighborhood of the first critical speed $\omega_{ng}'(K)=0.351$, not depending on the amount of unbalance. In upper region above this, the unstable region caused by the interaction appears clearly. Thus, the increase of the unbalance e' , expands the unstable region. An increase of R' in the characteristics of the driving torque, shrinks the region. In Fig.12, R' equals 22 times that in Fig.11. In this case, it is known that unstable regions don't exist in the lower range of the neighborhood $\phi'=0.53$ except of the first critical speed and even a little internal damping causes a self-excited vibration in the upper range of $\phi'=0.53$. Similarly, Fig.13 shows the case of $i_p=2$. In Fig.11 to Fig.13, the coefficient of external damping c_s' is 0.00632. The larger the value of c_s' , the wider the stable regions.

According to these results, in the presence of an internal damping, the unstable region increases with a decreasing of i_p and the self-excited vibrations are likely to occur with it.

Unstable vibrations in transition through the critical speed are shown in Fig.14. This is the result of calculation of the parameter $i_p=2, c_s'=0.00395, c_t'=0.16$ and shows that the speed increases with time, but the amplitude doesn't decrease.

6. Experiments

6.1 Test rig and experimental method

Fig.15 shows the layout of the experiments. The hydraulic motor has 2 PS and a variable speed range from 0 to 4000 rpm. (1) The constant angular acceleration is realized by the manual speed control under the full supply pressure; (2) the interaction between driver and vibration

Fig.12 Stability chart for

$i_p=0.09$

Fig.13 Stability chart for $i_p=2$

system is introduced by operating the changing valve instantaneously. The displacement and the inclination of the rotor are measured by a capacitive sensor and are recorded on an electromagnetic oscillograph, time mark of rotating shaft being recorded also. The accelerometers mounted on the upper and lower base plates supporting the rotation shaft show the sufficient rigidity of the test rig. The rotor is supported by two bearings with span length of 400 and has an overhang of 100 for introducing a sufficient gyroscopic effect.

Table.1 shows the dimensions of the three kinds of rotors. Every rotor is of soft steel and its weight is 1.57 kg. They are mounted on the shaft by the shrinkage fit or the force fit. For getting the same bending stiffness of the shaft, the rotors No.2 and No.3 are hollowed to the center of the gravity.

6.2 Experimental results

6.2.1 Vibration of inclination

Comparing the lateral displacement with the inclination in transition through a critical speed, their characteristics coincide exactly. This fact shows the

Fig.14 Case of the system with an internal damping

Fig.15 Test rig

Table.1 Dimensions of rotors

	No.1	No.2	No.3
I_p kg mms ²	0.8	0.088	0.032
I kg mms ²	0.4	0.088	0.357
ω_{ng} rad/s	167	141	125
$D\phi$	200	66.5	40
B	6.4	58	160

synchronous motion of the whirling of the rotor.

6.2.2 Influence of gyroscopic moment

Fig.16 shows the experimental results of the rotor No.1 of which the gyroscopic moment is positive and of the rotor No.2 of which the gyroscopic moment is zero during acceleration. It shows that the maximum amplitude of the No.1 is larger than that of the No.2 and the instantaneous angular velocity of the No.1 is also larger than that of the No.2. Fig.17 shows the dynamical behavior of the rotor No.2 with a gyroscopic moment being zero and of the rotor No.3 with the gyroscopic moment being negative during deceleration. The lateral displacement is shown in Fig.17. It is seen that the damping of amplitude of the No.3 is poor, while the maximum amplitude of the No.2 is larger than that of the No.3. The tendency of these curves agrees with that of analytical results of Fig.3.

Fig.18 shows that a self-excited and unstable vibration occurs immediately after passing through its critical speed for the angular acceleration of 85 rad/s. It seems that these phenomena are caused by the poor damping due to a gyroscopic effect and the internal damping of the rotating shaft system.

6.2.3 Case of interaction

Fig.19 shows the test examples of the rotor No.1 passing through its critical speed for the supply pressure $P=35$ kg/cm, 60 kg/cm, 90 kg/cm. For $P=90$, the angular velocity variation doesn't occur, in the range up to the maximum speed. For $P=60$ and $P=35$, the angular velocity variation occurs. In particular, for $P=35$, the

Fig.16 Influence of a gyroscopic moment

interaction appears remarkably. In this case, the maximum amplitude is larger than that for $P=90$ and it requires longer time for reaching the maximum amplitude.

7. Conclusions

In this paper, the analysis and experiments in transition through a critical speed are made. They are summarized as follows.

- (1) The characteristics in passing through a critical speed of the rotating shaft system with a gyroscopic effect are similar to those of the linear system without a gyroscopic effect. But, under the condition of constant acceleration, its characteristics depend on the moment of inertia ratio i . With a decreasing i , its maximum amplitude in passing through a critical speed decreases and its instantaneous speed decreases, too. Mean while, its phase difference increases and the damping decreases along with a decreasing i .
- (2) Comparing between the system with a gyroscopic effect and the system without it under the identical condition, they are identical for $i=1$. For $i<1$, the maximum amplitude of the former is smaller than that of the latter, and the damped degree

Fig.17 Influence of a gyroscopic moment

Fig.18 Unstable vibration for the rotor of the No.3

Fig.19 Case of interaction

of amplitude of the former is larger. For $i_p > 1$, their characteristics become inverse the case of $i_p < 1$.

(3) Under the condition that the variation in driving torque characteristics equals the variation of i_p , with a decrease of i_p , the allowable unbalance quantity which can pass through a critical speed becomes small.

(4) In the system with an internal damping, unstable regions increase with a decrease of i_p , and a self-excited vibration is apt to happen.

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